Some commutative neutrix convolution products of functions

Brian Fisher, Adem Kiliçman

Abstract. The commutative neutrix convolution product of the locally summable functions $\cos_-(\lambda x)$ and $\cos_+(\mu x)$ is evaluated. Further similar commutative neutrix convolution products are evaluated and deduced.

Keywords: neutrix, neutrix limit, neutrix convolution product

Classification: 46F10

In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The convolution product $f*g$ of two distributions f and g in $\mathcal{D}'$ is then usually defined by the equation

$$\langle (f*g)(x),\phi\rangle = \langle f(y),\langle g(x),\phi(x+y)\rangle\rangle$$

for arbitrary ϕ in $\mathcal{D}$, provided f and g satisfy either of the conditions

(a) either f or g has bounded support,

(b) the supports of f and g are bounded on the same side,

see Gel'fand and Shilov [5].

Note that if f and g are locally summable functions satisfying either of the above conditions then

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt. \tag{1}$$

It follows that if the convolution product $f*g$ exists by this definition then

$$f*g = g*f, \tag{2}$$

$$(f*g)' = f*g' = f'*g. \tag{3}$$

This definition of the convolution product is rather restrictive and can only be used for a small class of distributions. In order to extend the convolution product to a larger class of distributions, the commutative neutrix convolution product was introduced in [3] and was extended in [2]. In the following, we give a further generalization. We first of all let τ be a function in $\mathcal{D}$ satisfying the following conditions:

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \leq \tau(x) \leq 1$,

(iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$,

(iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function τ_ν is now defined by

$$\tau_\nu(x) = \begin{cases} 1, & |x| \leq \nu, \\ \tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu, \\ \tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu, \end{cases}$$

for $\nu > 0$.

We now define the extended neutrix convolution product.

Definition 1. *Let f and g be distributions in $\mathcal{D}'$ and let $f_\nu(x) = f(x)\tau_\nu(x)$ and $g_\nu(x) = g(x)\tau_\nu(x)$ for $\nu > 0$. Then the neutrix convolution product $f \boxed{*} g$ is defined as the neutrix limit of the sequence $\{f_\nu * g_\nu\}$, provided that the limit h exists in the sense that*

$$\underset{\nu\to\infty}{\mathrm{N-lim}}\langle f_\nu * g_\nu, \phi\rangle = \langle h, \phi\rangle,$$

for all ϕ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], *having domain N' the positive reals and range N'' the real numbers, with negligible functions finite linear sums of the functions*

$$\nu^\lambda \ln^{r-1}\nu,\ \ln^r\nu,\ \nu^\mu e^{\lambda\nu},\ \nu^\mu\cos\lambda\nu,\ \nu^\mu\sin\lambda\nu \quad (\lambda \neq 0,\ r = 1, 2, \dots)$$

and all functions which converge to zero in the usual sense as ν tends to infinity.

Note that in this definition the convolution product $f_\nu * g_\nu$ is defined in Gel'fand and Shilov's sense, the distribution f_ν and g_ν having bounded support. It is clear that if the neutrix convolution product $f \boxed{*} g$ exists then the neutrix convolution product $g \boxed{*} f$ exists and $f \boxed{*} g = g \boxed{*} f$.

In the original definition of the neutrix convolution product, the domain of the neutrix N was the set of positive integers $N' = \{1, 2, \dots, n, \dots\}$ and the negligible functions were finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n \quad (\lambda > 0,\ r = 1, 2, \dots)$$

and all functions which converge to zero in the usual sense as n tends to infinity. In [2], the set of negligible functions was extended to include finite linear sums of the functions

$$n^\lambda e^{\mu n} \quad (\mu > 0).$$

It is easily seen that any results proved with the original definition hold with the new definition. The following theorem proved in [3] therefore holds.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition* (a) *or condition* (b) *of Gel'fand and Shilov's definition. Then the neutrix convolution product $f \boxast g$ exists and*

$$f \boxast g = f * g.$$

A number of neutrix convolution products have been evaluated. For example, $x_-^\lambda \boxast x_+^\mu$ see [3], $\ln x_- \boxast \ln x_+$ see [6] and $\ln x_- \boxast x_+^r$ see [4].

