On the sequence of integer parts of a good sequence for the ergodic theorem

Emmanuel Lesigne

Abstract. If (u_n) is a sequence of real numbers which is good for the ergodic theorem, is the sequence of the integer parts $([u_n])$ good for the ergodic theorem ? The answer is negative for the mean ergodic theorem and affirmative for the pointwise ergodic theorem.

Keywords: ergodic theorem along subsequences, Banach principle

Classification: 28D

Introduction

Let us specify at once the notion of good sequence for the ergodic theorem.

Definition 1. *A sequence $u = (u_n)_{n\geq 0}$ of real positive numbers is a good sequence for the mean ergodic theorem if, given a probability space $(\Omega, \mathcal{T}, \mu)$ and a measure preserving flow $(S_t)_{t\geq 0}$ on Ω, for all $f \in L^2(\mu)$, the sequence*

$$\Big(\frac{1}{N}\sum_{n=0}^{N-1} f \circ S_{u_n}\Big)_{N>0}$$

converges in $L^2(\mu)$.

(In this definition the space L^2 does not play a particular role. The exponent 2 can be replaced by any exponent in $[1, +\infty[$.)

Definition 2. *Let $p \in [1, +\infty]$. A sequence $u = (u_n)_{n\geq 0}$ of real positive numbers is a good sequence for the pointwise ergodic theorem in L^p if, given a probability space $(\Omega, \mathcal{T}, \mu)$ and a measure preserving flow $(S_t)_{t\geq 0}$ on Ω, for all $f \in L^p(\mu)$, the sequence*

$$\Big(\frac{1}{N}\sum_{n=0}^{N-1} f(S_{u_n}\omega)\Big)_{N>0}$$

converges for μ-almost all ω.

Examples

1. Numerous and interesting examples of sequences of integers good for the ergodic theorem can be found in the literature. If (a_n) is such a sequence, then, for all reals α and β, the sequence $(\alpha a_n+\beta)$ is also a good sequence for the ergodic theorem.

2. For all real number $\alpha > 0$, the sequence (n^α) is good for the mean ergodic theorem (see for example [1]).

3. For all real numbers α except perhaps a countable family, and in particular for all numbers α rational non integer, the sequence (n^α) is not a good sequence for the pointwise ergodic theorem in L^∞. This is proved in [1].

Any good sequence for the pointwise ergodic theorem in one space L^p is a good sequence for the mean ergodic theorem. This can be easily justified, using the density of the space of bounded measurable functions in L^p and Lebesgue dominated convergence theorem.

Christian Mauduit and the author wondered if the sequence of integer parts of a good sequence for the ergodic theorem is still a good sequence. The answer is surprising: it is negative for the mean ergodic theorem but positive for the pointwise ergodic theorem!

Theorem 1. *Let $p \in [1,+\infty[$. If a sequence $u = (u_n)_{n\geq 0}$ of real positive numbers is good for the pointwise ergodic theorem in L^p, then the sequence $[u] := ([u_n])_{n\geq 0}$ of its integer parts is good for the pointwise ergodic theorem in L^p.*

Remark 1. *There exists a good sequence for the mean ergodic theorem whose sequence of integer parts is not good for the mean ergodic theorem.*

This remark is easy to justify; an example can be constructed by perturbation of a good sequence for example the sequence of all integers (see Section 1).

Proof of Theorem 1 is based on the following deep result which is due to J. Bourgain, answering a question posed by A. Bellow.

Theorem 2 ([3]). *Let $(a_n)_{n\geq 0}$ be a sequence of non zero real numbers which converges to zero.*
There exists a bounded measurable function f on the torus $\mathbb{T}$ such that the sequence

$$\Big(\frac{1}{N}\sum_{n=0}^{N-1} f(x+a_n)\Big)_{N>0}$$

diverges for all x in a set of positive Lebesgue measure.

1. On the mean theorem

Good sequences for the mean ergodic theorem are characterized by the next proposition which is well known as a consequence of the spectral theorem.

Proposition 1. *A sequence (u_n) is good for the mean ergodic theorem if and only if, for all $t \in \mathbb{R}$, the sequence $\left(\frac{1}{N}\sum_{n=0}^{N-1}\exp(itu_n)\right)$ converges.*

As a direct consequence we have the following result on perturbations of good sequences.

