Minimal generators for aperiodic endomorphisms

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Abstract. Every aperiodic endomorphism f of a nonatomic Lebesgue space which possesses a finite 1-sided generator has a 1-sided generator β such that $k_f \leq \operatorname{card} \beta \leq k_f + 1$. This is the best estimate for the minimal cardinality of a 1-sided generator. The above result is the generalization of the analogous one for ergodic case.

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0. Introduction

Let f be an aperiodic endomorphism of a nonatomic Lebesgue space $(X, \mathcal{B}, \mu)$. Let $f^{-1}\varepsilon$ denote the partition $\{f^{-1}(x) : x \in X\}$ and let $\{m_{f^{-1}(x)}\}_{x \in X}$ be the canonical system of measures. Denote by $h(f)$ the entropy of f. If $H(\varepsilon \mid f^{-1}\varepsilon) = h(f) < \infty$, then the canonical measures are purely atomic. In this case we define a number k_f in the following way:

$$k_f = \min\big\{k : \operatorname{card}\{y : y \in f^{-1}(x) \ \text{ and } \ m_{f^{-1}(x)}(y) > 0\} \leq k \text{ a.e.}\big\}.$$

The aim of this paper is to prove Theorem 1 [2] without assumptions of ergodicity of f.

Theorem 1. *An aperiodic endomorphism f has a finite 1-sided generator iff $H(\varepsilon \mid f^{-1}\varepsilon) = h(f) < \infty$ and $k_f < \infty$. Moreover, if f admits a finite 1-sided generator, then there exists a 1-sided generator β such that $k_f \leq \operatorname{card} \beta \leq k_f + 1$.*

We prove out theorem by using the ergodic decomposition of μ and some ideas of [1] and [2].

1. Preparation for the proof

We say that a set $B \in \mathcal{B}$ is invariant if $\mu(f^{-1}B \bigtriangleup B) = 0$. Let $\mathcal{A}$ be the σ-field of invariant sets and let α be the measurable partition of X determined by $\mathcal{A}$. Let $\{\mu_A\}_{A \in \alpha}$ be the canonical system of measures for α. The family of dynamical systems $(A, \mathcal{B}_A, \mu_A, f \mid A)_{A \in \alpha}$ is called the ergodic decomposition of (X, μ, f). For the next considerations we need the following lemma.

Lemma 1. *Let γ and β be measurable partitions such that $\gamma < \beta$. Moreover, let $\{\mu_G\}_{G\in\gamma}$, $\{\mu_B\}_{B\in\beta}$ be the canonical system of measures for γ, β respectively. If $\{\widetilde{\mu}_B\}_{B\in\beta\cap G}$ denotes the canonical system of measures with respect to μ_G then $\widetilde{\mu}_B = \mu_B$ a.e. for a.e. G.*

Proof: Let $\mathbb{G}$ and $\overline{\mathcal{B}}$ be the σ-fields for γ and β respectively. When we ignore a set of atoms of μ_γ measure zero then

$$\int f\,d\mu_G = E(f \mid \mathbb{G}) \mid G \quad \text{and} \quad \int f\,d\mu_B = E(f \mid \overline{\mathcal{B}}) \mid B,$$

for every $f \in L^1(\mu)$. We have also

$$E(E(f \mid \overline{\mathcal{B}}) \mid \mathbb{G}) = E(f \mid \mathbb{G}),$$

by $\mathbb{G} \subset \overline{\mathcal{B}}$. Therefore

$$\int f\,d\mu_G = E(f \mid \mathbb{G}) \mid G = E(E(f \mid \overline{\mathcal{B}}) \mid \mathbb{G}) \mid G = \int E(f \mid \overline{\mathcal{B}})\,d\mu_G = \\ = \int E(f \mid \overline{\mathcal{B}}) \mid B\,d\mu_G(B) = \int \int f\,d\mu_B\,d\mu_G(B).$$

The above implies that $\{\mu_B\}_{B\in\beta\cap G}$ is the canonical system of measures with respect to μ_G and therefore $\widetilde{\mu}_B = \mu_B$ a.e. □

Let β denote the finite partition of X and $h(\beta, f)$ the entropy of f with respect to β. Besides, let us denote by $h_A(\beta, f)$ the entropy of $f|A$ with respect to $\beta|A$.

