An existence theorem of positive solutions to a singular nonlinear boundary value problem

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Abstract. In this note we consider the boundary value problem $y'' = f(x, y, y')$ $(x \in [0, X]; X > 0)$, $y(0) = 0$, $y(X) = a > 0$; where f is a real function which may be singular at $y = 0$. We prove an existence theorem of positive solutions to the previous problem, under different hypotheses of Theorem 2 of L.E. Bobisud [J. Math. Anal. Appl. **173** (1993), 69–83], that extends and improves Theorem 3.2 of D. O'Regan [J. Differential Equations **84** (1990), 228–251].

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Let f be a real function defined on $[0, X] \times (0, \infty) \times (-\infty, \infty)$; $L^1([0, X])$ the space of all (equivalence classes of) measurable functions $\psi : [0, X] \to \mathbb{R}$ such that $\|\psi\|_{L^1([0,X])} = \int_0^X |\psi(x)|\, dx < \infty$; $W^{2,1}([0, X])$ the space of all $u \in C^1([0, X])$ such that u' is absolutely continuous in $[0, X]$ and $u'' \in L^1([0, X])$.

Consider the problem

$$\text{(P)} \qquad \begin{cases} y'' = f(x, y, y') \\ y(0) = 0 \\ y(X) = a > 0\,. \end{cases}$$

A function $u : [0, X] \to [0, \infty)$ is said to be a generalized solution to (P) if $u \in W^{2,1}([0, X])$, $u(0) = 0$, $u(X) = a$ and, for almost every $x \in [0, X]$, one has $u''(x) = f(x, u(x), u'(x))$. When the function f is continuous in $[0, X] \times (0, \infty) \times (-\infty, \infty)$, any generalized solution to problem (P) is a classical one, that is $u \in C^1([0, X]) \cap C^2((0, X])$ and $u''(x) = f(x, u(x), u'(x))$ for every $x \in (0, X]$.

Positive solutions to singular nonlinear boundary value problems appear in a variety of applications. Consequently, they have been studied by many authors (see, for instance, [2], [4] and the references given there). In particular, among the latest contributions, there are the following two theorems.

Theorem A ([2, Theorem 2]). *Let $X \geq 1$ be fixed. Assume the following hypotheses.*

(H$_1$) $f \in C([0, X] \times (0, \infty) \times (-\infty, \infty))$ *and $f(x, y, z)$ is locally Lipschitz in y and z on* $[0, X] \times (0, \infty) \times (-\infty, \infty)$.

(H$_2$) *$zf(x,y,z) \le 0$ on $[0,X] \times (0,\infty) \times (-\infty,\infty)$.*

(H$_3$) *There exist a nonnegative function f_1 continuous on $[0,1]$, a nonnegative, nonincreasing function g_1 continuous on $(0,a]$, and a function h_1 positive and continuous on $(a,\infty]$ such that*

- (i) *$f(x,y,z) \ge -f_1(x)g_1(y)h_1(z)z$ on $[0,X] \times (0,a] \times [a,\infty)$,*
- (ii) *$f_1(s)g_1(\frac{a}{X}s) \in L^1([0,1])$,*
- (iii) *$\int_a^\infty dv/vh_1(v) > \int_0^1 f_1(s)g_1(\frac{a}{X}s)\,ds$*

hold.

(H$_4$) *Put*

$$H(z) = \int_a^z \frac{1}{h_1(v)}\,dv; \quad \text{and} \quad M_1 = H^{-1}\left(\int_0^a g_1(u)\,du\right),$$

there exist a constant $k > M_1$ and a measurable function F on $[0,X]$ satisfying

- (i) *$|f(x,y,z)| \le F(x)$ for $0 \le x \le X$, $\frac{a}{X}x \le y \le k$, and $|z| \le k$,*
- (ii) *$\int_0^X F(x)\,dx < \infty$.*

Then, the problem (P) has at least one solution $u \in C^1([0,X]) \cap C^2((0,X])$ such that $u(x) > 0$ for every $x \in (0,X]$.