We now define the locally summable functions $e_+^{\lambda x}$, $e_-^{\lambda x}$, $\cos_+(\lambda x)$, $\cos_-(\lambda x)$, $\sin_+(\lambda x)$ and $\sin_-(\lambda x)$ by

$$e_+^{\lambda x} = \begin{cases} e^{\lambda x}, & x > 0, \\ 0, & x < 0, \end{cases} \qquad e_-^{\lambda x} = \begin{cases} 0, & x > 0, \\ e^{\lambda x}, & x < 0, \end{cases}$$

$$\cos_+(\lambda x) = \begin{cases} \cos(\lambda x), & x > 0, \\ 0, & x < 0, \end{cases} \qquad \cos_-(\lambda x) = \begin{cases} 0, & x > 0, \\ \cos(\lambda x), & x < 0, \end{cases}$$

$$\sin_+(\lambda x) = \begin{cases} \sin(\lambda x), & x > 0, \\ 0, & x < 0, \end{cases} \qquad \sin_-(\lambda x) = \begin{cases} 0, & x > 0, \\ \sin(\lambda x), & x < 0. \end{cases}$$

It follows that

$$\cos_-(\lambda x) + \cos_+(\lambda x) = \cos(\lambda x), \quad \sin_-(\lambda x) + \sin_+(\lambda x) = \sin(\lambda x).$$

The following theorem was proved in [2]

Theorem 2. *The neutrix convolution product $(x^r e_-^{\lambda x}) \boxast (x^s e_+^{\mu x})$ exists and*

$$(x^r e_-^{\lambda x}) \boxast (x^s e_+^{\mu x}) = D_\lambda^r D_\mu^s \frac{e_+^{\mu x} + e_-^{\lambda x}}{\lambda - \mu},$$

where $D_\lambda = \partial/\partial\lambda$ and $D_\mu = \partial/\partial\mu$, for $\lambda \neq \mu$ and $r, s = 0, 1, 2, \dots$, these neutrix convolution products existing as convolution products if $\lambda > \mu$ and

$$(x^r e_-^{\lambda x}) \boxast (x^s e_+^{\lambda x}) = -B(r+1, s+1) x^{r+s+1} \operatorname{sgn} x \cdot e^{\lambda x},$$

for all λ and $r, s = 0, 1, 2, \dots$, where B denotes the Beta function.

We now prove the following theorem.

Theorem 3. *The neutrix convolution products $\cos_-(\lambda x) \boxast \cos_+(\mu x)$, $\cos_-(\lambda x) \boxast \sin_+(\mu x)$, $\sin_-(\lambda x) \boxast \cos_+(\mu x)$ and $\sin_-(\lambda x) \boxast \sin_+(\mu x)$ exist and*

$$\cos_-(\lambda x) \boxast \cos_+(\mu x) = \frac{\lambda \sin_-(\lambda x) + \mu \sin_+(\mu x)}{\lambda^2 - \mu^2}, \tag{4}$$

$$\cos_-(\lambda x) \boxast \sin_+(\mu x) = -\frac{\mu \cos_-(\lambda x) + \mu \cos_+(\mu x)}{\lambda^2 - \mu^2}, \tag{5}$$

$$\sin_-(\lambda x) \boxast \cos_+(\mu x) = -\frac{\lambda \cos_-(\lambda x) + \lambda \cos_+(\mu x)}{\lambda^2 - \mu^2}, \tag{6}$$

$$\sin_-(\lambda x) \boxast \sin_+(\mu x) = -\frac{\mu \sin_-(\lambda x) + \lambda \sin_+(\mu x)}{\lambda^2 - \mu^2}, \tag{7}$$

for $\lambda \neq \pm\mu$.

