Upper and lower estimates in Banach sequence spaces

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Abstract. Here we study the existence of lower and upper ℓ_p-estimates of sequences in some Banach sequence spaces. We also compute the sharp ℓ_p estimates in their basis. Finally, we give some applications to weak sequential continuity of polynomials.

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The existence of ℓ_p-estimates in the sequences in a Banach space is of great interest in the study of the structure of the space. It is also relevant in some questions of non linear analysis such the behaviour of polynomials (see [4], [8], [11], [22], [25]) and the smoothness of a Banach space, in the sense of the existence of real bump functions with higher order of differentiability (see [11]). Here we study how these estimates are in some well known Banach sequence spaces.

Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $1 < p, q < \infty$. Recall that a sequence $\{x_n\}$ is said to have an **upper p-estimate** (respectively, a **lower q-estimate**) if there exists a constant $C > 0$ such that for all scalars $a_1, \dots, a_n \in \mathbb{K}$ and $n \in \mathbb{N}$ we have

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \le C\Big(\sum_{i=1}^{n} |a_i|^p\Big)^{1/p}$$

$$\Big(\text{respectively}\quad C\Big(\sum_{i=1}^{n} |a_i|^q\Big)^{1/q} \le \Big\|\sum_{i=1}^{n} a_i x_i\Big\|\Big).$$

A Banach space X has **S_p-property** [18], $1 < p < \infty$, if every weakly null normalized sequence in X has a subsequence with an upper p-estimate. Analogously, a Banach space X has **T_q-property** [10], $1 < q < \infty$, if every weakly null normalized sequence has a lower q-estimate. In the same way, X has **US_p-property** [18] (respectively **UT_q-property**) if there exists a constant $C > 0$ such that every weakly null normalized sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ with constant C, i.e. for all $a_1, \dots, a_n \in \mathbb{K}$, $n \in \mathbb{N}$,

$$\Big\|\sum_{i=1}^{n} a_i x_{n_i}\Big\| \le C\Big(\sum_{i=1}^{n} |a_i|^p\Big)^{1/p}$$

$$\Big(\text{respectively}\quad C\Big(\sum_{i=1}^{n} |a_i|^q\Big)^{1/q} \le \Big\|\sum_{i=1}^{n} a_i x_{n_i}\Big\|\Big).$$

In [18] it is proved that S_p and US_p-property coincide. The properties T_q and UT_q-property are not equivalent, even in reflexive spaces, as we shall see in Example 1.3.

Related to these properties the following indexes have been defined in [10]:

$$l(X) = \sup\{p \geq 1;\ \ X \text{ has } S_p\text{-property}\} \in [1, \infty]$$
$$u(X) = \inf\{q \geq 1;\ \ X \text{ has } T_q\text{-property}\} \in [1, \infty].$$

It is clear that for $X = \ell_p$, $1 < p < \infty$, we have $l(\ell_p) = u(\ell_p) = p$. In Section 1 we compute these indexes for Orlicz sequence spaces; we also compute the exact lower and upper estimates in the usual basis of these spaces.

The same is done for Lorentz sequence spaces in Section 2. We also compute the exact upper estimate for the canonical basis. By using these results we obtain a very easy way to find examples of non superreflexive Lorentz spaces.

Section 3 is devoted to the study of these indexes for James spaces, and we also compute S_p-property for the James type spaces introduced by Lohman-Cassaza [19].

In the last section we obtain some applications of these results regarding to weak sequential continuity of polynomials on these spaces.

1. Orlicz sequence spaces

Recall that an Orlicz function M is a continuous, convex, and non decreasing function defined for all $t \geq 0$ and such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. The Orlicz sequence space h_M with Orlicz function M is the Banach space consisting of all sequences $\{a_k\}$ of scalars such that

$$\sum_{n=1}^{\infty} M(|a_n|/\lambda) < \infty \quad \text{for all} \quad \lambda > 0,$$

endowed with the norm

$$\|\{a_n\}\|_{h_M} = \inf\{\lambda > 0;\ \sum_{n=1}^{\infty} M(|a_n|/\lambda) \leq 1\}.$$

The lower and upper Boyd indexes associated to M (see e.g. [20]) are defined by

$$\alpha_M = \sup\{p \geq 1;\ \sup_{0<u,v\leq 1} \frac{M(uv)}{u^p M(v)} < \infty\}$$
$$\beta_M = \inf\{q \geq 1;\ \inf_{0<u,v\leq 1} \frac{M(uv)}{u^p M(v)} > 0\}.$$

Concerning the S_p-property the following result was proved in [16].

