Some (new) counterexamples of parabolic systems

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Abstract. We give examples of parabolic systems (in space dimension $n \geq 3$) having a solution with real analytic initial and boundary values which develops the discontinuity in the interior of the parabolic cylinder.

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1. Introduction

The aim of this note is to give examples showing that parabolic systems do not conserve, in general, the regularizing property of the heat equation.

All the examples presented here have the same feature — the system has a solution with smooth initial and boundary values which develops the discontinuity (or even L_∞ blow-up) in the interior of the parabolic cylinder.

In comparison to the results published earlier ([1], [2]) we tried to make the results "sharper" (real analyticity of the coefficients) and such that the calculations are verifiable more easily. We are indebted to M. Wiegner for suggesting the problem of higher smoothness of coefficients (than that obtained in [1]) and for many stimulating conversations.

The idea of the construction goes back to M. Giaquinta and J. Souček, its modification to the parabolic case was described in [1].

All examples work in case of $n \geq 3$, where n is the number of space variables.

As for $n = 2$, there are various regularity results (see [7], [8], [9], [11]). Even in this case it still makes sense to ask for a counterexample or for a positive answer about Hölder continuity of solutions provided the coefficients of linear parabolic system are L_∞ only.

2. Notation

Let Ω be a nonempty open set in $\mathbb{R}^n$, $(n \in \mathbb{N}, n \geq 3)$, and let $T \in (0,\infty)$. Denote $Q = \Omega \times (0,T)$. We shall consider the case when space dimension n

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equals to the number of components of a solution only, i.e.

$$u = [u^1, u^2, ..., u^n] : Q \to \mathbb{R}^n,$$
$$u_t = \left[\frac{\partial u^1}{\partial t}, \frac{\partial u^2}{\partial t}, ..., \frac{\partial u^n}{\partial t}\right],$$
$$D_\alpha u^i = \frac{\partial u^i}{\partial x_\alpha}, \nabla u = \left[\frac{\partial u^i}{\partial x_\alpha}\right]_{i,\alpha=1,...,n}.$$

As $\langle u, v\rangle$ we denote the scalar product in any finite dimensional space $\mathbb{R}^p$, $p \in \mathbb{N}$, $|u| = \langle u, u\rangle^{\frac{1}{2}}$.

The summation convention is used throughout the paper.

3. An example of a quasi linear parabolic system with the coefficients depending smoothly on u

At the beginning we give an example of a quasilinear parabolic system. Its coefficients, depending on u only, are real analytic on a neighbourhood of the closure of $B(0,1) = \{v \in \mathbb{R}^n; |v| < 1\}$. Together with the system we construct its solution u which has real analytic initial data, develops discontinuity in the interior of Q and its values are contained in $\overline{B(0,1)}$.

Theorem 3.1. *Let $n \geq 3$; $\kappa \in (0, 2(n-1)(n-2))$. For $x \in \mathbb{R}^n$, $t \in (-\infty, 1)$ put*

$$u(x,t) = \frac{x}{\sqrt{\kappa(1-t) + |x|^2}}.$$

Then u is real analytic on $\mathbb{R}_n \times (-\infty, 1)$ and solves a quasilinear system

$$u_t^i = D_\alpha(A_{ij}^{\alpha\beta}(u) D_\beta u^j), \quad (i = 1, ..., n) \tag{3.1}$$

with $A_{ij}^{\alpha\beta}(u)$ real analytic on a neighbourhood of $\overline{B(0,1)}$ and infinitely differentiable on $\mathbb{R}^n$, satisfying an ellipticity condition

$$\begin{gathered} \exists\ \lambda_0,\ \lambda_1\ \in (0,\infty)\ \forall \xi\ \in\ \mathbb{R}^{n^2};\ u \in \mathbb{R}^n \\ \lambda_0 |\xi|^2 \leq\ \langle A(u)\xi, \xi\rangle\ \leq \lambda_1 |\xi|^2; \end{gathered} \tag{3.2}$$

with

$$\lambda_0 = \frac{\mu}{\lambda}(\mu - \sqrt{\mu^2 - \lambda^2}),$$
$$\lambda_1 = \frac{\mu}{\lambda}(\mu + \sqrt{\mu^2 - \lambda^2}),$$
$$\mu = n-1,\ \lambda = n - 2 - \frac{\kappa}{2(n-1)}.$$

PROOF: Denote

$$r = |x|;\ \nu = \frac{x}{r};\ \xi = \frac{r}{\sqrt{\kappa(1-t)}}\,. \tag{3.3}$$

