Existence and non-existence of global solutions for nonlinear hyperbolic equations of higher order*

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Abstract. The existence and uniqueness of classical global solution and blow up of non-global solution to the first boundary value problem and the second boundary value problem for the equation

$$u_{tt} - \alpha u_{xx} - \beta u_{xxtt} = \varphi(u_x)_x$$

are proved. Finally, the results of the above problem are applied to the equation arising from nonlinear waves in elastic rods

$$u_{tt} - \left[a_0 + na_1(u_x)^{n-1}\right] u_{xx} - a_2 u_{xxtt} = 0.$$

Keywords: nonlinear hyperbolic equation, initial boundary value problem, classical global solution, blow up of solutions

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1. Introduction

In the study of strain solitary waves in nonlinear elastic rods there exists a longitudinal wave equation [1], [2]

$$u_{tt} - \left[a_0 + na_1(u_x)^{n-1}\right] u_{xx} - a_2 u_{xxtt} = 0, \tag{1.1}$$

where $a_0, a_2 > 0$ are constants, a_1 is an arbitrary real number, n is a natural number. In [1], [2] the equation (1.1) is reduced approximately to KdV equation

$$u_t + uu_x + \mu u_{xxx} = 0, \tag{1.2}$$

where μ is a constant. In [2], authors study the strain solitary waves of equation (1.2), but about the equation (1.1) there has not been any discussion. Obviously, the equation (1.1) is different from the equation (1.2). There are few results in dealing with the equation (1.1). The existence and uniqueness of the local classical solutions for the initial value problems and the first boundary value problems of the equation (1.1) have been proved by Galerkin's method in [3].

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In the present paper, we are going to consider the following initial boundary value problem

$$u_{tt} - \alpha u_{xx} - \beta u_{xxtt} = \varphi(u_x)_x, \qquad x \in (0,1),\ t > 0, \tag{1.3}$$

$$u(0,t) = u(1,t) = 0, \qquad t \geq 0, \tag{1.4}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in [0,1], \tag{1.5}$$

or the initial boundary value problem for the equation (1.3)

$$u_x(0,t) = u_x(1,t) = 0, \qquad t \geq 0, \tag{1.6}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in [0,1], \tag{1.7}$$

where $u(x,t)$ is an unknown function, $\alpha, \beta > 0$ are real constants, $\varphi(s)$ is a given nonlinear function, $u_0(x)$ and $u_1(x)$ are given initial functions. Obviously, the equation (1.1) is a special case of the equation (1.3).

First of all, we reduce the problem (1.3)–(1.5) to an equivalent integral equation by Green's function of a boundary value problem for a second order ordinary differential equation, then making use of the contraction mapping principle we prove the existence and uniqueness of the local classical solutions for the integral equation in Section 2. In Section 3, under some conditions by use of a priori estimations of the solution we prove that the integral equation has a unique classical global solution, i.e. the problem (1.3)–(1.5) has a unique classical global solution. In Section 4 the conditions of non-existence of global solutions are given. The existence and non-existence theorems for the problem (1.3), (1.6), (1.7) are given in Section 5. In Section 6 the existence and non-existence theorems for the problem (1.1), (1.4), (1.5) and the problem (1.1), (1.6), (1.7) are given.

2. Existence and uniqueness of local solution for the problem (1.3)–(1.5)

Let $K(x,\xi)$ be the Green's function of the boundary value problem for the ordinary differential equation

$$y - \beta y'' = 0, \qquad y(0) = y(1) = 0,$$

where $\beta > 0$ is a real number, i.e.

$$K(x,\xi) = \frac{1}{\sqrt{\beta}\sinh\left[\frac{1}{\sqrt{\beta}}\right]} \begin{cases} \sinh\left[\frac{1}{\sqrt{\beta}}(1-\xi)\right]\sinh\left[\frac{1}{\sqrt{\beta}}x\right], & 0 \leq x \leq \xi, \\ \sinh\left[\frac{1}{\sqrt{\beta}}\xi\right]\sinh\left[\frac{1}{\sqrt{\beta}}(1-x)\right], & \xi \leq x \leq 1. \end{cases} \tag{2.1}$$

Suppose that $u_0(x)$ and $u_1(x)$ are appropriately smooth and satisfy the boundary condition (1.4), $u(x,t)$ is the classical solution of the problem (1.3)–(1.5), then the solution of the equation (1.3) satisfying the condition (1.4) satisfies the integral equation

$$u_{tt}(x,t) = \alpha \int_0^1 K(x,\xi) u_{\xi\xi}(\xi,t)\, d\xi + \int_0^1 K(x,\xi)\varphi(u_\xi(\xi,t))_\xi \, d\xi. \tag{2.2}$$