PROOF: We first of all note that since

$$\sin(\lambda x + \mu\nu) = \sin(\lambda x)\cos(\mu\nu) + \cos(\lambda x)\sin(\mu\nu),$$

it follows that

$$\operatorname*{N-lim}_{\nu\to\infty} \sin(\lambda x + \mu\nu) = \operatorname*{N-lim}_{\nu\to\infty} \nu\sin(\lambda x + \mu\nu) = 0 \tag{8}$$

for $\mu \neq 0$. Similarly

$$\operatorname*{N-lim}_{\nu\to\infty} \cos(\lambda x + \mu\nu) = \operatorname*{N-lim}_{\nu\to\infty} \nu\cos(\lambda x + \mu\nu) = 0 \tag{9}$$

for $\mu \neq 0$.

We now put $[\cos_-(\lambda x)]_\nu = \cos_-(\lambda x)\tau_\nu(x)$ and $[\cos_+(\mu x)]_\nu = \cos_+(\mu x)\tau_\nu(x)$. Since $[\cos_+(\mu x)]_\nu$ and $[\cos_-(\lambda x)]_\nu$ are locally summable functions with $[\cos_-(\lambda x)]_\nu$ and $[\cos_+(\mu x)]_\nu$ having compact support, the convolution product $[\cos_-(\lambda x)]_\nu * [\cos_+(\mu x)]_\nu$ is defined by equation (1) and so

$$[\cos_-(\lambda x)]_\nu * [\cos_+(\mu x)]_\nu = \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\cos_+(\mu(x-t))]_\nu \, dt. \tag{10}$$

When $-\nu \leq x \leq 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\cos_+(\mu(x-t))]_\nu \, dt &= \int_{-\nu}^{x} \cos(\lambda t)\cos[\mu(x-t)]\, dt + \\ &+ \int_{-\nu-\nu^{-\nu}}^{-\nu} \cos(\lambda t)\cos[\mu(x-t)]\tau_\nu(t)\tau_\nu(x-t)\, dt \\ &= \frac{\sin(\lambda x) - \sin[\mu x - (\lambda-\mu)\nu]}{2(\lambda-\mu)} + \\ &+ \frac{\sin(\lambda x) + \sin[\mu x + (\lambda+\mu)\nu]}{2(\lambda+\mu)} + O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$\operatorname*{N-lim}_{\nu\to\infty} \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\cos_+(\mu(x-t))]_\nu \, dt = \frac{\lambda\sin(\lambda x)}{\lambda^2 - \mu^2}, \tag{11}$$

on using equation (8).

When $\nu \geq x \geq 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\cos_+(\mu(x-t))]_\nu \, dt &= \int_{x-\nu}^{0} \cos(\lambda t)\cos[\mu(x-t)]\, dt + \\ &+ \int_{x-\nu-\nu^{-\nu}}^{x-\nu} \cos(\lambda t)\cos[\mu(x-t)]\tau_\nu(t)\tau_\nu(x-t)\, dt \\ &= \frac{\sin(\mu x) - \sin[\mu x + (\lambda-\mu)(x-\nu)]}{2(\lambda-\mu)} + \\ &- \frac{\sin(\mu x) + \sin[\mu x - (\lambda+\mu)(x-\nu)]}{2(\lambda+\mu)} + O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$\text{(12)} \qquad \mathop{\mathrm{N-lim}}_{\nu\to\infty} \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\cos_+(\mu(x-t))]_\nu \, dt = \frac{\mu \sin(\mu x)}{\lambda^2 - \mu^2},$$

on using equation (8).

