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Differential equations at resonance

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Abstract. New existence results are presented for the two point singular "resonant" boundary value problem $\frac{1}{p}(py')' + ry + \lambda_m qy = f(t, y, py')$ a.e. on $[0,1]$ with y satisfying Sturm Liouville or Periodic boundary conditions. Here λ_m is the $(m+1)^{st}$ eigenvalue of $\frac{1}{pq}[(pu')' + rpu] + \lambda u = 0$ a.e. on $[0,1]$ with u satisfying Sturm Liouville or Periodic boundary data.

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1. Introduction

In this paper we derive some existence results for the second order equation

$$(1.1)\quad \frac{1}{p(t)}(p(t)y'(t))' + r(t)y(t) + \lambda_m q(t)y(t) = f(t, y(t), p(t)y'(t)) \text{ a.e. on } [0,1]$$

with y satisfying either

(i) (Sturm Liouville)

$$\text{(SL)}\qquad \begin{cases} -\alpha y(0) + \beta \lim_{t\to 0^+} p(t)y'(t) = 0, & \alpha \ge 0, \beta \ge 0, \alpha^2 + \beta^2 > 0 \\ ay(1) + b \lim_{t\to 1^-} p(t)y'(t) = 0, & a \ge 0, b \ge 0, a^2 + b^2 > 0 \end{cases}$$

or

(ii) (Periodic)

$$\text{(P)}\qquad \begin{cases} y(0) = y(1) \\ \lim_{t\to 0^+} p(t)y'(t) = \lim_{t\to 1^-} p(t)y'(t). \end{cases}$$

Remarks. (i) λ_m will be described later.

(ii) The Neumann condition $\lim_{t\to 0^+} p(t)y'(t) = \lim_{t\to 1^-} p(t)y'(t) = 0$ is included in (SL) with $\alpha = a = 0$.

(iii) If a function $u \in C[0,1] \cap C^1(0,1)$ with $pu' \in C[0,1]$ satisfies boundary condition (i) we write $u \in$ (SL). A similar remark applies for the boundary condition (ii).

Throughout the paper $p \in C[0,1] \cap C^1(0,1)$ together with $p > 0$ on $(0,1)$. Also $pf : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is an L^1-Carathéodory function. By this we mean:

(i) $t \to p(t)f(t, y, q)$ is measurable for all $(y, q) \in \mathbf{R}^2$;

(ii) $(y, q) \to p(t)f(t, y, q)$ is continuous for a.e. $t \in [0,1]$;

(iii) for any $r > 0$ there exists $h_r \in L^1[0,1]$ such that $|p(t)f(t,y,q)| \le h_r(t)$ for a.e. $t \in [0,1]$ and for all $|y| \le r$, $|q| \le r$.

For notational purposes let w be a weight function. By $L^1_w[0,1]$ we mean the space of functions u such that $\int_0^1 w(t)|u(t)|\,dt < \infty$. $L^2_w[0,1]$ denotes the space of functions u such that $\int_0^1 w(t)|u(t)|^2\,dt < \infty$; also for $u, v \in L^2_w[0,1]$ define $\langle u, v\rangle = \int_0^1 w(t)u(t)\overline{v(t)}\,dt$. Let $AC[0,1]$ be the space of functions which are absolutely continuous on $[0,1]$.

Before we discuss the boundary value problem (1.1) and its appropriate literature we first gather together some facts on second order differential equations ([12], [16]). Consider the linear equation

$$\begin{cases} \frac{1}{p}(py')' + \tau y = g(t) & \text{a.e. on } [0,1] \\ y \in (\mathrm{SL}) \text{ or } (\mathrm{P}). \end{cases} \tag{1.2}$$

By a solution to (1.2) we mean a function $y \in C[0,1] \cap C^1(0,1)$ with $py' \in AC[0,1]$ which satisfies the differential equation in (1.2) a.e. on $[0,1]$ and the stated boundary conditions.

Theorem 1.1. *Suppose*

$$p \in C[0,1] \cap C^1(0,1) \text{ with } p > 0 \text{ on } (0,1) \text{ and } \int_0^1 \frac{ds}{p(s)} < \infty \tag{1.3}$$

and

$$\tau, g \in L^1_p[0,1] \tag{1.4}$$

are satisfied. If

$$\begin{cases} \frac{1}{p}(py')' + \tau y = 0 & \text{a.e. on } [0,1] \\ y \in (\mathrm{SL}) \text{ or } (\mathrm{P}) \end{cases} \tag{1.5}$$

has only the trivial solution, then (1.2) *has exactly one solution y given by*

$$y(t) = d_0 u_1(t) + d_1 u_2(t) + \int_0^t \frac{[u_2(t)u_1(s) - u_1(t)u_2(s)]}{W(s)} g(s)\,ds$$