Theorem 1.1 ([16]). *Consider $1 < p < \infty$ and let h_M be an Orlicz sequence space that does not contain ℓ_1. The following statements are equivalent:*

(i) *h_M has S_p-property.*
(ii) *The function M satisfies*

$$\sup_{0<u,v\leq 1} \frac{M(uv)}{u^p M(v)} < \infty.$$

In a similar way we have the following result:

Theorem 1.2. *Consider $1 < q < \infty$ and let h_M be an Orlicz sequence space that does not contain ℓ_1. The following statements are equivalent:*

(i) *h_M has UT_q-property.*
(ii) *The function M satisfies*

$$\inf_{0<u,v\leq 1} \frac{M(uv)}{u^q M(v)} > 0.$$

Proof: (i) $\Rightarrow$ (ii). Consider $C > 0$ such that every weakly null normalized sequence admits a subsequence which has a lower q-estimate with constant C. For each $k \in \mathbb{N}$ consider $1/\lambda_k = \|\sum_{i=1}^k e_i\|$, and we construct the following sequence for each k as in [16]:

$$b_m^{(k)} = \lambda_k \sum_{i=1}^k e_{mk+i}.$$

Since h_M does not contain ℓ_1, $\{b_m^{(k)}\}_m$ is a weakly null sequence, and therefore a subsequence of $\{b_m^{(k)}\}_m$ has a lower q-estimate with constant C. Then it is easy to see that for any k

$$\Big\|\sum_{i=1}^n b_i^{(k)}\Big\| \geq C n^{1/q}.$$

Now, consider $0 < s, t \leq 1$ and choose $k \in \mathbb{N}$ such that $\lambda_{k+1} \leq s < \lambda_k$ and $n \in \mathbb{N}$ verifying

$$\frac{1}{C(n+1)^{1/q}} \leq t < \frac{1}{Cn^{1/q}}.$$

We may assume that $s < \lambda_1$ and $t < 1/C$. Then

$$\frac{M(st)}{M(s)t^q} \geq \frac{M(\lambda_{k+1}/C(n+1)^{1/q})}{M(\lambda_k)(1/Cn^{1/q})^q} \geq \frac{C^q n}{(k+1)(n+1)M(\lambda_k)} =$$

and since $M(\lambda_k)k = 1$,

$$= \frac{C^q nk}{(k+1)(n+1)} \geq \frac{1}{4}C^q,$$

as we required.

(ii) $\Rightarrow$ (i). If (ii) holds, by [20, p. 139] there exists $C > 0$ such that for every choice of pairwise disjoint elements $x_1, \dots, x_n \in h_M$ in X, we have

$$C\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{1/p} \leq \Big\|\sum_{i=1}^{n} x_i\Big\|.$$

Then since every weakly null sequence has a subsequence equivalent to a block basis sequence, the result follows. □

Remark 1.3. It is not true in general that T_q and UT_q-properties are equivalent; indeed, there is a reflexive Banach space X which has T_q-property but fails UT_q-property. In order to see this we consider the example which appears in [15] (see Example 1). Let $1 < p < 2$. For each $0 < \lambda < 1$ let X_λ be the subspace of $(\ell_p \oplus \ell_p)_{\ell_p}$ generated by $\{\lambda e_n + \delta_n\}_n$ where $\{e_n\}_n$ (respectively $\{\delta_n\}_n$) is the unit vector basis of ℓ_p (respectively in ℓ_2). Each X_λ is isomorphic to ℓ_p. Consider $X = (\oplus_n X_{1/n})_{\ell_p}$. By using Proposition 3.1 in [15], every weakly null normalized sequence in X contains a subsequence that is equivalent to the unit vector basis of ℓ_p. In particular X has T_p-property. Fix $M > 0$, and for m large enough we construct the following sequence

$$x_n = m^{-1}\big(1 + m^p\big)\big(\frac{1}{m}e_n + \delta_n\big);$$

which is a weakly null and normalized sequence in X and verifies that for any $k_1 < \cdots < k_n$

$$\Big\|\sum_{i=1}^{n} x_{k_i}\Big\| \leq \frac{M}{2} n^{1/p}.$$

If X had UT_p-property with constant $M > 0$ it would follow that

$$Mn^{1/p} \leq \Big\|\sum_{i=1}^{n} x_{k_i}\Big\| \leq \frac{M}{2} n^{1/p}$$

for a certain subsequence $\{x_{n_i}\}$ of $\{x_n\}$, which is not possible.