Hence the classical solution of the problem (1.3)–(1.5) should satisfy the integral equation

$$\begin{aligned} u(x,t) = u_0(x) + u_1(x)t &+ \alpha \int_0^t \int_0^1 (t-\tau) K(x,\xi) u_{\xi\xi}(\xi,\tau)\, d\xi\, d\tau \\ &+ \int_0^t \int_0^1 (t-\tau) K(x,\xi) \varphi(u_\xi(\xi,\tau))_\xi \, d\xi\, d\tau. \end{aligned} \tag{2.3}$$

Therefore any classical solution of the initial boundary value problem (1.3)–(1.5) is the solution of the integral equation (2.3). By use of the properties of Green's function $K(x,\xi)$, it is easy to prove the following lemma.

Lemma 2.1. *Suppose that* $u_0(x), u_1(x) \in C^2[0,1]$, $u_0(0) = u_0(1) = u_1(0) = u_1(1) = 0$, $\varphi(s) \in C^1(R)$, *and* $u(x,t) \in C([0,T]; C^2[0,1])$ *is the solution of* (2.3), *then* $u(x,t)$ *must be the classical solution of the initial boundary value problem* (1.3)–(1.5).

Now we are going to prove the existence of local classical solution for the integral equation (2.3) by the contraction mapping principle. For this purpose, we define the function space

$$X(T) = \Big\{ u(x,t) \,|\, u \in C([0,T]; C^2[0,1]),\ u(0,t) = u(1,t) = 0 \Big\},$$

equipped with the norm defined by

$$\begin{aligned} \|u\|_{X(T)} &= \max_{0 \le t \le T} \left\{ \max_{0 \le x \le 1} |u_x(\cdot,t)| + \max_{0 \le x \le 1} |u_{xx}(\cdot,t)| \right\} \\ &= \|u\|_{C([0,T];C^2[0,1])}, \qquad \forall\, u \in X(T). \end{aligned}$$

It is easy to see that $X(t)$ is a Banach space.

Let $U = \|u_{0x}\|_{C^1[0,1]} + \|u_{1x}\|_{C^1[0,1]}$. We define the set

$$P(U,T) = \Big\{ u \,|\, u \in X(T),\ \|u\|_{X(T)} \le 2U + 1 \Big\}.$$

Obviously, $P(U,T)$ is nonempty bounded closed convex set for each $U, T > 0$. We define the map S as follows

$$\begin{aligned} Sw = u_0(x) + u_1(x)t &+ \alpha \int_0^t \int_0^1 (t-\tau) K(x,\xi) w_{\xi\xi}(\xi,\tau)\, d\xi\, d\tau \\ &+ \int_0^t \int_0^1 (t-\tau) K(x,\xi) \varphi(w_\xi(\xi,\tau))_\xi \, d\xi\, d\tau, \qquad \forall\, w \in P(U,T). \end{aligned} \tag{2.4}$$

Obviously, S maps $X(T)$ into $X(T)$. Our goal is to show that S has a unique fixed point in $P(U,T)$ for appropriately chosen T. For this purpose we employ the contraction mapping principle.

Lemma 2.2. *Suppose that $u_0, u_1 \in C^2[0,1]$, $u_0(0) = u_0(1) = u_1(0) = u_1(1) = 0$ and $\varphi(s) \in C^2(R)$, then S maps $P(U,T)$ into $P(U,T)$ and $S : P(U,T) \to P(U,T)$ is strictly contractive if T is appropriately small relative to U.*

PROOF: Integrating by parts with respect to ξ in (2.4), we get

$$\tag{2.5} \begin{aligned} Sw = u_0(x) + u_1(x)t - \alpha \int_0^t \int_0^1 (t-\tau) K_\xi(x,\xi) w_\xi(\xi,\tau)\, d\xi\, d\tau \\ - \int_0^t \int_0^1 (t-\tau) K_\xi(x,\xi) \varphi(w_\xi(\xi,\tau))\, d\xi\, d\tau. \end{aligned}$$

Differentiating (2.5) with respect to x, we have

$$\tag{2.6} \begin{aligned} (Sw)_x = u_{0x}(x) + u_{1x}(x)t - \frac{\alpha}{\beta} \int_0^t (t-\tau) w_x(x,\tau)\, d\tau \\ - \alpha \int_0^t \int_0^1 (t-\tau) K_{\xi x}(x,\xi) w_\xi(\xi,\tau)\, d\xi\, d\tau \\ - \frac{1}{\beta} \int_0^t (t-\tau) \varphi(w_x(x,\tau))\, d\tau \\ - \int_0^t \int_0^1 (t-\tau) K_{\xi x}(x,\xi) \varphi(w_x(x,\tau))\, d\xi\, d\tau. \end{aligned}$$