Proposition 2. *If (u_n) is a good sequence for the mean ergodic theorem and if (ϵ_n) is a real sequence which tends to zero, then the sequence $(u_n+\epsilon_n)$ is still a good sequence for the mean ergodic theorem.*

It is now easy to justify the Remark 1: let (a_n) be a sequence of 0 and -1's such that the sequence

$$\frac{1}{N}\sum_{n=0}^{N-1}(-1)^{n+a_n}$$

diverges. Consider the sequence $u_n := n + \frac{a_n}{n+1}$. By Proposition 2, the sequence (u_n) is good. By construction, the sequence of its integer parts is not good.

It is of course possible to wonder to which dynamical systems these counterexamples apply. We can prove the following result: let $(\Omega, \mathcal{T}, \mu, (S_t)_{t\geq 0})$ be a measure preserving system; if there exists a subset A of $\mathbb{N}$, with positive density, and a function f in $L^2(\mu)$ such that the sequence $\left(\frac{1}{N}\sum_{n\in A\cap[0,N[} f\circ S^n\right)$ does not converge in the mean, then there exists a sequence (ϵ_n) tending to zero and a function g in L^∞ such that the sequence $\left(\frac{1}{N}\sum_{n\in[0,N[} g\circ S^{[n+\epsilon_n]}\right)$ does not converge in the mean.

2. On the pointwise theorem

Bourgain's proof of Theorem 2 is based on his "entropy criteria" and on the following lemma.

Lemma 1. *Let (a_n) be a sequence of non zero real numbers converging to zero. Given a positive integer r, there are integers $J_1 < J_2 < \ldots < J_r$ satisfying the following condition:*
given any sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \{0,1\}^r$, there is an integer $n = n(\alpha)$ such that,
for each integer s between 1 and r,

$$\left|\, 1 - \frac{1}{J_s}\sum_{j<J_s}\exp(2\pi i a_j n)\,\right| \begin{cases} < \frac{1}{10} & \text{if } \alpha_s = 0 \\ > \frac{1}{2} & \text{if } \alpha_s = 1. \end{cases}$$

In fact the finite sequences $(J_s)_{1\leq s\leq r}$ appearing in this lemma can be chosen in any fixed infinite subset of $\mathbb{N}$. Therefore J. Bourgain proved the following result.

Theorem 3. *Let $(a_n)_{n\geq 0}$ be a sequence of non zero real numbers converging to zero and $(N_k)_{k\geq 0}$ a non bounded sequence of positive integers.*

There exists a bounded measurable function f on the torus $\mathbb{T}$ such that the sequence

$$\Big(\frac{1}{N_k}\sum_{n=0}^{N_k-1} f(x+a_n)\Big)_{k\geq 0}$$

is not almost everywhere convergent.

This theorem will be used in the proof of the following proposition, in which we denote by $\overline{x} = x - [x]$ the fractional part of a real x.

Proposition 3. *Let $p \in [1, +\infty[$. Let (u_n) be a good sequence for the pointwise ergodic theorem in L^p. For all $h \in]0, 1]$,*

$$\lim_{\delta\to 0^+} \limsup_{N\to+\infty} \frac{1}{N} \operatorname{card}\big\{n \in [0, N[\,\big|\; \overline{u_n} \in]h-\delta, h[\big\} = 0.$$

Let (u_n) be a good sequence for the pointwise ergodic theorem. It is easy to verify that this sequence has an asymptotic distribution modulo 1, that is to say the sequence of probabilities $\big(\frac{1}{N}\sum_{n<N}\delta_{\overline{u_n}}\big)$ converges on $\mathbb{T}$. Denote by ν this asymptotic distribution. Proposition 3 says that point masses of the probability ν can only appear along constant subsequences of the sequence $(\overline{u_n})$. More precisely, for all $h \in [0, 1[$,

$$\nu\big(\{h\}\big) = \lim_{N\to+\infty} \frac{1}{N} \operatorname{card}\big\{n \in [0, N[\,\big|\; \overline{u_n} = h\big\}.$$

Proof of Proposition 3: The only dynamical system we shall consider here is $\Omega = \mathbb{T}$ with the uniform probability μ and the measure preserving flow $S_t(x) = x + t$ modulo 1.

Let $(a_n)_{n\geq 0}$ be a real sequence. If f is a function on $\mathbb{T}$, we note

$$A_N f(x) := \frac{1}{N}\sum_{n<N} f(x+a_n).$$

Banach's principle (see for example [4]) states that if for all $f \in L^p(\mu)$ the sequence $(A_N f)_{N>0}$ converges almost everywhere, then

$$\lim_{\lambda\to+\infty} \sup_{\|f\|_p\leq 1} \mu\Big\{\sup_{N>0} |A_N f| > \lambda\Big\} = 0. \tag{1}$$

Reciprocally, if the sequence (a_n) has an asymptotic distribution modulo 1 and if (1) is true, then, for all $f \in L^p(\mu)$, the sequence $(A_N f)$ converges almost everywhere. (Indeed, if (a_n) has an asymptotic distribution modulo 1, then, for all continuous function f, the sequence $(A_N f)$ converges everywhere, and property (1) ensures that the set of functions f such that $(A_N f)$ converges almost everywhere is closed in $L^p(\mu)$.)