It now follows from equations (10), (11) and (12) that for arbitrary ϕ in $\mathcal{D}$

$$\mathop{\mathrm{N-lim}}_{\nu\to\infty} \langle [\cos_-(\lambda x)]_\nu * [\cos_+(\mu x)]_\nu, \phi(x)\rangle = \frac{\lambda}{\lambda^2-\mu^2}\langle \sin_-(\lambda x), \phi(x)\rangle + \\ + \frac{\mu}{\lambda^2-\mu^2}\langle \sin_+(\mu x), \phi(x)\rangle$$

and equation (4) follows.

We now prove equation (5). Putting $[\sin_+(\mu x)]_\nu = \sin_+(\mu x)\tau_\nu(x)$, we have as above

$$\text{(13)} \qquad [\cos_-(\lambda x)]_\nu * [\sin_+(\mu x)]_\nu = \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\sin_+((\mu(x-t))]_\nu \, dt.$$

When $-\nu \le x \le 0$,

$$\int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\sin_+(\mu(x-t))]_\nu \, dt = \int_{-\nu}^{x} \cos(\lambda t)\sin[\mu(x-t)]\, dt + \\ + \int_{-\nu-\nu^{-\nu}}^{-\nu} \cos(\lambda t)\sin[\mu(x-t)]\tau_\nu(t)\tau_\nu(x-t)\, dt \\ = -\frac{\cos(\lambda x) - \cos[\mu x - (\lambda-\mu)\nu]}{2(\lambda-\mu)} + \\ + \frac{\cos(\lambda x) - \cos[\mu x + (\lambda+\mu)\nu]}{2(\lambda+\mu)} + O(\nu^{-\nu}),$$

and it follows that

$$\text{(14)} \qquad \mathop{\mathrm{N-lim}}_{\nu\to\infty} \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\sin_+(\mu(x-t))]_\nu \, dt = -\frac{\mu \cos(\lambda x)}{\lambda^2 - \mu^2},$$

on using equations (9).

When $\nu \ge x \ge 0$,

$$\int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\sin_+(\mu(x-t))]_\nu \, dt = \int_{x-\nu}^{0} \cos(\lambda t)\sin[\mu(x-t)]\, dt + \\ + \int_{x-\nu-\nu^{-\nu}}^{x-\nu} \cos(\lambda t)\sin[\mu(x-t)]\tau_\nu(t)\tau_\nu(x-t)\, dt \\ = -\frac{\cos(\mu x) - \cos[\mu x + (\lambda-\mu)(x-\nu)]}{2(\lambda-\mu)} + \\ + \frac{\cos(\mu x) - \cos[\mu x - (\lambda+\mu)(x-\nu)]}{2(\lambda+\mu)} + O(\nu^{-\nu}),$$

and it follows that

$$\text{(15)}\qquad \mathop{\text{N}-\lim}_{\nu\to\infty}\int_{-\infty}^{\infty}[\cos_-(\lambda t)]_\nu[\sin_+(\mu(x-t))]_\nu\,dt = -\frac{\mu\cos(\lambda x)}{\lambda^2-\mu^2},$$

on using equations (9).

Equation (5) now follows as above on using equations (13), (14) and (15).

Replacing x by $-x$ in equation (5) and interchanging λ and μ we get

$$-\cos_+(\mu x)\boxed{*}\sin_-(\lambda x) = -\frac{\lambda\cos_+(\mu x)+\lambda\cos_-(\lambda x)}{\mu^2-\lambda^2}.$$

Equation (6) now follows since the convolution is commutative.

We finally prove equation (7). Putting $[\sin_-(\lambda x)]_\nu = \sin_-(\lambda x)\tau_\nu(x)$, we have as above

$$\text{(16)}\qquad [\sin_-(\lambda x)]_\nu * [\sin_+(\mu x)]_\nu = \int_{-\infty}^{\infty}[\sin_-(\lambda t)]_\nu[\sin_+(\mu(x-t))]_\nu\,dt.$$