Differentiating (2.6) also with respect to x, we obtain

$$\tag{2.7} \begin{aligned} (Sw)_{xx} = u_{0xx}(x) + u_{1xx}(x)t - \frac{\alpha}{\beta} \int_0^t (t-\tau) w_{xx}(x,\tau)\, d\tau \\ - \alpha \int_0^t \int_0^1 (t-\tau) K_{\xi xx}(x,\xi) w_\xi(\xi,\tau)\, d\xi\, d\tau \\ - \frac{1}{\beta} \int_0^t (t-\tau) \varphi(w_x(x,\tau))_x\, d\tau \\ - \int_0^t \int_0^1 (t-\tau) K_{\xi xx}(x,\xi) \varphi(w_\xi(\xi,\tau))\, d\xi\, d\tau. \end{aligned}$$

Let us define $\phi : [0,\infty) \to [0,\infty)$ by

$$\varphi(\eta) = \max_{|s| \le \eta} \left[|\varphi(s)| + |\varphi'(s)| + |\varphi''(s)| \right], \qquad \forall\, \eta \ge 0.$$

Observe that ϕ is continuous and nondecreasing on $[0,\infty)$. Using the boundedness of the Green's function $K(x,\xi)$ and its derivatives which appear in (2.6) and (2.7), when $T \le \frac{1}{2}$, we get

$$\begin{aligned} \|Sw\|_{X(T)} &\le U + UT + \left(\frac{\alpha}{\beta} + C_1 \alpha \right) \frac{T^2}{2} (2U+1) \\ &\quad + \left[\frac{1}{\beta}(2U+1) + C_2 \right] \frac{T^2}{2} \phi(2U+1) \\ &\le U + T[C_3 + C_4 \phi(2U+1)](2U+1), \end{aligned}$$

where C_1, C_2, C_3 and C_4 are constants. If T satisfies

$$T \le \min\left(\frac{1}{2}, \frac{1}{2[C_3 + C_4\phi(2U+1)]}\right), \tag{2.8}$$

then $\|Sw\|_{X(T)} \le 2U + 1$. Therefore, if (2.8) holds, then S maps $P(U,T)$ into $P(U,T)$.

Now we are going to prove that $S : P(U,T) \to P(U,T)$ is strictly contractive.

Let $T > 0$ and $w_1, w_2 \in P(U,T)$ be given. We have

$$\begin{aligned}
&(Sw_1 - Sw_2)_x = \\
&= -\frac{\alpha}{\beta}\int_0^t (t-\tau)[w_{1x}(x,\tau) - w_{2x}(x,\tau)]\,d\tau \\
&- \alpha\int_0^t\int_0^1 (t-\tau)K_{\xi x}(x,\xi)[w_{1\xi}(\xi,\tau) - w_{2\xi}(\xi,\tau)]\,d\xi\,d\tau \\
&- \frac{1}{\beta}\int_0^t (t-\tau)[\varphi(w_{1x}(x,\tau)) - \varphi(w_{2x}(x,\tau))]\,d\tau \\
&- \int_0^t\int_0^1 (t-\tau)K_{\xi x}(x,\tau)[\varphi(w_{1\xi}(\xi,\tau)) - \varphi(w_{2\xi}(\xi,\tau))]\,d\xi\,d\tau
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&(Sw_1 - Sw_2)_{xx} = \\
&= -\frac{\alpha}{\beta}\int_0^t (t-\tau)[w_{1xx}(x,\tau) - w_{2xx}(x,\tau)]\,d\tau \\
&- \alpha\int_0^t\int_0^1 (t-\tau)K_{\xi xx}(x,\xi)[w_{1\xi}(\xi,\tau) - w_{2\xi}(\xi,\tau)]\,d\xi\,d\tau \\
&- \frac{1}{\beta}\int_0^t (t-\tau)[\varphi'(w_{1x}(x,\tau))w_{1xx}(x,\tau) - \varphi'(w_{2x}(x,\tau))w_{2xx}(x,\tau)]\,d\tau \\
&- \int_0^t\int_0^1 (t-\tau)K_{\xi xx}(\xi,\tau)[\varphi(w_{1\xi}(\xi,\tau)) - \varphi(w_{2\xi}(\xi,\tau))]\,d\xi\,d\tau.
\end{aligned} \tag{2.10}$$

From (2.9) and (2.10) it follows that

$$\|Sw_1 - Sw_2\|_{X(T)} \le \{C_5 + C_6\phi(2U+1)\}\frac{T^2}{2}\|w_1 - w_2\|_{X(T)},$$

where C_5 and C_6 are constants.

