# On essential sets of function algebras in terms of their orthogonal measures

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*Abstract.* In the present note, we characterize the essential set of a function algebra defined on a compact Hausdorff space $X$ in terms of its orthogonal measures on $X$.

*Keywords:* compact Hausdorff space $X$, the sup-norm algebra $C(X)$ of all complex-valued continuous functions on $X$, its closed subalgebras (called function algebras), measure orthogonal to a function algebra

*Classification:* 46J10

Let $X$ be a compact Hausdorff topological space. Denote by $C(X)$ the commutative Banach algebra, consisting of all continuous complex-valued functions on $X$ (with respect to usual point-wise algebraic operations) endowed with the sup-norm.

By a *function algebra* on $X$ we mean any closed subalgebra of $C(X)$ which contains constant functions on $X$ and which separates points of $X$.

**Definition.** A function algebra $A$ on $X$ is said to be a *maximal* one if it is a proper subset (i.e., a proper subalgebra) of $C(X)$ and has the following property: whenever $B$ is a function algebra on $X$, $B \supset A$, then either $B = A$ or $B = C(X)$.

$A$ being a function algebra on $X$, the closed subset $E$ is said to be an *essential set of* $A$ if the following conditions are fulfilled:

(*) $A$ consists of all continuous prolongations of functions in the algebra of restrictions $A/E$ (i.e., the algebra of all restrictions of functions in $A$ from the set $X$ to its subset $E$).

(**) Whenever a closed subset $F$ of $X$ has the same property as $E$ in (*), then $E \subset F$ (or, $E$ is a unique minimal closed subset of $X$ satisfying the condition (*)).

The notion "essential set" is due to Bear, who proved in [1] that any maximal algebra on $X$ has an essential set.

Hoffman and Singer in [2] found an essential set of any, not necessarily maximal, function algebra on $X$.

Denote by $M(X)$ the space of all complex Borel regular measures on $X$, i.e., by the Riesz Representation Theorem, the dual space of $C(X)$.

The *annihilator* $A^{\perp}$ of a function algebra $A$ is defined to be the set of all measures $m \in M(X)$ such that $\int f\,dm = 0$ for any $f \in A$, or the set of all measures *orthogonal* to $A$. The dual space $A'$ of $A$ is then canonically isomorphic to the quotient space $M(X)/A^{\perp}$.

Now endow $M(X)$ with the weak-star topology: it is well known that $M(X)$ becomes a locally convex topological linear space with the dual space $C(X)$.

Our aim here is to characterize the essential set of a function algebra $A$ by means of the properties of the measures in $A^{\perp}$. Remark that our construction is rather simpler than the classical one.

**Theorem.** *Let $A$ be a function algebra on $X$. Denote by $E$ the closure of the union of all closed supports of measures in $A^{\perp}$. Then $E$ is the essential set of $A$.*

PROOF: Let $f \in C(X)$, $g \in A$ and let $f/E = g/E$, where $f/E$ denotes the restriction of the function $f$ from $X$ to $E$. If for $m \in A^{\perp}$ we denote $M = \operatorname{spt}(m)$, then

$$\int f\,dm = \int_M f\,dm = \int_M g\,dm = \int g\,dm = 0,$$

hence $f$ is orthogonal to $A^{\perp}$ and, by Banach theorem, $f \in A$. It means that $E$ has the property (*) from Definition.

Now let a closed subset $K$ have the property (*); we shall prove that $K \supset E$. Suppose that $K \not\supset E$. Then there is a measure $m \in A^{\perp}$, whose closed support is not a subset of $K$. Take $x \in \operatorname{spt}(m) \setminus K$. Let $V$ be an open neighbourhood of $x$ in $X$ such that its closure $\overline{V}$ is disjoint with $K$. We shall find a function $f \in C(\overline{V})$ which fulfills the following two conditions:

$$\operatorname{spt}(f) \subset V, \ \int_V f\,dm \neq 0,$$

where $\operatorname{spt}(f)$ means the closed support of $f$. Denote by $g$ such a function in $C(X)$, which is equal to $f$ on $\overline{V}$ and equal to 0 off $\overline{V}$. Then $g/K = 0 \in A/K$, but

$$\int g\,dm = \int_V g\,dm = \int_V f\,dm \neq 0$$

and then $g \not\perp A^{\perp}$, so $g \notin A$. It follows that $K$ has not the property (*). □

Now the following question arises: whether the word "closure" in Theorem may be omitted, or whether the essential set $E$ of a function algebra $A$ on $X$ is composed of the union of closed supports of all measures in $A^{\perp}$, without closure. We shall show that it is true if $X$ is a metric space (Proposition), but in general it is not the case (Example).

**Proposition.** *Let $X$ be a compact metric space, $A$ a function algebra on $X$. Then the essential set $E$ of $A$ is equal to the union of closed supports of all measures in $A^{\perp}$.* (*Especially, the union of closed supports of all orthogonal measures is a closed set.*)

PROOF: Let $x \in E$. We shall find the measure $m \in A^{\perp}$ such that $\operatorname{spt}(m) \ni x$. Denote by $U_n$, $n = 1, 2, \dots$, the open balls in $X$ with centres at $x$ and radii $\frac{1}{n}$. We shall construct a finite or infinite sequence of measures $m_n \in A^{\perp}$ such that

$$|m_n|(X) \leq 1, \tag{1}$$

$$(\operatorname{spt}(m_n) \setminus \bigcup_{k=1}^{n-1} \operatorname{spt}(m_k)) \cap U_n \stackrel{def}{=} M_n \neq \emptyset \text{ and then } |m_n|(M_n) > 0, \tag{2}$$

$$|m_n|(X) < \min_{1 \leq k \leq n-1} |m_k|(M_k), \tag{3}$$

where $|m|$ means a total variation of a measure $m$.