When $-\nu \le x \le 0$,

$$\begin{aligned}\int_{-\infty}^{\infty}[\sin_-(\lambda t)]_\nu[\sin_+(\mu(x-t))]_\nu\,dt &= \int_{-\nu}^{x}\sin(\lambda t)\sin[\mu(x-t)]\,dt+\\ &+\int_{-\nu-\nu^{-\nu}}^{-\nu}\sin(\lambda t)\sin[\mu(x-t)]\tau_\nu(t)\tau_\nu(x-t)\,dt\\ &= \frac{\sin(\lambda x)+\sin[\mu x+(\lambda+\mu)\nu]}{2(\lambda+\mu)}+\\ &-\frac{\sin(\lambda x)-\sin[\mu x-(\lambda-\mu)\nu]}{2(\lambda-\mu)}+O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$\text{(17)}\qquad \mathop{\text{N}-\lim}_{\nu\to\infty}\int_{-\infty}^{\infty}[\sin_-(\lambda t)]_\nu[\sin_+(\mu(x-t))]_\nu\,dt = -\frac{\mu\sin(\lambda x)}{\lambda^2-\mu^2},$$

on using equations (9).

When $\nu \ge x \ge 0$,

$$\begin{aligned}\int_{-\infty}^{\infty}[\sin_-(\lambda t)]_\nu[\sin_+(\mu(x-t))]_\nu\,dt &= \int_{x-\nu}^{0}\sin(\lambda t)\sin[\mu(x-t)]\,dt+\\ &+\int_{x-\nu-\nu^{-\nu}}^{x-\nu}\sin(\lambda t)\sin[\mu(x-t)]\tau_\nu(t)\tau_\nu(x-t)\,dt\\ &= -\frac{\sin(\mu x)-\sin[\mu x-(\lambda+\mu)(x-\nu)]}{2(\lambda+\mu)}+\\ &-\frac{\sin(\mu x)-\sin[\mu x+(\lambda-\mu)(x-\nu)]}{2(\lambda-\mu)}+O(\nu^{-\nu}),\end{aligned}$$

and it follows that

$$\operatorname*{N-lim}_{\nu\to\infty} \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu [\sin_+((x-t))]_\nu \, dt = -\frac{\lambda \sin(\mu x)}{\lambda^2 - \mu^2} \tag{18}$$

on using equations (9).

Equation (7) now follows as above on using equations (16), (17) and (18).

Corollary. *The neutrix convolution products* $[1-H(x)] \boxed{*} \cos_+(\mu x)$, $\cos_-(\lambda x) \boxed{*} H(x)$, $[1-H(x)] \boxed{*} \sin_+(\mu x)$ *and* $\sin_-(\lambda x) \boxed{*} H(x)$ *exist and*

$$[1-H(x)] \boxed{*} \cos_+(\mu x) = -\mu^{-1} \sin_+(\mu x), \tag{19}$$

$$\cos_-(\lambda x) \boxed{*} H(x) = \lambda^{-1} \sin_-(\lambda x), \tag{20}$$

$$[1-H(x)] \boxed{*} \sin_+(\mu x) = \mu^{-1}[1 - H(x) + \cos_+(\mu x)], \tag{21}$$

$$\sin_-(\lambda x) \boxed{*} H(x) = -\lambda^{-1}[H(x) + \cos_-(\lambda x)], \tag{22}$$

for $\lambda, \mu \neq 0$, *where* H *denotes Heaviside's function.*

Proof: Equations (19) and (20) follow from equations (4) and (5) respectively on putting $\lambda = 0$ and equations (20) and (21) follow from equations (4) and (6) respectively on putting $\mu = 0$. □

Further results can be easily deduced. For example, it is easily proved that

$$\cos_+(\lambda x) * \cos_+(\mu x) = \frac{\lambda \sin_+(\lambda x) - \mu \sin_+(\mu x)}{\lambda^2 - \mu^2},$$

for $\lambda \neq \pm\mu$, and it follows that

$$\begin{aligned} \cos(\lambda x) \boxed{*} \cos_+(\mu x) &= \cos_-(\lambda x) \boxed{*} \cos_+(\mu x) + \cos_+(\lambda x) * \cos_+(\mu x) \\ &= \frac{\lambda \sin(\lambda x)}{\lambda^2 - \mu^2}. \end{aligned}$$