If T satisfies

$$T \le \min\left(\frac{1}{2}, \frac{1}{2[C_3 + C_4\phi(2U+1)]}, \frac{1}{\sqrt{C_5 + C_6\phi(2U+1)}}\right), \tag{2.11}$$

then

$$\|Sw_1 - Sw_2\|_{X(T)} \le \frac{1}{2}\|w_1 - w_2\|_{X(T)}.$$

The lemma is proved. □

Theorem 2.1. *Let the assumptions of Lemma 2.2 hold. Then the integral equation* (2.3) *has a unique solution* $u(x,t) \in C([0,T_0);C^2[0,1])$*, where* $[0,T_0)$ *is a maximal time interval. Moreover, if*

$$\sup_{t\in[0,T_0)} \left(\|u_x\|_{C^1[0,1]} + \|u_{xt}\|_{C^1[0,1]}\right) < \infty, \tag{2.12}$$

then $T_0 = \infty$.

Proof: It follows from Lemma 2.2 and the contraction mapping principle that, for appropriately chosen $T > 0$, S has a unique fixed point $u(x,t) \in P(U,T)$ which is obviously a solution of the integral equation (2.3). It is easy to prove that for each $T' > 0$, the equation (2.3) has at most one solution which belongs to $X(T')$.

Let $[0,T_0)$ be the maximal time interval of existence for $u \in X(T_0)$. It only remains to show that if (2.12) is satisfied, then $T_0 = \infty$. This can be done in the usual way: If (2.12) holds and $T_0 < \infty$, we can reapply the contraction mapping principle extending the solution to an interval $[0,T_0+\delta]$, $\delta > 0$, which contradicts the assumption that $[0,T_0)$ is maximal.

Suppose that (2.12) holds and $T_0 < \infty$. For any $T' \in [0,T_0)$, we consider the integral equation

$$\begin{aligned} v(x,t) &= u(x,T') + u_t(x,T')t + \alpha \int_0^t \int_0^1 (t-\tau)K(x,\xi)v_{\xi\xi}(\xi,\tau)\,d\xi\,d\tau \\ &\quad + \int_0^t \int_0^1 (t-\tau)K(x,\xi)\varphi(v_\xi(x,\tau))_\xi\,d\xi\,d\tau. \end{aligned} \tag{2.13}$$

By virtue of (2.12), $\|u_x(\cdot,T')\|_{C^1[0,1]} + \|u_{xt}(\cdot,T')\|_{C^1[0,1]}$ is uniformly bounded in $T' \in [0,T_0)$, which allows us to choose $T^* \in (0,T_0)$, such that for each $T' \in [0,T_0)$, the integral equation (2.13) has a unique solution $v(x,t) \in X(T^*)$. The existence of such a T^* follows from Lemma 2.2 and the contraction mapping principle. In particular (2.8) and (2.11) reveal that T^* can be selected independently of $T' \in [0,T_0)$. Set $T' = T_0 - \frac{T^*}{2}$, let v denote the corresponding solution of (2.13), and define $\widehat{u}(x,t) : [0,1] \times [0,T_0 + \frac{T^*}{2}] \to R$ by

$$\widehat{u}(x,t) = \begin{cases} u(x,t), & t \in [0,T'], \\ v(x,t-T'), & t \in \left[T', T_0 + \frac{T^*}{2}\right]. \end{cases} \tag{2.14}$$

By construction, $\widehat{u}(x,t)$ is a solution of (2.3) on $[0,T_0 + \frac{T^*}{2}]$, and by local uniqueness, $\widehat{u}$ extends u. This violates the maximality of $[0,T_0)$. Hence if (2.12) holds, $T_0 = \infty$.

This completes the proof of the theorem. $\square$

3. The classical global solution of the problem (1.3)–(1.5)

Lemma 3.1. *Suppose that* $u_0(x), u_1(x) \in H_0^1[0,1]$, $\varphi(s) \in C^1(R)$, *then the classical solution of the problem* (1.3)–(1.5) *satisfies the following identity*

$$
\begin{aligned}
E(t) &\equiv \|u_t\|^2_{L_2[0,1]} + \alpha\|u_x\|^2_{L_2[0,1]} + \beta\|u_{xt}\|^2_{L_2[0,1]} \\
&+ 2\int_0^1 \int_0^{u_x} \varphi(s)\,ds\,dx \\
&= \|u_1\|^2_{L_2[0,1]} + \alpha\|u_{0x}\|^2_{L_2[0,1]} + \beta\|u_{xt}\|^2_{L_2[0,1]} \\
&+ 2\int_0^1 \int_0^{u_{0x}} \varphi(s)\,ds\,dx \\
&\equiv E(0), \qquad \forall\, t \in [0,T].
\end{aligned}
\tag{3.1}
$$