Theorem B ([4, Theorem 3.2 and subsequent remark])**.** *Consider the problem*

$$(\mathrm{P}_0) \qquad \begin{cases} y'' + \Psi(x)h(x,y) = 0 & 0 < x < 1 \\ y(0) = 0 \\ y(1) = a > 0\,. \end{cases}$$

where h and Ψ satisfy

(K$_1$)

- (i) *h is continuous on $[0,1] \times (0,\infty)$;*
- (ii) *$\lim_{y\to 0^+} h(x,y) = \infty$ for each $x \in [0,1]$;*
- (iii) *$0 < h(x,y) \le g(y)$ on $[0,1]$, where g is continuous and nonincreasing on $(0,\infty)$.*
- (iv) *In addition $1/\Psi \in C([0,1])$ with $\Psi > 0$ on $(0,1)$.*

(K$_2$) *There exist $p > 1$, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ together with $\int_0^1 \Psi^p(z)\,dz < \infty$ and $\int_0^1 g^q(u)\,du < \infty$.*

(K$_3$) *For each constant $M > 0$ there exists $\eta(x)$ continuous and positive on $[0,1]$ such that $h(x,y) \ge \eta(x)$ on $[0,1] \times (0,M]$.*

Then, the problem (P$_0$) has at least one solution $u \in C([0,1]) \cap C^2((0,1))$ such that $u(x) > 0$ for every $x \in (0,1]$.

The purpose of this note is to establish Theorem 1 below. We remark that our result extends and improve Theorem B (see Remark 3) and is independent of Theorem A. In particular, contrary to (H_1), we assume that f is continuous in y and z. Moreover, the condition $f(x, y, 0) \equiv 0$, which is implied by (H_2), does not follow from our assumptions.

Let $r > 0$, $X > 0$ and $x \in [0, X]$. Here and in the sequel, $W(r, x)$ stands for the set $\left\{(y, z) \in (0, \infty) \times (-\infty, \infty) : \frac{a}{X}x \le y \le a + Xr; |z| \le \frac{a}{X} + 2r\right\}$. Let now f be a real function defined on $[0, X] \times (0, \infty) \times (-\infty, \infty)$. For every $x \in [0, X]$, we put

$$M_r(x) = \sup_{(y,z)\in W(r,x)} |f(x, y, z)| \quad \text{and} \quad m_r(x) = \sup_{(y,z)\in W(r,x)} f(x, y, z).$$

Theorem 1. *Let f be a real function defined in $[0, X] \times (0, \infty) \times (-\infty, \infty)$. Assume that*

(a) *the function $(y, z) \to f(x, y, z)$ is continuous for almost every $x \in [0, X]$;*
(b) *the function $x \to f(x, y, z)$ is measurable for every $(y, z) \in (0, \infty) \times (-\infty, \infty)$;*
(c) *there exists $r > 0$ such that the function M_r belongs to $L^1([0, X])$ and one has*
$$\|M_r\|_{L^1([0,X])} \le r;$$
(d) *for almost every $x \in [0, X]$, one has*
$$m_r(x) < 0.$$

Then, the problem (P) *has at least one generalized solution $u \in W^{2,1}([0, X])$ such that $u(x) > 0$ for every $x \in (0, X]$.*

Proof: Consider the set

$$K = \left\{v \in L^1([0, X]) : -m_r(x) \le v(x) \le M_r(x) \ \text{ a.e. in } \ [0, X]\right\}.$$

Of course, K is nonempty and convex. By the Dunford-Pettis theorem (see, for instance, [3, Theorem 1, p. 101]), it is also weakly compact. For every $v \in L^1([0, X])$ and every $x \in [0, X]$, we put

$$\begin{aligned} \phi_1(v)(x) &= \frac{a}{X}x + \frac{X - x}{X}\int_0^x sv(s)\, ds + \frac{x}{X}\int_x^X (X - s)v(s)\, ds; \\ \phi_2(v)(x) &= \frac{a}{X} - \frac{1}{X}\int_0^X sv(s)\, ds + \int_x^X v(s)\, ds; \end{aligned} \tag{1}$$