Then

$$u(x,t) = \frac{x}{\sqrt{\kappa(1-t)+|x|^2}} = \nu\varphi(\xi), \tag{3.4}$$

where

$$\varphi(\xi) = \frac{\xi}{\sqrt{1+\xi^2}}\,. \tag{3.5}$$

Then we obtain

$$c^i_\alpha = D_\alpha u^i = \frac{1}{r}[\varphi(\delta_{i\alpha} - \nu_i\nu_\alpha) + \varphi'\xi\nu_i\nu_\alpha], \quad (\alpha, i = 1, ..., n),$$

and

$$u^i_t = \frac{\kappa}{2}\frac{1}{r^2}\nu_i\varphi'\xi^3, \quad (i = 1, ..., n).$$

Putting

$$b^i_\alpha = \frac{1}{r}\{\varphi[(n-2)\delta_{i\alpha}+\nu_i\nu_\alpha]+\varphi'\xi(1-\frac{\kappa\xi^2}{2(n-1)})(\delta_{i\alpha}-\nu_i\nu_\alpha),\ (\alpha, i = 1, ..., n), \tag{3.6}$$

we verify easily that

$$\begin{aligned} &\text{(i)}\quad D_\alpha b^i_\alpha = u^i_t, \quad (i = 1, ..., n),\\ &\text{(ii)}\ \ |b|^2 \le \mu^2|c|^2 \text{ with } \mu = n-1\\ &\text{(iii)}\ \langle b, c\rangle \ge \lambda|c|^2 \text{ with } \lambda = n-2-\frac{\kappa}{2(n-1)}\,. \end{aligned} \tag{3.7}$$

Following [6] we put for $\theta \in (0, \lambda)$

$$A^{\alpha\beta}_{ij} = \theta\delta_{ij}\delta_{\alpha\beta} + \langle b - \theta c, c\rangle^{-1}(b^i_\alpha - \theta c^i_\alpha)(b^j_\beta - \theta c^j_\beta) \tag{3.8}$$

and we obtain ellipticity condition with

$$\lambda_0 = \theta,\ \lambda_1 = \frac{\mu^2 - \theta\lambda}{\lambda - \theta}\,.$$

Choosing $\theta = \frac{\mu^2-\mu\sqrt{\mu^2-\lambda^2}}{\lambda}$, we get finally (3.2). (The choice of θ was done to keep $\frac{\lambda_0}{\lambda_1}$ maximal.)

The relations (3.7)(i) immediately imply that u solves system (3.1) with $A_{ij}^{\alpha\beta}$ given by (3.8). Using the equality

$$|u| = \varphi(\xi) = \frac{\xi}{\sqrt{1+\xi^2}}$$

we can express terms

$$A_{i\alpha} = \frac{b_\alpha^i - \theta c_\alpha^i}{\sqrt{\langle b - \theta c, c\rangle}}$$

as a function of u. Thus we get

$$A_{ij}^{\alpha\beta}(u) = \theta\delta_{ij}\delta_{\alpha\beta} + A_{i\alpha}(u)A_{j\beta}(u)$$

with

$$A_{i\alpha}(u) = \frac{\{n-1-\theta-|u|^2(1+\frac{\kappa}{2(n-1)})\}\delta_{i\alpha} + (1+\theta+\frac{\kappa}{2(n-1)})u^i u^\alpha}{\sqrt{n(n-1-\theta) - \{2(n-1-\theta)+\frac{\kappa}{2}\}|u|^2 - \theta|u|^4}}. \tag{3.9}$$

The coefficients $A_{i\alpha}$ are real analytic on a neighbourhood of $\overline{B(0,1)} = \{v \in \mathbb{R}^n; |v| \le 1\}$ where the solution u given by (3.3) takes its values.

Really, putting $f(|u|)$ for the expression under the root in the denominator of (3.9), we get

$$f(1) = (n-1)(n-2-\theta) - \frac{\kappa}{2}.$$

Because of the fact that $0 < \theta < n-2-\frac{\kappa}{2(n-1)} = \lambda$ we can estimate $f(1) > 0$. Thus $f(|u|)$ is positive on

$$B(0, 1+\epsilon) = \{u; |u| < 1+\epsilon\} \tag{3.10}$$

with an ϵ positive.

Moreover, let $\phi \in C^\infty(\mathbb{R})$ be any function for which $0 \le \phi(s) \le 1$ everywhere in $\mathbb{R}$, $\phi(s) = 0$ if $|s| \ge 1+\epsilon$ and $\phi(s) = 1$ for $|s| < 1+\frac{\epsilon}{2}$, where ϵ is positive number from (3.10). Denote

$$\tilde{A}_{ij}^{\alpha\beta}(u) = \begin{cases} \theta\delta_{ij}\delta_{\alpha\beta} + \phi(|u|)A_{i\alpha}(u)A_{j\beta}(u), & |u| < 1+\epsilon, \\ \theta\delta_{ij}\delta_{\alpha\beta}, & \text{otherwise.} \end{cases}$$

Then $\tilde{A}_{ij}^{\alpha\beta} \in C^\infty(\mathbb{R}^n)$ and the system (3.1) with the coefficients $\tilde{A}_{ij}^{\alpha\beta}$ satisfies ellipticity condition (3.2) with the same λ_0, λ_1. □

In the following remark we suppose $n = 3$. Nevertheless, it can be reformulated for $n \ge 3$.