where u_1 *is the unique solution to*

$$\begin{cases} \frac{1}{p}(pu')' + \tau u = 0 \quad \text{a.e. on } [0,1] \\ u(0) = 1, \ \lim_{t\to 0^+} p(t)u'(t) = 0 \end{cases}$$

and u_2 *is the unique solution to*

$$\begin{cases} \frac{1}{p}(pu')' + \tau u = 0 \quad \text{a.e. on } [0,1] \\ u(0) = 0, \ \lim_{t\to 0^+} p(t)u'(t) = 1 \end{cases}$$

and d_0 *and* d_1 *are uniquely determined from the boundary condition;* W *of course denotes the Wronskian. In fact*

$$y(t) = \int_0^1 G(t,s)g(s)\,ds$$

with

$$G(t,s) = \begin{cases} \frac{y_1(s)y_2(t)}{W(s)}, & 0 < s \le t \\ \frac{y_1(t)y_2(s)}{W(s)}, & t \le s < 1 \end{cases}$$

where y_1 *and* y_2 *are the two "usual" linearly independent solutions i.e. choose* $y_1 \neq 0$, $y_2 \neq 0$ *so that* y_1, y_2 *satisfy* $\frac{1}{p}(py')' + \tau y = 0$ *a.e. on* $[0,1]$ *with* y_1 *satisfying the first boundary condition and* y_2 *satisfying the second boundary condition.*

We now state an existence principle ([16]), which was established using fixed point methods, for the second order nonresonant boundary value problem

$$(1.6) \qquad \begin{cases} \frac{1}{p}(py')' + \tau y = f(t,y,py') \quad \text{a.e. on } [0,1] \\ y \in (\text{SL}) \text{ or } (\text{P}). \end{cases}$$

Theorem 1.2. *Let* $pf : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ *be an* L^1*-Carathéodory function and assume* (1.3) *and*

$$(1.7) \qquad \tau \in L^1_p[0,1]$$

hold. In addition suppose (1.5) *has only the trivial solution. Now assume there is a constant* M_0, *independent of* λ, *with*

$$\|y\|_* = \max\{\sup_{[0,1]} |y(t)|, \ \sup_{(0,1)} |p(t)y'(t)|\} \le M_0$$

for any solution y *to*

$$\begin{cases} \frac{1}{p}(py')' + \tau y = \lambda f(t,y,py') \quad \text{a.e. on } [0,1] \\ y \in (\text{SL}) \text{ or } (\text{P}) \end{cases}$$

for each $\lambda \in (0,1)$. *Then* (1.6) *has at least one solution* $u \in C[0,1] \cap C^1(0,1)$ *with* $pu' \in AC[0,1]$.

Next we gather together some results on the Sturm Liouville eigenvalue problem

$$\begin{cases} Lu = \lambda u \quad \text{a.e. on } [0,1] \\ u \in (\mathrm{SL}) \text{ or } (\mathrm{P}) \end{cases} \tag{1.8}$$

where $Lu = -\frac{1}{pq(t)}[(pu')' + r(t)pu]$. Assume (1.3) and

$$r, q \in L^1_p[0,1] \quad \text{with} \quad q > 0 \quad \text{a.e. on } [0,1] \tag{1.9}$$

hold. Let

$$D(L) = \{w \in C[0,1] : w, pw' \in AC[0,1] \text{ with } w \in (\mathrm{SL}) \text{ or } (\mathrm{P})\}.$$

Then L has a countably infinite number ([1], [12], [16]) of real eigenvalues λ_i with corresponding eigenfunctions $\psi_i \in D(L)$. The eigenfunctions ψ_i may be chosen so that they form a orthonormal set and we may also arrange the eigenvalues so that

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots \ . \tag{1.10}$$

Remark. The λ_i's may be estimated numerically ([2]) using SLEIGN.

In addition the set of eigenfunctions ψ_i form a basis for $L^2_{pq}[0,1]$ and if $h \in L^2_{pq}[0,1]$ then h has a Fourier series representation and h satisfies Parseval's equality i.e.

$$h = \sum_{i=0}^{\infty} \langle h, \psi_i \rangle \psi_i \quad \text{and} \quad \int_0^1 pq|h|^2\, dt = \sum_{i=0}^{\infty} |\langle h, \psi_i \rangle|^2.$$