Corollary 1.4. *Let h_M be an Orlicz sequence space. Then $l(h_M) = \alpha_M$ and $u(h_M) = \beta_M$.*

Proof: From Theorems 1.1 and 1.2 and since h_M contains an isomorphic copy of ℓ_{α_M} and ℓ_{β_M}, the result follows. □

Now we compute the exact upper and lower estimates in the canonical basis in Orlicz sequence spaces. We define

$$\gamma_M = \sup\{p \geq 1 : \sup_{0<t\leq 1} \frac{M(t)}{t^p} < \infty\}$$

$$\delta_M = \inf\{q < \infty : \inf_{0<t\leq 1} \frac{M(t)}{t^q} > 0\}.$$

Proposition 1.5. *Let h_M be the Orlicz sequence space associated to an Orlicz function M that does not contain ℓ_1. Let $\{e_n\}$ be the unit vector basis in h_M. Then the following statements are equivalent:*

(i) $\sup_{0<t\leq 1} \frac{M(t)}{t^p} < \infty$ (*respectively* $\inf_{0<t\leq 1} \frac{M(t)}{t^q} > 0$).

(ii) *The sequence $\{e_n\}$ satisfies an upper p-estimate* (*respectively a lower q-estimate*). *Besides*

$$\gamma_M = \sup\{p \geq 1 : \{e_n\} \ \textit{satisfies an upper } p\textit{-estimate}\}$$

$$\delta_M = \inf\{q < \infty : \{e_n\} \ \textit{satisfies a lower } q\textit{-estimate}\}.$$

Proof: We shall only prove the assertion concerning γ_M; the assertion concerning δ_M can be proved in an entirely similar way.

(i) $\Rightarrow$ (ii). Assume that $M(t) \leq At^p$ for all $0 < t \leq 1$. Consider $x = \sum_{i=1}^{\infty} a_i e_i$, $\|x\| > 0$. Then

$$1 = \sum_{n=1}^{\infty} M\left(\frac{|a_n|}{\|x\|}\right) \leq A^{1/p} \sum_{n=1}^{\infty} \frac{|a_n|^p}{\|x\|^p}$$

and therefore $\{e_n\}$ has an upper p-estimate.

(ii) $\Rightarrow$ (i). If (i) does not hold, there is a decreasing to zero real sequence $\{t_n\}$ verifying:

$$\frac{M(t_n)}{t_n^p} \to \infty \quad \text{when} \quad n \to \infty.$$

As it is done in [21, Corollary 2.4], for each n, consider $\lambda_n = \| \sum_{i=1}^{n} \|$. Since $\{e_n\}$ has an upper p-estimate, there exists a constant $C > 0$ such that $\lambda_n \leq Cn^{1/p}$ for all n. Consider a subsequence $\{t_{n_j}\}$ of $\{t_n\}$ such that

$$n_j \geq \left(\frac{1}{Ct_{nj}}\right)^p \geq n_{j-1}.$$

Then

$$\frac{1}{C(n_j)^{1/p}} \leq t_{n_j} \leq \frac{1}{C(n_{j-1})^{1/p}}\,.$$

Therefore, since $nM(1/\lambda_n) = 1$,

$$1 \geq (n_{j-1}) M\Big(\frac{1}{C(n_{j-1})^{1/p}}\Big) \geq (n_{j-1}) \frac{M(t_{n_j})}{t^p_{n_j}} t^p_{n_j} \geq$$
$$\geq \frac{(n_{j-1}) M(t_{n_j})}{C n_j t^p_{n_j}} \geq \frac{1}{2C} \frac{M(t_{n_j})}{t^p_{n_j}} \to \infty$$

which is not possible. □

From the above proposition we easily deduce the following well known result:

Corollary 1.5. *Let M be an Orlicz function. Then if $1 < p < \infty$, the following statements are equivalent:*

(i) *ℓ_M is isomorphic to ℓ_p.*
(ii) *$M \cong t^p$ at 0, i.e there exist constants $A, B > 0$ such that $At^p \leq M(t) \leq Bt^p$ when $t \in [0,1]$.*

Remark 1.6. As a consequence of Proposition 1.5 we obtain that the existence of an upper p-estimate (or lower q-estimate) in a basis, even in a symmetric basis, does not imply the existence of upper p-estimates (or lower q-estimates) in the sequences of the space, since α_M is, in general, different from γ_M (the same for β_M and δ_M), see [20, p. 140].