Proof: Multiplying both sides of the equation (1.3) by u_t, integrating the product with respect to x over $[0,1]$ and integrating by parts we get

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|u_t\|^2_{L_2[0,1]} + \alpha\|u_x\|^2_{L_2[0,1]} + \beta\|u_{xt}\|^2_{L_2[0,1]} \\
&+ 2\int_0^1 \int_0^{u_x} \varphi(s)\,ds\,dx\Big) = 0.
\end{aligned}
\tag{3.2}
$$

Integrating (3.2) with respect to t, we obtain (3.1). The lemma is proved. □

Theorem 3.1. *Suppose that the condition of Theorem* 2.1 *and the following condition*

$$
|\varphi(s)| \le A \int_0^s \varphi(y)\,dy + B \tag{3.3}
$$

hold, where A and B are positive constants. Then the problem (1.3)–(1.5) *has a unique classical global solution* $u(x,t)$.

Remark 3.1. The function $\varphi(s)$ satisfying (3.3) exists. For example, $\varphi(s) = e^s$ satisfies the inequality (3.3). $\varphi(s) = rs^n$ is the second example, where $r > 0$ is a real number and n is a natural number. When n is an odd number, $\varphi(s) = rs^n$ satisfies the inequality (3.3), i.e.

$$
|rs^n| \le n \int_0^s ry^n\,dy + \frac{r}{n+1}\,. \tag{3.4}
$$

In fact, taking $p = \frac{n+1}{n}$, $p' = n+1$ and using Young's inequality we have

$$
\begin{aligned}
|rs^n| = r|s^n| &\le r\left(\frac{|s|^{np}}{p} + \frac{1}{p'}\right) = r\frac{n}{n+1}s^{n+1} + \frac{r}{n+1} \\
&= n\int_0^s ry^n\,dy + \frac{r}{n+1}\,.
\end{aligned}
$$

Proof of Theorem 3.1: By virtue of Theorem 2.1, we are only required to prove that (2.12) holds. Integrating by parts in (2.3), we obtain

$$\begin{aligned} u(x,t) = u_0(x) + u_1(x)t - \alpha \int_0^t \int_0^1 (t-\tau) K_\xi(x,\xi) u_\xi(\xi,\tau)\, d\xi\, d\tau \\ - \int_0^t \int_0^1 (t-\tau) K_\xi(x,\xi) \varphi(u_\xi(\xi,\tau))\, d\xi\, d\tau. \end{aligned} \tag{3.5}$$

It follows from (3.5) that

$$\begin{aligned} u_x(x,t) = u_{0x}(x) + u_{1x}(x)t - \frac{\alpha}{\beta} \int_0^t (t-\tau) u_x(x,\tau)\, d\tau \\ - \alpha \int_0^t \int_0^1 (t-\tau) K_{\xi x}(x,\xi) u_\xi(\xi,\tau)\, d\xi\, d\tau \\ - \frac{1}{\beta} \int_0^t (t-\tau) \varphi(u_x(x,\tau))\, d\tau \\ - \int_0^t \int_0^1 (t-\tau) K_{\xi x}(x,\xi) \varphi(u_\xi(\xi,\tau))\, d\xi\, d\tau, \end{aligned} \tag{3.6}$$

$$\begin{aligned} u_{xtt}(x,t) = -\frac{\alpha}{\beta} u_x(x,t) - \alpha \int_0^1 K_{\xi x}(x,\xi) u_\xi(\xi,t)\, d\xi \\ - \frac{1}{\beta} \varphi(u_x(x,t)) - \int_0^1 K_{\xi x}(x,\xi) \varphi(u_\xi(\xi,t))\, d\xi. \end{aligned} \tag{3.7}$$

Multiplying both sides of (3.7) by u_{xt} we get

$$\begin{aligned} \frac{d}{dt}\Big[u_{xt}^2 + \frac{\alpha}{\beta} u_x^2 + \frac{2}{\beta} \int_0^{u_x} \varphi(s)\, ds\Big] = 2\Big[-\alpha \int_0^1 K_{\xi x}(x,\xi) u_\xi(\xi,t)\, d\xi \\ - \int_0^1 K_{\xi x}(x,\xi) \varphi(u_\xi(\xi,t))\, d\xi\Big] u_{xt}. \end{aligned} \tag{3.8}$$