Theorem 2 [1]. $h(\beta, f) = \int h_A(\beta, f)\,d\mu_\alpha(A)$,
$h(f) = \int h_A(f)\,d\mu_\alpha(A)$.

Let $J_f(x)$ denote the Jacobian for f, i.e.

$$J_f(x) = (m_{f^{-1}(f(x))}(x))^{-1} \quad (\textit{see } [5]).$$

Then

$$H(\varepsilon \mid f^{-1}\varepsilon) = \int \log J_f\,d\mu = \int \int_A \log J_f\,d\mu_A\,d\mu_\alpha \\ = \ (\textit{by Lemma } 1) \ = \int H_A(\varepsilon \mid f^{-1}\varepsilon)\,d\mu.$$

If $h(f) < \infty$ then due to Theorem 2 we get the following equivalence

$$(1) \qquad H(\varepsilon \mid f^{-1}\varepsilon) = h(f) \ \text{ iff } \ H_A(\varepsilon \mid f^{-1}\varepsilon) = h_A(f) \text{ a.e.}$$

2. Proof of Theorem 1

We are in the position to prove the part of Theorem 1 (the necessity). Namely, if f possesses a 1-sided finite generator β then $h(f) < \infty$ and $\beta|A$ is the 1-sided generator for a.e. $A \in \alpha$. Therefore $H_A(\varepsilon \mid f^{-1}\varepsilon) = h_A(f) < \infty$ for a.e. A ([3, p. 97]) and by (1) $H(\varepsilon \mid f^{-1}\varepsilon) = h(f) < \infty$. Since β is the 1-sided generator, $f|B$ is 1–1 for every $B \in \beta$. Therefore $k_f \leq \operatorname{card} \beta$.

In order to prove the sufficiency part of Theorem 1 we show that the conditions $H(\varepsilon \mid f^{-1}\varepsilon) = h(f) < \infty$ and $k_f < \infty$ imply that f possesses a 1-sided generator β such that $k_f \leq \operatorname{card} \beta \leq k_f + 1$. Let $\beta = \{B_1, \dots, B_{k_f}\}$ be a partition such that $\beta \vee f^{-1}\varepsilon = \varepsilon$. We obtain the partition as above by using the following construction of Rohlin [4]: $B_1 \cap f^{-1}(x)$ consists of an atom of the greatest $m_{f^{-1}(x)}$ measure, next $B_2 \cap f^{-1}(x)$ consists of an atom of the greatest measure in $f^{-1}(x) - B_1$, etc.

If $\beta \vee f^{-1}\varepsilon = \varepsilon$ then $\beta|A$ has the same property for a.e. A. By Lemma 1 [2] we have also

$$h_A(\beta, f) = h_A(f) \text{ a.e.} \tag{2}$$

Due to (1) and by our assumptions

$$h_A(\beta, f) = h_A(f) = H_A(\varepsilon \mid f^{-1}\varepsilon) \text{ a.e.}$$

Let $\overline{f}$ denote the natural extension of f to an automorphism. The transformation $\overline{f}$ is an aperiodic automorphism of the measurable space $(\overline{X}, \overline{\mathcal{B}}, \overline{\mu})$, where $\mathcal{B}$ is an exhaustive σ-algebra of $\overline{\mathcal{B}}$. The ergodic decomposition $\{\mu_A\}_{A \in \alpha}$ lifts to the ergodic decomposition of $\overline{f}$. We will denote it by $\{\overline{\mu}_x\}_{x \in \overline{X}}$. Here $\overline{\mu}_x = \overline{\mu}_A$ for $x \in \overline{A}$. To obtain the sufficiency part of Theorem 1 we need (as in [2]) the following lemma.

Lemma 2. *Let $\beta = \{B_1, \dots, B_{k_f}\}$ be a partition such that $h(\beta, \overline{f}) = h(\overline{f})$ and $\beta \subseteq \mathcal{B}$. Then there exists a partition $\{A_1, A_2\}$ of B_1 such that $\{A_1, A_2\} \subseteq \mathcal{B}$ and $\gamma = \{A_1, A_2, B_2, \dots, B_{k_f}\}$ is a generator for $\overline{f}$.*