Remark 1 ($n = 3, \kappa = 1$)**.** In our example, the ratio $\frac{\lambda_0}{\lambda_1}$ is approximately 0.04. It follows from the results of E. Kalita [3] that if $\frac{\lambda_0}{\lambda_1}$ in parabolic case is sufficiently big ($\frac{\lambda_0}{\lambda_1} > 0.33$), then all its solutions are locally Hölder continuous. In elliptic case, as it follows from the (sharp) results of A.I. Koshelev ([4], [5]), the local Hölder continuity of all solutions is guaranteed if only $\frac{\lambda_0}{\lambda_1} > \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} \doteq 0.11$.

Recalculating our example with

$$\varphi(\xi) = \frac{\xi}{\sqrt{K + |\xi|^2}},\ K > 0$$

instead of (3.4), we get

$$\frac{\lambda_0}{\lambda_1} > \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}} - \epsilon$$

for any given $\epsilon > 0$ choosing the constant K properly.

Remark 2. Inserting values of u in A^i_α we can define the coefficients $A^{\alpha\beta}_{ij}(x,t)$ by

$$A^{\alpha\beta}_{ij}(x,t) = \theta\delta_{ij}\delta_{\alpha\beta} + A_{i\alpha}(x,t)A_{j\beta}(x,t), \tag{3.11}$$

where

$$A_{i\alpha}(x,t) = \frac{[\kappa(1-t)(n-1-\theta) + |x|^2(n-2-\theta-\frac{\kappa}{2(n-1)})]\delta_{i\alpha} + x_i x_\alpha(1+\theta+\frac{\kappa}{2(n-1)})}{\sqrt{\mathcal{A}\kappa^2(1-t)^2 + \mathcal{B}\kappa(1-t)|x|^2 + \mathcal{C}|x|^4}}$$

for $t \in (-\infty, 1)$, $x \in \mathbb{R}^n$ with

$$\begin{aligned} \mathcal{A} &= n(n-1-\theta), \\ \mathcal{B} &= [2(n-1)(n-1-\theta) - \frac{\kappa}{2}], \\ \mathcal{C} &= [(n-1)(n-2-\theta) - \frac{\kappa}{2}]. \end{aligned}$$

(Constants $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are positive.) Thus we get that the coefficients $A^{\alpha\beta}_{ij} \in L_\infty(\mathbb{R}^n \times (-\infty, 1))$ and they are analytic on $\mathbb{R}^n \times (-\infty, 1)$. We can define them in various manners for $t > 1$. For example, let

$$A^{\alpha\beta}_{ij} = \delta_{ij}\delta_{\alpha\beta}$$

for $t > 1$. In this way, for t tending to 1 from below, we reach the solution $u^i(x,1) = \frac{x_i}{|x|}$. Taking this function as the initial values for $t = 1$ and solving the diagonal heat equation system for $t > 1$, we get finally the weak solution on $\mathbb{R}^n \times (0, \infty)$ which looses continuity at the point $[x, t] = [0, 1]$ and which is infinitely smooth on $\mathbb{R}^n \times (1, \infty)$.

Another of the possible choices is to take the coefficients

$$A_{ij}^{\alpha\beta}(x,t) = \mathcal{A}_{ij}^{\alpha\beta}(x,t), t > 1,$$

where $\mathcal{A}_{ij}^{\alpha\beta}$ are the coefficients of the well known De Giorgi's example [10]. In this case we get the solution u such that

$$u^i(x,t) = \frac{x_i}{|x|},$$

for all $t > 1, x \in \mathbb{R}^n$; i.e. the solution develops one point of discontinuity for $t = 1$ and it is stationary for $t > 1$.