This remark is also true for the convergence of subsequences of (A_N) and it allows us to deduce from Theorem 3 the following lemma.

Lemma 2. *Let $(a_n)_{n\geq 0}$ be a sequence of non zero real numbers converging to zero and $(N_k)_{k\geq 0}$ be an unbounded sequence of positive integers.*
There exists $\epsilon > 0$ such that, for all $\lambda > 0$, there exists $f \in L^p(\mu)$ satisfying

$$\|f\|_p \leq 1 \quad \textit{and} \quad \mu\big\{\sup_{N_k>0} |A_{N_k} f| > \lambda\big\} > \epsilon.$$

Replacing the function f by its absolute value, we can also suppose that this function is positive.

We can now prove Proposition 3.

Let (u_n) be a real sequence and h a fixed number in $]0,1]$. Let us suppose that

$$\lim_{\delta\to 0^+} \limsup_{N\to+\infty} \frac{1}{N} \operatorname{card}\big\{n \in [0,N[\,\big|\ \overline{u_n} \in]h-\delta, h[\big\} > 0.$$

We want to show that (u_n) is not a good sequence for the pointwise ergodic theorem; replacing u_n by $u_n - h + 1$, we can suppose that $h = 1$. There exists $\rho > 0$ such that, for all $\delta > O$

$$\limsup_{N\to+\infty} \frac{1}{N} \operatorname{card}\big\{n \in [0,N[\,\big|\ \overline{u_n} > 1-\delta\big\} > \rho. \tag{2}$$

This implies that there is an increasing sequence of integers $(n_j)_{j\geq 0}$ such that

$$\lim_{j\to\infty} \overline{u_{n_j}} = 1 \quad \text{and} \quad \limsup_{j\to\infty} \frac{j}{n_j} \geq \rho > 0.$$

$\big($ This sequence (n_j) can be constructed as follows: by (2) there is an integer sequence (N_p) such that $N_0 = 0$, $N_{p+1} > N_p$ and, for $p > 0$,

$$\frac{1}{N_p} \operatorname{card}\big\{n \in [0,N_p[\,\big|\ \overline{u_n} > 1 - \frac{1}{p}\big\} > \rho;$$

we put

$$\{n_j\} := \bigcup_{p>0} \big\{n \in [N_{p-1}, N_p[\,\big|\ \overline{u_n} > 1 - \frac{1}{p}\big\}.\big)$$

Let $(j_k)_{k\geq 0}$ be an increasing sequence of integers such that, for all k, $\dfrac{j_k}{n_{j_k}} > \dfrac{\rho}{2}$.

Let f be a positive function on $\mathbb{T}$. We have:

$$\begin{aligned}
\sup_N\Big(\frac{1}{N}\sum_{n<N} f(x+u_n)\Big) &\geq \sup_j\Big(\frac{1}{n_j}\sum_{n<n_j} f(x+u_n)\Big)\\
&\geq \sup_j\Big(\frac{j}{n_j}\frac{1}{j}\sum_{i<j} f(x+u_{n_i})\Big)\\
&\geq \frac{\rho}{2}\sup_k\Big(\frac{1}{j_k}\sum_{i<j_k} f(x+u_{n_i})\Big).
\end{aligned}$$

Using notations $u_{n_i} = a_i$ and $j_k = N_k$, we can apply Lemma 2. There exists $\epsilon > 0$ such that, for all $\lambda > 0$, there is $f \in L^p$ satisfying

$$\|f\|_p \leq 1 \quad \text{and} \quad \mu\{x \mid \sup_N \big(\frac{1}{N} \sum_{n<N} f(x+u_n)\big) > \lambda\} > \epsilon.$$

By Banach's principle, this implies that the sequence (u_n) is not good for the pointwise ergodic theorem. Proof of Proposition 3 is complete. □

Proof of Theorem 1: Let (u_n) be a real sequence, good for the pointwise ergodic theorem. Denote by $d_n := [u_n]$ the integer part of u_n. In order to prove that (d_n) is a good sequence, it is enough to prove that, if $(\Omega, \mathcal{T}, \mu)$ is a probability space and T a measure preserving transformation on this space, then, for all $f \in L^p(\mu)$, the sequence $\big(\frac{1}{N}\sum_{n<N} f \circ T^{d_n}\big)$ converges μ-almost everywhere.