Let us denote $u_{1x}^2(x) + \frac{\alpha}{\beta} u_{0x}^2(x) + \frac{2}{\beta} |\int_0^{u_{0x}(x)} \varphi(s)\, ds|$ by $E_1(x)$. Integrating both sides of (3.8) with respect to t and making use of the conditions (3.3) and (3.1), we can obtain

$$\begin{aligned} & u_{xt}^2 + \frac{\alpha}{\beta} u_x^2 + \frac{2}{\beta} \int_0^{u_x} \varphi(s)\, ds \\ & \le E_1(x) + 2 \int_0^t \Big[-\alpha \int_0^1 K_{\xi x}(x,\xi) u_\xi(\xi,\tau)\, d\xi \\ & - \int_0^1 K_{\xi x}(x,\xi) \varphi(u_\xi(\xi,\tau))\, d\xi\Big] u_{xt}\, d\tau \end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&\le E_1(x) + \int_0^t \Big\{ \Big[\int_0^1 4\alpha K_{\xi x}^2(x,\xi)\,d\xi \Big]^{\frac{1}{2}} \Big[\int_0^1 \alpha u_x^2(x,\tau)\,dx \Big]^{\frac{1}{2}} \\
&+ C_7 \int_0^1 |\varphi(u_x(x,\tau))|\,dx \Big\} |u_{xt}|\,d\tau \\
&\le E_1(x) + \int_0^t \Big\{ C_8 + \alpha \int_0^1 u_x^2(x,\tau)\,dx \\
&+ C_7 \int_0^1 \Big[A \int_0^{u_x(x,\tau)} \varphi(s)\,ds + B \Big]\,dx \Big\} |u_{xt}|\,d\tau \\
&\le E_1(x) + \int_0^t \{ C_9 + C_{10} E(0) \} |u_{xt}|\,d\tau \\
&\le E_1(x) + \frac{1}{4}[C_9 + C_{10}E(0)]^2 T + \int_0^t u_{xt}^2\,d\tau.
\end{aligned}
$$

Multiplying both sides of (3.9) by A, adding the product to $\frac{2B}{\beta}$ and using (3.3), we get

$$
Au_{xt}^2 + \frac{A\alpha}{\beta}u_x^2 + \frac{2}{\beta}|\varphi(u_x)| \le M_1(T) + A\int_0^t u_{xt}^2\,d\tau, \tag{3.10}
$$

where $M_1(T)$ is a constant dependent on T.

It follows from (3.10) by Gronwall's inequality that

$$
Au_{xt}^2 + \frac{A\alpha}{\beta}u_x^2 + \frac{2}{\beta}|\varphi(u_x)| \le M_1(T)e^{AT}.
$$

Therefore

$$
\sup_{0\le t\le T} \|u_x\|_{C[0,1]} + \sup_{0\le t\le T} \|u_{xt}\|_{C[0,1]} + \sup_{0\le t\le T} \|\varphi(u_x)\|_{C[0,1]} \le M_2(T). \tag{3.11}
$$

Differentiating (3.6) with respect to x, we obtain

$$
\begin{aligned}
u_{xx}(x,t) &= u_{0xx}(x) + u_{1xx}(x)t - \frac{\alpha}{\beta}\int_0^t (t-\tau)u_{xx}(x,\tau)\,d\tau \\
&- \alpha \int_0^t \int_0^1 (t-\tau)K_{\xi xx}(x,\xi)u_\xi(\xi,\tau)\,d\xi\,d\tau \\
&- \frac{1}{\beta}\int_0^t (t-\tau)\varphi'(u_x(x,\tau))u_{xx}(x,\tau)\,d\tau \\
&- \int_0^t \int_0^1 (t-\tau)K_{\xi xx}(x,\xi)\varphi(u_\xi(\xi,\tau))\,d\xi\,d\tau.
\end{aligned} \tag{3.12}
$$

It follows from (3.12) that

$$
|u_{xx}(x,t)| \le \max_{0\le x\le 1} |u_{0xx}(x)| + \max_{0\le x\le 1} |u_{1xx}|T + C_{11}T^2 + C_{12}T \int_0^t |u_{xx}(x,\tau)|\,d\tau.
$$

Making use of Gronwall's inequality, we have

$$\sup_{0\le t\le T} \|u_{xx}(\cdot,t)\|_{C[0,1]} \le M_3(T). \tag{3.13}$$

Differentiating (3.12) with respect to t, we obtain

$$\begin{aligned} u_{xxt}(x,t) &= u_{1xx}(x) - \frac{\alpha}{\beta}\int_0^t u_{xx}(x,\tau)\,d\tau \\ &\quad - \alpha\int_0^t\int_0^1 K_{\xi xx}(x,\xi)u_\xi(\xi,\tau)\,d\xi\,d\tau \\ &\quad - \frac{1}{\beta}\int_0^t \varphi'(u_x(x,\tau))u_{xx}(x,\tau)\,d\tau \\ &\quad - \int_0^t\int_0^1 K_{\xi xx}(x,\xi)\varphi(u_\xi(\xi,\tau))\,d\xi\,d\tau. \end{aligned} \tag{3.14}$$