4. L_∞ blow-up of solutions to linear parabolic systems

Theorem 4.1. *Let $n \geq 3$; $\gamma \in (0, \min\{\sqrt{n-1} - 1, \frac{1}{2}\})$; $\kappa \in (0, 2(n-1)(n-2-2\gamma))$. For $x \in \mathbb{R}^n$, $t \in (-\infty, 1)$ put*

$$u(x,t) = \frac{x}{|x|^\gamma \sqrt{\kappa(1-t) + |x|^2}}. \tag{4.1}$$

Then u is Hölder continuous on $\mathbb{R}^n \times (-\infty, 1)$ and it is a weak solution of a linear parabolic system

$$u_t^i = D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j), \quad (i = 1, ..., n) \tag{4.2}$$

with $A_{ij}^{\alpha\beta} \in L_\infty(\mathbb{R}^n \times (-\infty, 1))$ satisfying an ellipticity condition

$$\begin{aligned} &\exists\, \lambda_0,\ \lambda_1\ \in (0,\infty)\ \forall\, \xi \in \mathbb{R}^{n^2}, \forall\, x \in \mathbb{R}^n, \forall\, t \in (-\infty, 1) \\ &\lambda_0 |\xi|^2 \leq\ \langle A(x,t)\xi, \xi\rangle\ \leq \lambda_1 |\xi|^2. \end{aligned} \tag{4.3}$$

Nevertheless,

$$\lim_{t \to 1-} \|u(.,t)\|_{L_\infty(\mathbb{R}^n)} = \infty. \tag{4.4}$$

Proof: The proposed solution u obviously satisfies (4.4) and smoothness properties formulated in Theorem 4.1. We shall define now coefficients of a corresponding parabolic system. Using the notation (3.3) we have

$$u(x,t) = \nu\phi(\xi)\psi(r)$$

where $\psi(r) = r^{-\gamma}$. Analogously to Section 3 we get

$$c^i_\alpha = D_\alpha u^i = \frac{1}{r}[\psi\varphi(\delta_{i\alpha} - \nu_i\nu_\alpha) + (\psi' r\varphi + \psi\varphi'\xi)\nu_i\nu_\alpha], \quad (\alpha, i = 1, ..., n),$$

and

$$u^i_t = \frac{\kappa}{2}\frac{1}{r^2}\nu_i\psi\varphi'\xi^3, \quad (i = 1, ..., n).$$

Putting this time

$$\begin{aligned} b^i_\alpha = &\frac{1}{r}\{\psi\varphi[(n-2)\delta_{i\alpha} + \nu_i\nu_\alpha] + \\ &[\psi' r\varphi + \psi\varphi'\xi(1 - \frac{\kappa\xi^2}{2(n-1)})](\delta_{i\alpha} - \nu_i\nu_\alpha\}), \ (\alpha, i = 1, ..., n), \end{aligned} \tag{4.5}$$

we verify easily that

$$\begin{aligned} &\text{(i)} \quad D_\alpha b^i_\alpha = u^i_t, (i = 1, ..., n), \\ &\text{(ii)} \quad |b|^2 \le \mu^2|c|^2 \\ &\text{(iii)} \quad \langle b, c\rangle \ge \lambda|c|^2 \end{aligned} \tag{4.6}$$

with

$$\mu = \sqrt{n^2 - n - 1 + (1 + \gamma + \frac{\kappa}{2(n-1)})^2},$$

$$\lambda = \frac{(n-1)(n - 2 - 2\gamma - \frac{\kappa}{2(n-1)})}{n - 1 - (1+\gamma)^2}.$$

Assumptions of Theorem 4.1 guarantee that both λ, μ are positive. Choosing $\theta \in (0, \lambda)$ small enough and repeating procedures of Section 3 we get finally that u given by (4.1) is a weak solution of a parabolic system (4.2) with $A^{\alpha\beta}_{ij}$ given by (3.11) where

$$A_{i\alpha}(x,t) = \frac{\delta_{i\alpha}\{\mathcal{A}\kappa(1-t) + \mathcal{B}|x|^2\} + \nu_i\nu_\alpha\{\mathcal{C}\kappa(1-t) + \mathcal{D}|x|^2\}}{\sqrt{\mathcal{E}|x|^4 + \mathcal{F}|x|^2\kappa(1-t) + \mathcal{G}\kappa^2(1-t)^2}} \tag{4.7}$$

and

$$\begin{aligned} \mathcal{A} &= n - 1 - \theta - \gamma, \\ \mathcal{B} &= n - 2 - \theta - \gamma - \frac{\kappa}{2(n-1)}, \\ \mathcal{C} &= \gamma(1+\theta), \\ \mathcal{D} &= (1+\theta)(1+\gamma) + \frac{\kappa}{2(n-1)}, \\ \mathcal{E} &= (n-1)(n - 2 - \theta - 2\gamma) - \frac{\kappa}{2} - \theta\gamma^2, \\ \mathcal{F} &= 2(n-1)(n - 1 - \theta - 2\gamma) - \frac{\kappa}{2} + 2\theta\gamma(1-\gamma), \\ \mathcal{G} &= (n-1)(n - \theta - \gamma) - \theta(1-\gamma)^2. \end{aligned} \tag{4.8}$$

(All the constants are positive for θ small enough.)

The construction of coefficients guarantees both L_∞ estimates and the ellipticity of the system.

Assertions analogous to Remarks 1, 2 of Section 3 can be formulated for this case, too. □

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