We are concerned with existence results for the nonlinear second order equation

$$\begin{cases} \frac{1}{p}(py')' + ry + \lambda_m qy = f(t, y, py') \quad \text{a.e. on } [0,1] \\ y \in (\mathrm{SL}) \text{ or } (\mathrm{P}) \end{cases} \tag{1.11}$$

where λ_m is the $(m+1)^{st}$ eigenvalue of (1.8). In recent years several authors ([4], [7]–[9], [11], [13], [18]–[19]) have examined the boundary value problems

$$\begin{cases} y'' + n^2\pi^2 y = f(t, y) \quad \text{a.e. on } [0,1] \\ y(0) = y(1) = 0 \end{cases}$$

and

$$\begin{cases} y'' + m^2\pi^2 y = f(t, y) \quad \text{a.e. on } [0,1] \\ y(0) = y(1),\ y'(0) = y'(1) \end{cases}$$

where $n \geq 1$, $m \geq 0$ are integers. Most of the papers in the literature ([3], [7], [11], [18]–[19]) concentrate on the first eigenvalue ($n = 1$ or $m = 0$). However over the last ten years or so ([6], [10]) the case when $n > 1$ or $m > 0$ has been discussed. This paper continues this study for the more general problem (1.11); also it provides a new approach to studying the above resonant type problems. We refer the reader to [6]–[9] for many of the motivating ideas in this paper. Finally it is of interest to note that in previous studies ([6], [8], [11]) the nonlinearity f is required to grow no more than linearly in y as $|y| \to \infty$ whereas in this paper solutions will exist provided f grows fast enough e.g. $yf(t,y,z) \geq A|y|^{\theta+1}$ for some $A > 0$ and $\theta > 0$.

2. Existence

Existence theory is developed for the second order boundary value problem

$$(2.1)\qquad \begin{cases} \frac{1}{p}(py')' + ry + \lambda_m qy = f(t,y,py') & \text{a.e. on } [0,1] \\ y \in (\mathrm{SL}) \text{ or } (\mathrm{P}) \end{cases}$$

where λ_m is the $(m+1)^{st}$ eigenvalue of

$$(2.2)\qquad \begin{cases} Lu = \lambda u & \text{a.e. on } [0,1] \\ u \in (\mathrm{SL}) \text{ or } (\mathrm{P}) \end{cases}$$

and $Lu = -\frac{1}{pq(t)}[(pu')' + r(t)pu]$.

Two types of existence results are presented, the first examines the problem on the "left" of the eigenvalue whereas the second discusses the problem on the "right" of the eigenvalue.

Existence theory I.

Throughout this subsection let

$$H_{\alpha_0,\theta}(u_1) = \begin{cases} |u_1|^{\theta+1}, & |u_1| \leq 1 \\ |u_1|^{\alpha_0+1}, & |u_1| > 1. \end{cases}$$

Theorem 2.1. *Let $pf : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ be an L^1-Carathéodory function with (1.3) and (1.9) satisfied. Suppose f has the decomposition $f(t,u_1,u_2) = g(t,u_1,u_2) + h(t,u_1,u_2)$ with $pg, ph : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ L^1-Carathéodory functions and*

$$(2.3)\qquad \begin{cases} \text{there exist constants } A > 0, 0 < \alpha_0 < 1 \text{ and a function} \\ \phi \in L^1_p[0,1],\ \phi > 0 \text{ a.e. on } [0,1] \text{ with } u_1 g(t,u_1,u_2) \geq A\phi(t)H_{\alpha_0,\theta}(u_1) \\ \text{for a.e. } t \in [0,1];\ \text{here } \alpha_0 \leq \theta \end{cases}$$

$$(2.4)\qquad \begin{cases} \textit{there exist } \phi_i \in L^1_p[0,1],\ i=1,2,3 \textit{ and constants } \beta_0 \textit{ and} \\ \sigma \textit{ with } |h(t,u_1,u_2)| \le \phi_1(t) + \phi_2(t)|u_1|^{\beta_0} + \phi_3(t)|u_2|^{\sigma} \textit{ for} \\ \textit{a.e. } t\in[0,1];\ \textit{here } \beta_0 < \alpha_0 \textit{ and } 0 \le \sigma < \frac{\alpha_0}{2} \textit{ and} \\ \phi_3 > 0 \textit{ a.e. on } [0,1] \textit{ or } \phi_3 \equiv 0 \textit{ on } [0,1] \end{cases}$$

$$(2.5)\qquad \begin{cases} \textit{there exist } \phi_i \in L^1_p[0,1],\ i=4,5 \textit{ and a constant } \gamma \le \alpha_0 \textit{ with} \\ |g(t,u_1,u_2)| \le \phi_4(t) + \phi_5(t)|u_1|^{\gamma} \textit{ for a.e. } t\in[0,1] \end{cases}$$