It follows from (3.14) that

$$\sup_{0\le t\le T} \|u_{xxt}(\cdot,t)\|_{C[0,1]} \le M_4(T). \tag{3.15}$$

From (3.11), (3.13) and (3.15) it follows that

$$\sup_{0\le t\le T} \left(\|u_x\|_{C^1[0,1]} + \|u_{xt}\|_{C^1[0,1]}\right) < \infty.$$

By virtue of Theorem 2.1 and Lemma 2.1 we know that the problem (1.3)–(1.5) has a unique classical global solution $u(x,t)$. Theorem 3.1 is proved. □

4. Blow-up of solutions of the problem (1.3)–(1.5)

In this section we are going to consider blow-up of solutions of the problem (1.3)-(1.5).

Theorem 4.1. *Suppose that the following conditions hold:*

(1) $\int_0^1 (u_0u_1 + \beta u_{0x}u_{1x})\,dx > 0$,
(2) $E(0) \le 0$,
(3) $\varphi \in C^1(R)$, $\varphi(s)s \le 2(2\delta+1)\int_0^s \varphi(y)\,dy + 2\delta\alpha s^2$,

where $\delta > 0$ is a constant. Then the classical solutions of the problem (1.3)–(1.5) *must blow up in finite time.*

PROOF: The proof is made by use of so called "concavity" arguments. Assume that $u(x,t)$ is the classical solution of the problem (1.3)–(1.5) on $[0,1]\times[0,T]$. Let

$$F(t) = \int_0^1 (u^2 + \beta u_x^2)\,dx.$$

We have

$$F'(t) = 2\int_0^1 (uu_t + \beta u_x u_{xt})\,dx,$$

$$\begin{aligned} F''(t) &= 2\int_0^1 (u_t^2 + \beta u_{xt}^2)\,dx + 2\int_0^1 (uu_{tt} + \beta u_x u_{xtt})\,dx \\ &= 2\int_0^1 (u_t^2 + \beta u_{xt}^2)\,dx + 2\int_0^1 u(u_{tt} - \beta u_{xxtt})\,dx. \end{aligned}$$

Using Cauchy's inequality, we see that

$$[F'(t)]^2 \le 4\Big[\int_0^1 (u^2 + \beta u_x^2)\,dx\Big]\Big[\int_0^1 (u_t^2 + \beta u_{xt}^2)\,dx\Big]. \tag{4.1}$$

Therefore, using (1.3) and (4.1), we find that

$$\begin{aligned} &FF'' - (1+\delta)(F')^2 \ge F\Big\{\Big[2\int_0^1 (u_t^2 + \beta u_{xt}^2)\,dx \\ &+ 2\int_0^1 u(\alpha u_{xx} + \varphi(u_x)_x)\,dx\Big] - 4(1+\delta)\Big[\int_0^1 (u_t^2 + \beta u_{xt}^2)\,dx\Big]\Big\} \\ &= 2F\Big[-\int_0^1 (\alpha u_x^2 + \varphi(u_x)u_x)\,dx - (2\delta+1)\int_0^1 (u_t^2 + \beta u_{xt}^2)\,dx\Big]. \end{aligned} \tag{4.2}$$

Thus from (4.2), (3.1) and the conditions (2), (3) it follows that

$$\begin{aligned} &FF'' - (1+\delta)(F')^2 \ge 2F\Big[-2(2\delta+1)\int_0^1\int_0^{u_x} \phi(s)\,ds\,dx \\ &- (2\delta+1)\alpha\int_0^1 u_x^2\,dx - (2\delta+1)\int_0^1 (u_t^2 + \beta u_{xt}^2)\,dx\Big] \\ &= -2F(2\delta+1)E(0) \ge 0, \qquad t \in [0,T]. \end{aligned}$$

We see that $F(t) > 0$ for all $t \in [0,T]$ and that from the condition (1), $F'(0) > 0$. From "concavity" arguments (see [4], [5]) we know that there exists a constant t_0 such that

$$\lim_{t\to t_0^-}\big(\|u\|^2_{L_2[0,1]} + \beta\|u_x\|^2_{L_2[0,1]}\big) = +\infty$$

and

$$T < t_0 = \frac{\|u_0\|^2_{L_2[0,1]} + \beta\|u_{0x}\|^2_{L_2[0,1]}}{2\delta\int_0^1 (u_0u_1 + \beta u_{0x}u_{1x})\,dx}.$$

Theorem 4.1 is proved. $\square$

5. Initial boundary value problem (1.3), (1.6), (1.7)

This section is concerned with the problem (1.3), (1.6), (1.7). We consider problem (1.3), (1.6), (1.7) by the method used in Sections 1–4. Observe that

$$K(x,\xi) = \frac{1}{\sqrt{\beta}\sinh\frac{1}{\sqrt{\beta}}}\begin{cases} \cosh\frac{x}{\sqrt{\beta}}\cosh\frac{1-\xi}{\sqrt{\beta}}, & x \le \xi, \\ \cosh\frac{1-x}{\sqrt{\beta}}\cosh\frac{\xi}{\sqrt{\beta}}, & x > \xi, \end{cases}$$

is the Green's function of the boundary value problem for the ordinary differential equation

$$y - \beta y'' = 0, \quad y'(0) = y'(1) = 0,$$

where $\beta > 0$ is a real number.