Proof: Let us begin with presentation of the general idea of the proof. Let $H_0 = B_1$. We take certain Rohlin tower in H_0 with basis from $\mathcal{B}$. The tower is given by induced transformation $\overline{f}_{H_0}$. We suitably label the part of levels of the tower by elements of the set $\{0, 1\}$. The union of remaining levels will be denoted by H_1. Next, we repeat the same reasoning with H_1 and etc. Here we care for the measure of H_i to tend to zero as i tends to infinity. For coding we use ergodic theorem, Shannon-McMillan-Breiman theorem and the equality $h_x(\beta, \overline{f}) = h(\overline{f})$ a.e. Consequently the set A_i is the union of levels with label i for $i = 0, 1$. This construction is modification of the proof of Theorem 30.1 [1] and in consequence of the proof of Theorem 28.1 [1]. The detailed presentation of the construction needs the reproduction of the proofs of these theorems. Therefore we enclose below only necessary modifications of the proofs of Theorems 28.1 and 30.1 [1].

At first we observe by (2) that $h_x(f) \leq \log k_f$ a.e. Now, we specify our modifications.

(i) For a sequence (γ_i) of partitions we assume additionally that $\beta < \gamma_1$ and $\gamma_i \subseteq \mathcal{B}$ for $i = 1, 2, \ldots$.

(ii) $Z_i = \emptyset$ for $i = 1, 2, \ldots$.

(iii) We start with $G_0 = \{B_2 \cup \cdots \cup B_{k_f}\}$, $H_0 = B_1$.

(iv) Let $M = \{0,1\}^Z$ and $K = \{1\}^Z$. There exists a subshift $\overline{M}$ of a finite type (see Lemma 26.17 [1]) such that $h(\overline{M}) \geq \frac{1}{2}h(M) = \frac{1}{2}\log 2$ and $\overline{M} \cap K = \emptyset$. Therefore there exists L such that $a = [\underbrace{1\ldots1}_{L}] \notin \overline{M}$. As the blocks U_p^1, U_p^2 (see [1, p. 283]) we take block [0]. We start with $\overline{f}_{H_0}$-Rohlin set $F_1 \subset H_0$ such that $F_1 \in \mathcal{B}$ and $\overline{\mu}_x(F_1) = \text{const.}$ a.e. We use the same coding method as in the proof of Theorem 28.1 [1] with respect to M, $\overline{M}$ and a, U_p^i, $i = 1, 2$, as above. By the definition of the first step we have $\gamma \cap G_i > \beta$ for $i = 1, 2, \ldots$ and hence $h_x(\gamma \cap G_i, \overline{f}) = h_x(\overline{f})$ a.e. Therefore the condition (a) ([1, p. 311]) is always satisfied.

(v) In the step (i), for $i \geq 2$, we take $\overline{f}_{H_{i-1}}$-Rohlin set $F_i \subset H_{i-1}$ such that $F_i \in \mathcal{B}$. For coding we use $\overline{f}_{H_{i-1}}^{-n_i} F_i$ instead of F_i.

For the proof it suffices to apply the reasoning from the proof of Theorem 30.1 with the above modifications (i)–(v). Consequently we get the generator $\gamma = \{A_0, A_1, B_2, \ldots, B_{k_f}\}$ for $\overline{f}$. It remains only to show that $\gamma \subseteq \mathcal{B}$. Assume that $\gamma \cap G_{i-1} \subseteq \mathcal{B}$ for some $i \geq 1$. Then $H_{i-1} \in \mathcal{B}$, $\bigvee_{i=0}^{q_i-1} \overline{f}^{-i}\gamma_i \subseteq \mathcal{B}$, $\bigvee_{i=0}^{q_i-1} \overline{f}^{-i}(\gamma \cap G_{i-1}) \subseteq \mathcal{B}$. We code $\overline{F}_i = \overline{f}_{H_{i-1}}^{-n_i} F_i$ by adjoining to every $\overline{f}_{H_{i-1}}^{-n_i} A' \cap \overline{F}_i \subset \overline{f}_{H_{i-1}}^{-n_i} A'' \cap \overline{F}_i$ a different $\overline{M}$-block of length $n_i - k_i - c$ for $A' \in S_i' \subset \mathcal{B}$, $A'' \in S_i'' \subset \mathcal{B}$. Therefore $\gamma \cap G_i \subseteq \mathcal{B}$. It follows that $\gamma \subseteq \mathcal{B}$. $\square$

By Lemma 2 we conclude (as in [2]) that γ such that $\gamma \vee f^{-1}\varepsilon = \varepsilon$ is a 1-sided generator for f_A a.e. and consequently is a 1-sided generator for f.

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