Replacing x by $-x$ in this equation we get

$$\cos(\lambda x) \boxed{*} \cos_-(\mu x) = -\frac{\lambda \sin(\lambda x)}{\lambda^2 - \mu^2}$$

and so

$$\cos(\lambda x) \boxed{*} \cos(\mu x) = \cos(\lambda x) \boxed{*} \cos_-(\mu x) + \cos(\lambda x) \boxed{*} \cos_+(\mu x) = 0.$$

Theorem 4. *The neutrix convolution products* $\cos_-(\lambda x) \boxed{*} \cos_+(\lambda x)$, $\cos_-(\lambda x) \boxed{*} \sin_+(\lambda x)$, $\sin_-(\lambda x) \boxed{*} \cos_+(\lambda x)$ *and* $\sin_-(\lambda x) \boxed{*} \sin_+(\lambda x)$ *exist and*

(23)
$$\cos_-(\lambda x) \boxed{*} \cos_+(\lambda x) = \frac{2\lambda x[\cos_-(\lambda x) - \cos_+(\lambda x)] + \sin_-(\lambda x) - \sin_+(\lambda x)}{4\lambda},$$

(24)
$$\cos_-(\lambda x) \boxed{*} \sin_+(\lambda x) = \frac{2\lambda x[\sin_-(\lambda x) - \sin_+(\lambda x)] + \cos(\lambda x)}{4\lambda},$$

(25)
$$\sin_-(\lambda x) \boxed{*} \cos_+(\lambda x) = -\frac{2\lambda x[\sin_+(\lambda x) - \sin_-(\lambda x)] + \cos(\lambda x)}{4\lambda},$$

(26)
$$\sin_-(\lambda x) \boxed{*} \sin_+(\lambda x) = \frac{2\lambda x[\cos_+(\lambda x) - \cos_-(\lambda x)] + \sin_-(\lambda x) - \sin_+(\lambda x)}{4\lambda},$$

for $\lambda \neq 0$.

Proof: We have

$$[\cos_-(\lambda x)]_\nu * [\cos_+(\lambda x)]_\nu = \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\cos_+(\lambda(x-t))]_\nu \, dt. \tag{27}$$

When $-\nu \leq x \leq 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\cos_+(\lambda(x-t))]_\nu \, dt &= \int_{-\nu}^{x} \cos(\lambda t)\cos[\lambda(x-t)]\, dt + \\ &+ \int_{-\nu-\nu^{-\nu}}^{-\nu} \cos(\lambda t)\cos[\lambda(x-t)]\tau_\nu(t)\tau_\nu(x-t)\, dt \\ &= \frac{(x+\nu)\cos(\lambda x)}{2} + \frac{\sin(\lambda x) + \sin(\lambda x + 2\lambda\nu)}{4\lambda} + O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$\operatorname*{N-lim}_{\nu\to\infty} \int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\cos_+(\lambda(x-t))]_\nu \, dt = \frac{2\lambda x \cos(\lambda x) + \sin(\lambda x)}{4\lambda}, \tag{28}$$

on using equation (8).

When $\nu \geq x \geq 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} [\cos_-(\lambda t)]_\nu [\cos_+(\lambda(x-t))]_\nu \, dt &= \int_{x-\nu}^{0} \cos(\lambda t)\cos[\lambda(x-t)]\, dt + \\ &+ \int_{x-\nu-\nu^{-\nu}}^{x-\nu} \cos(\lambda t)\cos[\lambda(x-t)]\tau_\nu(t)\tau_\nu(x-t)\, dt \\ &= -\frac{(x-\nu)\cos(\lambda x)}{2} - \frac{\sin(\lambda x) + \sin(\lambda x - 2\lambda\nu)}{4\lambda} + O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$\text{(29)} \qquad \underset{\nu\to\infty}{\text{N}-\lim}\int_{-\infty}^{\infty}[\cos(\lambda t)]_\nu[\cos_+(\lambda(x-t))]_\nu\,dt = -\frac{2\lambda x\cos(\lambda x)+\sin(\lambda x)}{4\lambda},$$

on using equation (8).