The following theorems can be proved analogously as Theorems 3.1 and 4.1 above.

Theorem 5.1. *Assume that the following conditions hold:*

(1) $u_0(x), u_1(x) \in C^2[0,1]$ *and* $u_{0x}(0) = u_{0x}(1) = u_{1x}(0) = u_{1x}(1) = 0$,
(2) $\varphi(s) \in C^2(R)$, $\varphi(0) = 0$ *and* $|\varphi(s)| \le A\int_0^s \varphi(y)\,dy + B$,

where $A, B > 0$ *are constants. Then the problem* (1.3), (1.6), (1.7) *has a unique classical global solution* $u(x,t)$.

Theorem 5.2. *Assume that the following conditions hold:*

(1) $\int_0^1 (u_0u_1 + \beta u_{0x}u_{1x})\,dx > 0$,
(2) $E(0) \le 0$,
(3) $\varphi \in C^1(R)$, $\varphi(0) = 0$ *and* $\varphi(s)s \le 2(2\delta+1)\int_0^s \varphi(s)\,ds + 2\delta\alpha s^2$,

where $\delta > 0$ *is a constant.*
Then the classical solutions of the problem (1.3), (1.6), (1.7) *must blow up in finite time.*

6. On the problems (1.1), (1.4), (1.5) and (1.1), (1.6), (1.7)

Here we apply the results of the problem (1.3), (1.4), (1.5) to the problem (1.1), (1.4), (1.5) and the results of the problem (1.3), (1.6), (1.7) to the problem (1.1), (1.6), (1.7).

Theorem 6.1. *Suppose that*

$$u_0(x), u_1(x) \in C^2[0,1],\ u_0(0) = u_0(1) = u_1(0) = u_1(1) = 0,\ a_0, a_2 > 0.$$

(1) *If* n *is an odd number,* $a_1 > 0$, *then the problem* (1.1), (1.4), (1.5) *has a unique classical global solution* $u(x,t)$.

(2) *If* n $(n \ne 1)$ *is an odd number,* $a_1 < 0$,

$$\|u_1\|^2_{L_2[0,1]} + a_0\|u_{0x}\|^2_{L_2[0,1]} + a_2\|u_{1x}\|^2_{L_2[0,1]} + \frac{2a_1}{n+1}\int_0^1 (u_{0x})^{n+1}\,dx \equiv \widehat{E}(0) \le 0$$

and the condition

$$\int_0^1 (u_0 u_1 + a_2 u_{0x} u_{1x})\, dx > 0$$

holds, then the classical solutions of the problem (1.1), (1.4), (1.5) *must blow up in finite time.*

(3) *If n is an even number, $a_1 \neq 0$, $\widehat{E}(0) \leq 0$ and the condition*

$$\int_0^1 (u_0 u_1 + a_2 u_{0x} u_{1x})\, dx > 0$$

holds, then the classical solutions of the problem (1.1), (1.4), (1.5) *must blow up in finite time.*

Theorem 6.2. *Suppose that $u_0(x), u_1(x) \in C^2[0,1]$, $u_{0x}(0) = u_{0x}(1) = u_{1x}(0) = u_{1x}(1) = 0$, $a_0, a_2 > 0$.*

(1) *If n is an odd number, $a_1 > 0$, then the problem* (1.1), (1.6), (1.7) *has a unique classical global solution $u(x,t)$.*

(2) *If n $(n \neq 1)$ is an odd number, $a_1 < 0$, $\widehat{E}(0) \leq 0$ and the condition*

$$\int_0^1 (u_0 u_1 + a_2 u_{0x} u_{1x})\, dx > 0$$

holds, then the classical solutions of the problem (1.1), (1.6), (1.7) *must blow up in finite time.*

(3) *If n is an even number, $a_1 \neq 0$, $\widehat{E}(0) \leq 0$ and the condition*

$$\int_0^1 (u_0 u_1 + a_2 u_{0x} u_{1x})\, dx > 0$$

holds, then the classical solutions of the problem (1.1), (1.6), (1.7) *must blow up in finite time.*

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