2. Lorentz sequence spaces

We now study lower and upper estimates in the Lorentz sequence spaces.

Let $1 \leq p < \infty$ and let $w = \{w_n\} \in c_0 \setminus \ell_1$, such that $1 \geq w_1 \geq w_2 \geq \dots$. The Banach space of all sequences of scalars $x = (x_1, x_2, \dots)$ for which

$$\|x\| = \sup_{\sigma \in \Pi} \Big(\sum_{i=1}^{\infty} |x_{\sigma(i)}|^p w_i\Big) < \infty$$

where Π ranges over all the permutations of the integers, is denoted by $d(w,p)$ and it is called the **Lorentz sequence space** associated to w and p. For the basic properties of these spaces see for instance [20].

Proposition 2.1 ([5]). *The Lorentz sequence space $d(w,p)$ has S_p-property, if $1 < p < \infty$.*

Corollary 2.2. *If $1 < p < \infty$, then $l(d(w,p)) = p$.*

Proof: The result follows from Proposition 2.1 and since $d(w,p)$ contains ℓ_p. □

In the sequel we compute upper estimates in the usual basis in the Lorentz sequence space. In order to get it we define the following index associated to space $d(w,p)$:

$$r = r(w) = \inf\{s \in [1, \infty];\ \{w_n\} \in \ell_s\},$$

(note that such an index only depends on w).

Proposition 2.3. *Let $\{e_n\}$ be the unit vector basis of $d(w,p)$. Then, if $\frac{1}{r}+\frac{1}{r^*}=1$, where $r^*=1$ (respectively $r^*=\infty$) if $r=\infty$ (respectively $r=1$), we have*

$$pr^* = \sup\{s>1 : \{e_n\} \text{ has an upper } s\text{-estimate}\}.$$

Proof: We first prove that if $q>p$, the following statements are equivalent:

(a) $\{e_n\}$ has an upper p-estimate.
(b) $\{w_n\} \in \ell_{(q/p)^*}$.

(a) $\Rightarrow$ (b). Let $M>0$ be verifying that for any $a_1,\dots,a_n \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\Big\|\sum_{i=1}^n a_i e_i\Big\| \le M\Big(\sum_{i=1}^n |a_i|^q\Big)^{1/q}.$$

Fix $b=\{b_n\} \in \ell_{q/p}$; then

$$\Big(\sum_{i=1}^n |b_i| w_i\Big)^{1/p} \le \Big\|\sum_{i=1}^n |b_i|^{1/p} e_i\Big\| \le M\Big(\sum_{i=1}^n |b_i|^{q/p}\Big)^{1/q} \le M\|b\|_{q/p}^{1/p}.$$

Hence

$$\sum_{i=1}^\infty |b_i| w_i \le M^p \|b\|_{q/p}$$

and therefore $\{w_n\} \in \ell_{(q/p)^*}$.

(b) $\Rightarrow$ (a). Assume that $\{w_n\} \in \ell_{(q/p)^*}$. Consider $a=\{a_n\} \in \ell_q$, $\|a\|_q \le 1$ and a permutation of integers π such that $\{|a_{\pi(n)}|\}$ is a decreasing to zero sequence. Then for each $n \in \mathbb{N}$

$$\Big\|\sum_{i=1}^n a_i e_i\Big\| \le \sum_{n=1}^\infty |a_{\pi(n)}|^p w_n \le \|\{w_n\}\|_{(q/p)^*},$$

hence

$$\Big\|\sum_{i=1}^n a_i e_i\Big\| \le \|w\|_{(q/p)^*} \|a\|_q$$

and therefore $\{e_n\}$ has an upper q-estimate.