Equation (23) now follows as above on using equations (27), (28) and (29).

We now prove equation (24). We have as above

$$\text{(30)} \qquad [\cos_-(\lambda x)]_\nu * [\sin_+(\lambda x)]_\nu = \int_{-\infty}^{\infty}[\cos_-(\lambda t)]_\nu[\sin_+(\lambda(x-t))]_\nu\,dt.$$

When $-\nu \le x \le 0$,

$$\begin{aligned}\int_{-\infty}^{\infty}[\cos_-(\lambda t)]_\nu[\sin_+(\lambda(x-t))]_\nu\,dt &= \int_{-\nu}^{x}\cos(\lambda t)\sin[\lambda(x-t)]\,dt+\\ &+\int_{-\nu-\nu^{-\nu}}^{-\nu}\cos(\lambda t)\sin[\lambda(x-t)]\tau_\nu(t)\tau_\nu(x-t)\,dt\\ &= \frac{(x+\nu)\sin(\lambda x)}{2}+\frac{\cos(\lambda x)-\cos(\lambda x+2\lambda\nu)}{4\lambda}+O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$\text{(31)} \qquad \underset{\nu\to\infty}{\text{N}-\lim}\int_{-\infty}^{\infty}[\cos_-(\lambda t)]_\nu[\sin_+(\lambda(x-t))]_\nu\,dt = \frac{2\lambda x\sin(\lambda x)+\cos(\lambda x)}{4\lambda},$$

on using equations (9).

When $\nu \ge x \ge 0$,

$$\begin{aligned}\int_{-\infty}^{\infty}[\cos_-(\lambda t)]_\nu[\sin_+(\lambda(x-t))]_\nu\,dt &= \int_{x-\nu}^{0}\cos(\lambda t)\sin[\lambda(x-t)]\,dt+\\ &+\int_{x-\nu-\nu^{-\nu}}^{x-\nu}\cos(\lambda t)\sin[\lambda(x-t)]\tau_\nu(t)\tau_\nu(x-t)\,dt\\ &= -\frac{(x-\nu)\sin(\lambda x)}{2}+\frac{\cos(\lambda x)-\cos(\lambda x-2\lambda\nu)}{4\lambda}+O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$\text{(32)} \qquad \underset{\nu\to\infty}{\text{N}-\lim}\int_{-\infty}^{\infty}[\cos_-(\lambda t)]_\nu[\sin_+(\lambda(x-t))]_\nu\,dt = \frac{-2\lambda x\sin(\lambda x)+\cos(\lambda x)}{4\lambda},$$

on using equations (9).

Equation (24) now follows as above on using equations (30), (31) and (32).

Replacing x by $-x$ in equation (24) we get

$$-\cos_+(\lambda x)\,\boxed{*}\,\sin_-(\lambda x) = \frac{2\lambda x[\sin_+(\lambda x)-\sin_-(\lambda x)]+\cos(\lambda x)}{4\lambda}$$

and equation (25) follows since the convolution is commutative.