Obviously, one has $\phi_1(v)(0) = 0$, $\phi_1(v)(X) = a$, $[\phi_1(v)]' = \phi_2(v)$; $[\phi_1(v)]'' = [\phi_2(v)]' = -v$; $\phi_1(v) \in W^{2,1}([0, X])$, moreover, if $v(x) > 0$ for almost $x \in [0, X]$, therefore $\phi_1(x) > 0$ for every $x \in (0, X]$. We now put

$$G(v)(x) = -f(x, \phi_1(v)(x), \phi_2(v)(x))$$

for every $v \in L^1([0,X])$ and for every $x \in (0,X]$.

Let us prove that $G(K) \subseteq K$. To this end, fix $v \in K$ and observe that, by (1) and (c), one has

$$\begin{aligned}\frac{a}{X}x \le \phi_1(v)(x) &\le a + \int_0^x Xv(s)\,ds + \int_x^X Xv(s)\,ds\\ &\le a + X\,\|M_r\|_{L^1([0,X])} \le a + Xr;\\ |\phi_2(v)(x)| &\le \frac{a}{X} + \frac{1}{X}\int_0^X Xv(s)\,ds + \int_0^X v(s)\,ds\\ &\le \frac{a}{X} + 2\,\|M_r\|_{L^1([0,X])} \le \frac{a}{X} + 2r.\end{aligned}$$

Therefore, $(\phi_1(v)(x), \phi_2(v)(x)) \in W(r,x)$ for every $x \in (0,X]$. Hence, for almost every $x \in [0,X]$, one has:

$$-m_r(x) \le -f(x, \phi_1(v)(x), \phi_2(v)(x)) \le M_r(x).$$

This implies that $G(v) \in K$.

Now, let us prove that the operator G is weakly sequentially continuous. Let $v \in K$ and let $\{v_n\}$ be a sequence in K weakly converging to v in $L^1([0,X])$. From (1) it follows that, for every $x \in [0,X]$, $\lim_{n\to\infty} \phi_1(v_n)(x) = \phi_1(v)(x)$; $\lim_{n\to\infty} \phi_2(v_n)(x) = \phi_2(v)(x)$. Therefore, by (a), the sequence $\{G(v_n)\}$ converges almost everywhere in $[0,X]$ to $G(v)$. Bearing in mind that for almost every $x \in [0,X]$ and every $n \in \mathbb{N}$ one has

$$|G(v_n)(x)| \le M_r(x),$$

the Lebesgue Dominated Convergence theorem yields $\lim_{n\to\infty} G(v_n) = G(v)$ in $L^1([0,X])$. So, $\{G(v_n)\}$ converges weakly to $G(v)$ in $L^1([0,X])$.

We now have proved that the function $G : K \to K$ verifies all that assumptions of Theorem 1 of [1]. Then, there is $v \in K$ such that $v = G(v)$. The function $u(x) = \phi_1(v)(x)$, $x \in [0,X]$, satisfies our conclusion. □

Remark 1. This theorem ensures the existence of positive solutions even if $f(x,y,z)$ is not locally Lipschitz in y and z. For example, the problem

$$(\mathrm{P}_1) \qquad \begin{cases} y'' = -(\operatorname{sen} y)^{1/3}|y'|^{1/3} - xy^{-1/2}|y'|^{1/2} - x^3\\ y(0) = 0\\ y(1) = a > 0\,, \end{cases}$$

owing to Theorem 1, has at least one positive solution $u \in C^1([0,X]) \cap C^2((0,X])$. Indeed, taking into account that

$$\int_0^X \sup_{(y,z)\in W(r,x)} |f(x,y,z)|\,dx \le \Big(\frac{a}{X} + 2r\Big)^{1/3} X + \frac{2}{3}\frac{X^2}{\sqrt{a}}\Big(\frac{a}{X} + 2r\Big)^{1/2} + \frac{X^4}{4}$$

and

$$\lim_{r\to\infty}\frac{r-\frac{X^4}{4}}{\left(\frac{a}{X}+2r\right)^{1/3}X+\frac{2}{3}\frac{X^2}{\sqrt{a}}\left(\frac{a}{X}+2r\right)^{1/2}}=\infty,$$

there exists $r>0$ such that $\|M_r\|_{L^1([0,X])}<r$. Hence, it is easily seen that all the assumptions of Theorem 1 hold.