By the Theorem, we can find a measure $m_1 \in A^{\perp}$ such that $|m_1|(X) = 1$ for which $\operatorname{spt}(m_1) \cap U_1 \neq \emptyset$. If $x \in \operatorname{spt}(m_1)$, the proof is finished. If it is not the case, then, by the Theorem, there exists the measure $m_2 \in A^{\perp}$ such that $(\operatorname{spt}(m_2) \setminus \operatorname{spt}(m_1)) \cap U_2 \neq \emptyset$; (2) follows. Multiplying $m_2$ by a small enough nonzero constant, we can reach fulfilling (1) and (3). If $x \in \operatorname{spt}(m_2)$, we are done. In the opposite case, we shall continue the construction ...

In the case the sequence $\{m_n\}$ is finite, the proof is finished. If it is not the case, put

$$m = \sum_{n=1}^{\infty} \frac{1}{2^n} m_n.$$

By (1), it is $m \in M(X)$. Also $m \in A^{\perp}$ because $m_n \perp A$.

Take an arbitrary $n$. By (2), it is $|m_n|(M_n) > 0$, while $|m_k|(M_n) = 0$ for $1 \leq k \leq n-1$. By (3), it is

$$|m|(M_n) = |\sum_{k=n}^{\infty} \frac{1}{2^k} m_k(M_n)| \geq \frac{1}{2^n}|m_n|(M_n) - \sum_{k=n+1}^{\infty} \frac{1}{2^k}|m_k|(X) \geq$$

$$\geq \frac{1}{2^n}|m_n|(M_n) - \sum_{k=n+1}^{\infty} \frac{1}{2^k}|m_k|(X) > \frac{1}{2^n}|m_n|(M_n) - \frac{1}{2^n}|m_n|(M_n) = 0$$

and then $\operatorname{spt}(m) \cap U_n \neq \emptyset$. Since $n$ was arbitrary, Proposition follows. □

Now, we shall construct a function algebra $A$ on $X$ such that there exists a point $x \in E$ which is not contained in the closed support of any measure in $A^{\perp}$.

**Example.** *Let us denote by $\omega_1$ the first uncountable ordinal number, put*

$$\Omega = \{\omega \text{ ordinal; } \omega \le \omega_1\},$$

*let $C$ be the closed unit disk in the complex plane. Denote by $Y$ the cartesian product $C \times \Omega$ and let $X$ arises from $Y$ by "collapsing" the "last disk" $C \times \{\omega_1\}$ into one point, say $x_1$, i.e., $X = Y/C \times \{\omega_1\}$. Let the algebra $A$ consist of all functions $f$ continuous on $X$ such that, for a fixed ordinal $\omega$, $\omega < \omega_1$, the function $z \mapsto f(z, \omega)$ is holomorphic in $|z| < 1$. Then the singleton $\{x_1\}$ does not meet the closed support of any measure from $A^\perp$, while the essential set $E$ of $A$ is whole $X$.*

PROOF: (1) Any function $f \in C(\Omega)$ is constant on a neighbourhood of $\omega_1$.

Let us suppose that $f(\omega_1) = 0$. Put, for natural $n$,

$$U_n = \{\omega \text{ ordinal}; \omega \le \omega_1 \ , |f(\omega)| < \frac{1}{n}\},$$
$$\omega^n = \sup \{\Omega \setminus U_n\},\ \omega^0 = \sup_n \omega^n.$$

It follows from the properties of ordinal numbers that $\omega^n < \omega_1$, so $\omega^0 < \omega_1$, and $f = 0$ identically on the "ordinal interval" $[\omega^0, \omega_1]$.

(2) Any function in $C(X)$ is constant on some neighbourhood of $x_1$: this follows from (1).

(3) If $m \in A^\perp$ then $\operatorname{spt}(m) \cap \{x_1\} = \emptyset$.
Let $m \in A^\perp$. Then the ordinal

$$\omega_2 = \sup\{\omega \text{ ordinal; } \omega < \omega_1, (z, \omega) \in \operatorname{spt}(m) \text{ for some } \ z,\ z \in C\}$$

is less than $\omega_1$. Now let $f \in A$ be a function which is equal to 0 on the set $S = (C \times [1, \omega_2])$ and equal to 1 on $X \setminus S$. If the measure $m$ contains a nonzero multiple of the one-point mass at $\{x_1\}$, it does not annihilate $f$, a contradiction. It follows that $\operatorname{spt}(m) \subset S$.

(4) Any "non-collapsed" disk supports the measure $m \in M(X)$ for which

$$\int f\, dm = \int_0^1 \int_{C_r(0)} f(z)\, dz\, dr$$

where $C_r(0) = re^{it}$ for $t \in [0, 2\pi]$, $0 < r \le 1$. But $\int f\, dm = 0$, by the classical Cauchy Integral Theorem, and $m \in A^\perp$. The union of such disks is $X \setminus \{x_1\}$.

$\square$

## References


[1] Bear H.S., *Complex function algebras*, Trans. Amer. Math. Soc. **90** (1959), 383–393.

[2] Hoffman K., Singer I.M., *Maximal algebras of continuous functions*, Acta Math. **103** (1960), 217–241.



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(*Received* December 21, 1994)