We finally prove equation (26). We have

$$[\sin_-(\lambda x)]_\nu * [\sin_+(\lambda x)]_\nu = \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu [\sin_+(\lambda(x-t))]_\nu \, dt. \tag{33}$$

When $-\nu \le x \le 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu [\sin_+(\lambda(x-t))]_\nu \, dt &= \int_{-\nu}^{x} \sin(\lambda t)\sin(\lambda(x-t))\, dt + \\ &+ \int_{-\nu-\nu^{-\nu}}^{-\nu} \sin(\lambda t)\sin[\lambda(x-t)]\tau_\nu(t)\tau_\nu(x-t)\, dt \\ &= \frac{\sin(\lambda x)+\sin(\lambda x + 2\nu x)}{4\lambda} - \frac{(x-\nu)\cos(\lambda x)}{2} + O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$\underset{\nu\to\infty}{\mathrm{N-lim}} \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu [\sin_+(\lambda(x-t))]_\nu \, dt = \frac{\sin(\lambda x) - 2\lambda x\cos(\lambda x)}{4\lambda}, \tag{34}$$

on using equation (8).

When $\nu \ge x \ge 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu [\sin_+(\lambda(x-t))]_\nu \, dt &= \int_{x-\nu}^{0} \sin(\lambda t)\sin[\lambda(x-t)]\, dt + \\ &+ \int_{x-\nu-\nu^{-\nu}}^{x-\nu} \sin(\lambda t)\sin[\lambda(x-t)]\tau_\nu(t)\, dt \\ &= \frac{(x-\nu)\cos(\lambda x)}{2} - \frac{\sin(\lambda x)+\sin(\lambda x - 2\lambda\nu)}{4\lambda} + O(\nu^{-\nu})\end{aligned}$$

and it follows that

$$\underset{\nu\to\infty}{\mathrm{N-lim}} \int_{-\infty}^{\infty} [\sin_-(\lambda t)]_\nu \sin_+[\lambda(x-t)] \, dt = \frac{2\lambda x\cos(\lambda x) - \sin(\lambda x)}{4\lambda}, \tag{35}$$

on using equation (8).

Equation (26) now follows as above on using equations (33), (34) and (35).

Further results can again be easily deduced. For example, since,

$$\cos_+(\lambda x) * \cos_+(\lambda x) = \frac{\sin_+(\lambda x) + \lambda x\cos_+(\lambda x)}{2\lambda},$$

for $\lambda \neq 0$, it follows as above that

$$\begin{aligned}\cos(\lambda x) \boxed{*} \cos_+(\lambda x) &= \cos_-(\lambda x) \boxed{*} \cos_+(\lambda x) + \cos_+(\lambda x) * \cos_+(\lambda x) \\ &= \frac{\sin(\lambda x) + 2\lambda x\cos_-(\lambda x)}{4\lambda}, \\ \cos(\lambda x) \boxed{*} \cos_-(\lambda x) &= -\frac{\sin(\lambda x) + 2\lambda x\cos_+(\lambda x)}{4\lambda}, \\ \cos(\lambda x) \boxed{*} \cos(\lambda x) &= \tfrac{1}{2}\, x\cos(\lambda x),\end{aligned}$$

for $\lambda \neq 0$. □

References

[1] van der Corput J.G., *Introduction to the neutrix calculus*, J. Analyse Math. **7** (1959-60), 291–398.

[2] Fisher B., Kiliçman A., *Commutative neutrix convolution products of functions*, Commentationes Math. Univ. Carolinae **35** (1994), 47–53.

[3] Fisher B., Li Chen Kuan,, *A commutative neutrix convolution product of distributions*, Univ. u Novom Sadu Zb. Rad. Privod.-Mat. Fak. Ser. Mat. **23** (1993), 13–27.

[4] Fisher B., Özçağ E., *Results on the commutative neutrix convolution product of distributions*, Arch. Math. **29** (1993), 105–117.

[5] Gel'fand I.M., Shilov G.E., *Generalized Functions*, Vol. I, Academic Press, 1964.

[6] Kiliçman A., Fisher B., Pehlivan S., *On the neutrix convolution product* $\ln x_- \boxed{*} \ln x_+$, submitted for publication.

Department of Mathematics and Computer Science, Leicester University, Leicester, LE1 7RH, England

(*Received* May 10, 1994)