We cannot apply Theorem A to the problem (P_1), even because $f(x,y,0)=x^3\not\equiv 0$.

We also observe that assumption (H_3) and (H_4) of Theorem A and assumption (c) of Theorem 1 are mutually independent.

Remark 2. We explicitly observe that in Theorem 1 f may be singular at some set $\Omega\subseteq[0,X]$, with $|\Omega|=0$ ($|\Omega|$ denotes the Lebesgue measure of Ω). Particularly, if $f\in C((0,X)\times(0,\infty)\times(-\infty,\infty))$ and the assumptions (c) and (d) of Theorem 1 hold, then there exists at least one function $u\in C^1([0,X])\cap C^2((0,X))$ such that $u(0)=0$, u(X)=a and, for every $x\in(0,X)$, $u''(x)=f(x,u(x),u'(x))$ and $u(x)>0$.

Remark 3. Theorem 1 extends and improves Theorem B. Indeed, the assumptions of Theorem B, even without the condition $\lim_{y\to 0^+}h(x,y)=\infty$, imply the ones of Theorem 1. Let us prove this. Of course, from (i) and (iv) of (K_1), (a) and (b) follow; (c) is verified by choosing $r=\|\Psi\|_{L^p([0,1])}\left(\frac{1}{a}\right)^{1/q}\|g\|_{L^q([0,a])}$, since, by (iii) of ($K_1$), ($K_2$) and Hölder inequality, one has

$$\int_0^1\sup_{\frac{a}{X}x\le y\le a+Xr}|\Psi(x)h(x,y)|\,dx\le$$

$$\le\int_0^1\Psi(x)g\left(\frac{a}{X}x\right)\,dx\le\|\Psi\|_{L^p([0,1])}\left(\frac{1}{a}\right)^{1/q}\|g\|_{L^q([0,a])};$$

(d) follows from (iv) of (K_1) and (K_3), since in $(0,a+Xr]$ one has $\Psi(x)h(x,y)\ge\Psi(x)\eta(x)>0$, therefore

$$-m_r(x)=\inf_{\frac{a}{X}x\le y\le a+Xr}\Psi(x)h(x,y)\ge\Psi(x)\eta(x)>0$$

for every $x\in(0,1)$. Hence, our claim is proved.

Now, consider the problem

$$(\mathrm{P}_2)\qquad\begin{cases}y''+x\left[\left|\operatorname{sen}\frac{1}{y}\right|^{1/2}+y^{1/2}+x\right]=0\\ y(0)=0\\ y(1)=a>0\,.\end{cases}$$

Owing to Theorem 1, the problem (P$_2$) has at least one positive solution $u \in C^1([0,X]) \cap C^2((0,X])$. Indeed, taking into account that

$$\int_0^X \sup_{\frac{a}{X}x \le y \le a+Xr} |f(x,y)|\,dx \le \frac{X^2}{2} + \frac{X^2}{2}(a+Xr)^{1/2} + \frac{X^3}{3}$$

and

$$\lim_{r\to\infty} \frac{r - \left(\frac{X^2}{2} + \frac{X^3}{3}\right)}{\frac{X^2}{2}(a+Xr)^{1/2}} = \infty,$$

there exists $r > 0$ such that $\|M_r\|_{L^1([0,X])} < r$. Hence, it is easily seen that all hypotheses of Theorem 1 hold and our claim is proved.

In the previous example the condition $\lim_{y\to 0^+} h(x,y) = \infty$ is not satisfied and moreover there is no function $g(y)$, nonincreasing in $(0,\infty)$, such that $h(x,y) \le g(y)$, as it is required by Theorem B.

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