Let us fix $(\Omega, \mathcal{T}, \mu, T, f)$, where f is a bounded measurable function on Ω.

We consider the special flow defined above the system $(\Omega, \mathcal{T}, \mu, T)$, under the constant ceiling function 1. Denoting by m the uniform probability on $[0,1[$, this flow $(S_t)_{t\geq 0}$ is defined on the space $(\Omega \times [0,1[, \mu \times m)$ by

$$S_t(\omega, x) = (T^{[t+x]}\omega, \overline{(t+x)}).$$

We denote by $\tilde{f}$ the trivial extension of f on $\Omega \times [0,1[$. It is defined by $\tilde{f}(\omega, x) := f(\omega)$.

By hypothesis, for $\mu \times m$-almost all (ω, x), the sequence

$$\Big(\frac{1}{N} \sum_{n<N} \tilde{f}\big(S_{u_n}(\omega, x)\big)\Big)$$

converges. Now

$$\frac{1}{N} \sum_{n<N} \tilde{f}\big(S_{u_n}(\omega, x)\big) = \frac{1}{N} \sum_{n<N} f(T^{[u_n+x]}\omega).$$

Fix $\delta > 0$. For μ-almost all ω, there exists $x \in [0, \delta[$ such that the sequence

$$\Big(\frac{1}{N} \sum_{n<N} f\big(T^{[u_n+x]}\omega\big)\Big)$$

converges. For such an x, we have $[u_n + x] = d_n$ except perhaps when $\overline{u_n} \in]1-\delta, 1[$. We pose $E_\delta = \big\{n \in \mathbb{N} \mid \overline{u_n} > 1 - \delta\big\}$.
If $x \in [0, \delta[$, we have

$$\Big| \frac{1}{N} \sum_{n<N} f(T^{d_n}\omega) - \frac{1}{M} \sum_{n<M} f(T^{d_n}\omega) \Big| \leq$$

$$\leq \Big| \frac{1}{N} \sum_{n<N} f(T^{[u_n+x]}\omega) - \frac{1}{M} \sum_{n<M} f(T^{[u_n+x]}\omega) \Big| +$$

$$+ 2\|f\|_\infty \Big(\frac{1}{N} \operatorname{card}\big([0,N[\cap E_\delta\big) + \frac{1}{M} \operatorname{card}\big([0,M[\cap E_\delta\big)\Big).$$

So

$$\limsup_{N,M\to\infty} \Big| \frac{1}{N}\sum_{n<N} f(T^{d_n}\omega) - \frac{1}{M}\sum_{n<M} f(T^{d_n}\omega) \Big| \leq$$
$$\leq 4\|f\|_\infty \limsup_{N\to\infty} \frac{1}{N}\operatorname{card}\big([0,N[\cap E_\delta\big).$$

Proposition 3 says that this last quantity tends to zero with δ. This proves that, for μ-almost all ω, $\big(\frac{1}{N}\sum_{n<N} f(T^{d_n}\omega)\big)$ is a Cauchy sequence.

This result has been obtained for bounded functions f. We shall now prove that the set of functions f in $L^p(\mu)$ such that the sequence $\big(\frac{1}{N}\sum_{n<N} f(T^{d_n}\omega)\big)$ converges almost everywhere is closed in $L^p(\mu)$. This is the direct consequence of a maximal inequality based on the following remark (where $\tilde{f}$ is the trivial extension of f to $\Omega\times[0,1[$).
For each $(\omega,x)\in\Omega\times[0,1[$, we have $f(T^{d_n}\omega)=\tilde{f}(S_{u_n}(\omega,x))$ or $\tilde{f}(S_{u_n-1}(\omega,x))$. This implies that

$$\Big| \frac{1}{N}\sum_{n<N} f\circ T^{d_n} \Big| \leq \frac{1}{N}\sum_{n<N} |\tilde{f}|\circ S_{u_n} \;+\; \frac{1}{N}\sum_{n<N} |\tilde{f}\circ S^{-1}|\circ S_{u_n}.$$

And maximal inequality for this last expression is a consequence of our hypothesis and Banach's principle. This completes the proof of Theorem 1. □

N.B.: After the writing of this paper, M. Wierdl informed the author that, in a common work with M. Boshernitzan and R. Jones, he had obtained recently a result similar to the main one of this note ([2]).

References

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Département de Mathématiques, Faculté des Sciences et Techniques, Université François Rabelais, Parc de Grandmont, 37200 Tours, France

E-mail: lesigne@balzac.univ-tours.fr