The result is now easily deduced. Indeed, if $q<pr^*$ then $(q/p)^*>r$ and $\{w_n\} \in \ell_{(q/p)^*}$. By the preceding, $\{e_n\}$ has an upper q-estimate. Conversely, if $\{e_n\}$ has an upper q-estimate then $\{w_n\} \in \ell_{(q/p)^*}$ and therefore $r \le (q/p)^*$ and $q \le pr^*$. $\square$

Corollary 2.4. *Let $d(w,p)$ be a Lorentz sequence space, for $1 < p < \infty$. Then $u(d(w,p)) \geq r^*p$.*

Proof: Assume the contrary, that is, $u(d(w,p)) < r^*p$. Choose s such that $u(d(w,p)) < s < r^*p$. Since $d(w,p)$ has T_s-property than there is a subsequence of the basis $\{e_n\}$ with a lower s-estimate. Since $\{e_n\}$ is a symmetric basis, $\{e_n\}$ itself would have lower s-estimate. Now, by using Proposition 2.3, $\{e_n\}$ admits an upper s-estimate; therefore $\{e_n\}$ is equivalent to the unit vector basis of ℓ_s and then $d(w,p)$ coincides with the space ℓ_s, which is not possible. $\square$

In [2] it is studied when a given function can be the modulus of convexity of a Lorentz sequence space $d(w,p)$. In general, such spaces need not be superreflexive. From the above result we obtain in a very simple way examples of Lorentz sequence spaces that are non superreflexive.

Corollary 2.5. *Let $d(w,p)$ be a Lorentz sequence space such that $r = \inf\{p \leq \infty : \{w_n\} \in \ell_p\} = 1$. Then $d(w,p)$ is not superreflexive.*

Proof: Since $r = 1$, by using Proposition 2.3 we have that $\{e_n\}$ admits an upper q-estimate for all $q > 1$. If $d(w,p)$ were superreflexive, it would have T_q-property for some $q < \infty$ (see [12], [10]). In particular, a subsequence of $\{e_n\}$ would admit a lower q-estimate, and therefore so would $\{e_n\}$; this is impossible. $\square$

3. James type spaces

We now study the James space J; J is the space of all sequences of real numbers, $x = (a_1, a_2, \dots)$, such that $\lim |a_n| = 0$ and

$$\|x\| = \sup\Big\{\Big[\sum_{i=1}^{n}\big[\big(a_{p_{2i-1}} - a_{p_{2i}}\big)^2\big]^{1/2}\Big\} < \infty$$

where the supremum is taken when we consider all possible choices of $n \in \mathbb{N}$ and of integers $p_1 < \cdots < p_{2n}$.

$(J, \|\cdot\|)$ is a Banach space. For an extensive treatment of the properties of this space, see for instance [9]; there, it is proved the following result:

Theorem 3.1 ([9]). *Let $y_j = \sum_{n=p_j}^{q_j} \alpha_n e_n$, $j = 1, 2, \dots$ be a block basic semi-normalized sequence of $\{e_n\}$ in J. If $p_{j+1} - q_j > 1$ then the sequence $\{y_j\}$ is equivalent to the unit vector basis of ℓ_2.*

Corollary 3.2. *The James space has S_2 and T_2-properties. Therefore $l(J) = u(J) = 2$.*

Proof: Let $\{x_n\}$ be a weakly null normalized basic sequence in J. Then there is a subsequence of $\{x_n\}$ which is equivalent to a block basis $\{y_n\}$ and $\{y_n\}$ may be assumed to be as in Theorem 3.1. Therefore $\{y_n\}$ is equivalent to the basis in ℓ_2, in particular has upper and lower 2-estimates. $\square$

Let J^* be the dual space of J. The sequence $\{e_n^*\}$ of biorthogonal functionals associated to $\{e_n\}$ in J is a basis in J^*.

Proposition 3.3. *The space J^* has S_2 and T_2-properties. Then $l(J^*) = u(J^*) = 2$.*

Proof: In the first place, by [10] since J has S_2-property the space J^* has T_2-property. We now see that J^* has S_2-property. Let $\{x_n^*\}$ be a weakly null basic normalized sequence in J^*. We may assume, by passing to a subsequence and relabeling again, that

$$x_n^* = \sum_{p_j+1}^{q_j} \alpha_i e_i^* \quad \text{where} \quad p_{j+1} - q_j > 1.$$

Fix $x = \sum_{i=1}^{\infty} a_i e_i$, $\|x\| \le 1$, and define $x_n = \sum_{p_j+1}^{q_j} a_i e_i$. By Theorem 3.1 $\{x_n/\|x_n\|\}$ is equivalent to the unit vector basis of ℓ_2. In particular, $\{x_n/\|x_n\|\}$ admits a lower 2-estimate, that is, there exists $M > 0$ such that for any $n \in \mathbb{N}$ and $b_1, \dots, b_n \in \mathbb{R}$

$$\Big(\sum_{i=1}^{n} |b_i|^2\Big)^{1/2} \le M \Big\| \sum_{i=1}^{n} b_i \frac{x_i}{\|x_i\|} \Big\|.$$

Consider $x^* = \sum_{n=1}^{\infty} b_n x_n^*$, then

$$|x^*(x)| = \Big| \sum_{n=1}^{\infty} b_n \langle x_n^*, x_n \rangle \Big| \le \sum_{n=1}^{\infty} |b_n| \, \|x_n^*\| \, \|x_n\| \le$$

$$\le \Big(\sum_{n=1}^{\infty} |b_n|^2 \|x_n^*\|^2\Big)^{1/2} \Big(\sum_{n=1}^{\infty} \|x_n\|^2\Big)^{1/2} \le C \Big(\sum_{n=1}^{\infty} |b_n|^2\Big)^{1/2}.$$

Therefore,

$$\|x^*\| = \Big\| \sum_{n=1}^{\infty} b_n x_n^* \Big\| \le C \Big(\sum_{n=1}^{\infty} |b_n|^2\Big)^{1/2}$$

and $\{x_n^*\}$ has an upper 2-estimate. □

In [19] the James' construction is generalized by considering norms determined by Banach spaces with basis in the following way:

Let X be a Banach space with normalized monotone basis $\{x_n\}$; then we define X_J as the space of all scalar sequences $\{a_n\}$ such that

$$\|\{a_n\}\|_{X_J} = \sup \Big\{ \Big\| \sum_{i=1}^{n} (a_{p_{2j-1}} - a_{p_{2j}}) x_j \Big\| \Big\} < \infty$$

where the supremum is taken when we consider all possible choices of $n \in \mathbb{N}$ and of integers $p_1 < \dots < p_{2n}$.

Recall that a basis $\{x_n\}$ in X is said to be **block p-Hilbertian** if there exists a constant $C > 0$ such that if $\{y_n\}$ is a normalized block sequence in X then $\{y_n\}$ has an upper p-estimate with constant C.

It is easy to see that if X has a block p-Hilbertian basis then X has S_p-property. By using the following result that is proved in [19] we obtain that some James type spaces have S_p-property.

Theorem 3.4 ([19]). *If X has a block p-Hilbertian basis then $\{e_n\}$ is block p-Hilbertian in X_j. In particular, X_j has S_p-property.*

Proposition 3.5. *The space $(lp)_J$ has S_p-property and T_p-property. Therefore $l((\ell_p)_J) = u((\ell_p)_J) = p$.*

Proof: (i) By Theorem 3.4 the space $(\ell_p)_J$ has S_p-property. We now see that $(\ell_p)_J$ has T_p-property. Let $\{y_n\}$ be a weakly null normalized sequence in $(\ell_p)_J$, by a standard perturbation argument we may assume that $\{y_n\}$ has a subsequence equivalent to a normalized sequence $\{x_n\}$, where

$$x_n = \sum_{i=l_n}^{p_n} a_i e_i$$

and $l_{n-1} < k_n < l_n < k_{n+1}$, $k_n - l_{n-1} > 2$ for all n.

For each $n \in \mathbb{N}$, consider a choice of integers $p_0^n < \cdots < p_{2j_n}^n$ such that

$$\|x_n\| = \Big(\sum_{i=1}^{j_n} |a_{p_{2i}^n} - a_{p_{2i-1}^n}|^p\Big)^{1/p} \geq 1,$$

where $p_0^n = k_n - 1$ if $a_{p_0^n} = 0$ and $p_{2j_n}^n = 0$ if $a_{p_{2j_n}^n} = 0$, and $k_n - 1 \leq p_0^n < \cdots < p_{2j_n}^n \leq l_n + 1$.

Now, if $\alpha_1, \dots, \alpha_m \in \mathbb{K}$, and if we consider the following choice of integers

$$p_0^1 < \cdots < p_{2j_1}^1 < p_0^2 < \cdots < p_{2j_2}^2 < \cdots < p_0^m < \cdots < p_{2j_m}^m,$$

we have that

$$\Big\| \sum_{n=1}^{m} \alpha_n x_n \Big\|^p \geq \sum_{n=1}^{m} |\alpha_n|^p \Big(\sum_{i=1}^{j_n} |a_{p_{2i}^n} - a_{p_{2i-1}^n}|^p \Big) \geq \frac{1}{2} \sum_{n=1}^{m} |\alpha_n|^p.$$

Therefore, $\{x_n\}$ has a lower p-estimate, and $(\ell_p)_J$ has T_p-property. □

Now we consider the original Tsirelson space T^* and the associated Tsirelson-James space T_J^* space constructed by Aron-Dineen in [3]. Then:

Proposition 3.6. *The space T_J^* has S_p-property for all $1 < p < \infty$. In particular, no weakly null normalized subsequence in T_J^* admits a lower q-estimate for any q, $1 < q < \infty$.*

Proof: By [3, Corollary 10], if $\{u_j\}$ is a normalized block basis in T_J^* then

$$\Big\| \sum_{n+1}^{2n} a_j u_j \Big\|_{T_J^*} \leq 4 \sup_{1+n \leq j \leq 2n} |a_j|.$$

Now, by the same procedure as in [6] it can be proved that $\{u_j\}$ has an upper p-estimate for any $1 < p < \infty$. Therefore, since every weakly null sequence in T_J^* has a subsequence equivalent to a block basis, we have that T_J^* has S_p-property for any $1 < p < \infty$. □

4. Some applications to polynomials

The upper and lower ℓ_p-estimates in sequences give us much information about weak continuity of polynomials, and this is of a great interest in some problems of smoothness ([11]) and in the study of reflexivity of the spaces of polynomials (see [1], [7], [21]). The relationship between the existence of these estimates and the behavior of polynomials has been studied in [10] and [11].

Proposition 4.1. *Let X be a Banach space.*

(i) *If a sequence $\{x_n\}$ in X has an upper p-estimate, then for every N-homogeneous polynomial P on X, with $N < p$, the sequence $\{P(x_n)\}$ is convergent to zero.*

(ii) *If X has a basis $\{e_n\}$ which satisfies a lower q-estimate, then there exists an N-homogeneous polynomial on X, with $N \geq q$, such that $P(e_n) \geq 1$ for all $n \in \mathbb{N}$.*

Proof: (i) is proved in [10]. In order to prove (ii), since $\{e_n\}$ has a lower p-estimate, there is $C > 0$ such that for all $a_1, \dots, a_n \in \mathbb{K}$

$$\Big(\sum_{i=1}^{n} |a_i|^q\Big)^{1/q} \leq C \Big\| \sum_{i=1}^{n} a_i e_i \Big\|.$$

Then the result follows by considering for $N \geq q$ the polynomial defined by

$$P\Big(\sum_{i=1}^{\infty} a_i e_i\Big) = \sum_{i=1}^{n} (a_i)^N.$$

□

By applying the above result to the spaces studied here we have:

Corollary 4.2.

(i) *On the Orlicz sequence space h_M, where h_M does not contain ℓ_1, all polynomials of degree N with $< \alpha_M$ are weakly sequentially continuous; moreover if $N < \gamma_M$ it follows from Corollary 1.4 and Proposition 4.1 that every N-homogeneous polynomial on h_M verifies that $\{P(e_n)\}_n$ is convergent to zero and if $N > \delta_M$ there exists an N-homogeneous polynomial which is not weakly sequentially continuous.*

(ii) *On the Lorentz sequence spaces $d(w,p)$ every N-homogeneous polynomial with $N < p$ is weakly sequentially continuous. From Proposition 2.3 and 4.1 it follows that if $N < r^*p$ and P is an N-homogeneous polynomial then $\{P(e_n)\}$ is convergent to zero. In the particular case that $r = 1$, the sequence $\{e_n\}$ is weak-polynomial convergent to zero, i.e. for all polynomials P with $P(0) = 0$ we have that $\{P(e_n)\}$ converges to zero; therefore $d(w,p)$ is not a Λ-space (see [13]).*

(iii) *On the space* $(\ell_p)_J$ *every* N*-homogeneous polynomial with* $N < p$ *is weakly sequentially continuous. Besides, since* $(\ell_p)_J$ *has a quotient isomorphic to* ℓ_p*, it is proved in* [14] *(see Remark* 3*) that there exists an* N*-homogeneous polynomial for* $N \geq p$ *which is not weakly sequentially continuous.*

(iv) *On* T_J^* *all polynomials are weakly sequentially continuous.*

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