$$(2.6)\qquad \begin{cases} \phi_4^2 q^{-1} \in L^1_p[0,1],\ \phi_1^2 q^{-1} \in L^1_p[0,1], \\ \left(\phi_5^{2(\alpha_0+1)} q^{-(\alpha_0+1)} \phi^{-2\gamma}\right)^{\frac{1}{\alpha_0+1-2\gamma}} \in L^1_p[0,1], \\ \left(\phi_2^{2(\alpha_0+1)} q^{-(\alpha_0+1)} \phi^{-2\beta_0}\right)^{\frac{1}{\alpha_0+1-2\beta_0}} \in L^1_p[0,1] \textit{ and} \\ \left(\phi^2 q^{-(\alpha_0+1)}\right)^{\frac{1}{1-\alpha_0}} \in L^1_p[0,1] \end{cases}$$

and

$$(2.7)\qquad \begin{cases} \left(\phi_1^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}} \in L^1_p[0,1],\ \left(\phi_2^{\alpha_0+1}\phi^{-(\beta_0+1)}\right)^{\frac{1}{\alpha_0-\beta_0}} \in L^1_p[0,1], \\ \left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}} \in L^1_p[0,1], \\ \left(\phi_5^{\alpha_0+1}\phi^{-\gamma}\right)^{\frac{1}{\alpha_0+1-\gamma}} \in L^1_p[0,1] \textit{ and} \\ \left(q^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}} \in L^1_p[0,1] \end{cases}$$

holding. Then (2.1) *has at least one solution* $y \in C[0,1] \cap C^1(0,1)$ *with* $py' \in AC[0,1]$.

Remark. Typical examples where (2.3) is satisfied are say (i) $g(t,u_1,u_2) = u_1^{\frac{m}{n}}$, m odd and n odd or (ii) $g(t,u_1,u_2) = u_1^{\frac{1}{2}}$, $u_1 \ge 0$ with $g(t,u_1,u_2) = -|u_1|^{\frac{1}{2}}$, $u_1 < 0$.

PROOF: Consider the family of problems

$$(2.8)_\lambda\qquad \begin{cases} \frac{1}{p}(py')' + ry + \mu q y = \lambda[f(t,y,py') + (\mu-\lambda_m)qy] \ \text{ a.e. on } [0,1] \\ y \in \text{(SL) or (P)} \end{cases}$$

where $0 < \lambda < 1$ and $\lambda_{m-1} < \mu < \lambda_m$; here $\lambda_{-1} = -\infty$ (for notational purposes) with λ_i as described in (1.10).

Notice $L^2_{pq}[0,1] = \Omega \bigoplus \Omega^{\perp}$ where $\Omega = span\,\{\psi_0, \psi_1, \dots, \psi_{m-1}\}$; here ψ_i are the eigenfunctions corresponding to the eigenvalues λ_i (see Section 1).

Let y be any solution to $(2.8)_\lambda$. Then $y = u + w$ where $u \in \Omega$ and $w \in \Omega^{\perp}$. Multiply $(2.8)_\lambda$ by $w - u$ and integrate from 0 to 1 to obtain

$$\int_0^1 (w-u)(py')'\,dt + \int_0^1 pr[w^2 - u^2]\,dt + \mu \int_0^1 pq[w^2 - u^2]\,dt \\ = \lambda \int_0^1 (w-u)pf(t,y,py')\,dt + \lambda(\mu - \lambda_m) \int_0^1 pq[w^2 - u^2]\,dt.$$

Integration by parts yields

$$\int_0^1 (w-u)(py')'\,dt = Q_0 - \int_0^1 p(w')^2\,dt + \int_0^1 p(u')^2\,dt$$

where

$$Q_0 = \begin{cases} -\frac{a}{b}[w^2(1) - u^2(1)] - \frac{\alpha}{\beta}[w^2(0) - u^2(0)] & \text{if } \ y \in (\mathrm{SL}) \\ 0 & \text{if } \ y \in (P); \end{cases}$$

here $y(0) = 0$ means $u(0) + w(0) = 0$ and so $u(0) = w(0) = 0$. Thus we have

$$\begin{aligned} &Q_0 + \int_0^1 [-p(w')^2 + prw^2 + \mu pqw^2]\,dt + \int_0^1 [p(u')^2 - pru^2 - \mu pqu^2]\,dt \\ &= \lambda \int_0^1 (w-u)pf(t,y,py')\,dt + \lambda(\mu - \lambda_m) \int_0^1 pqw^2\,dt \\ &- \lambda(\mu - \lambda_m) \int_0^1 pqu^2\,dt. \end{aligned} \tag{2.9}$$

Now since $u \in \Omega$, $w \in \Omega^{\perp}$ and $y = u + w$ we have

$$u = \sum_{i=0}^{m-1} c_i \psi_i \quad \text{and} \quad w = \sum_{i=m}^{\infty} c_i \psi_i \quad \text{where} \quad c_i = \langle y, \psi_i \rangle;$$

note $u = 0$ if $m = 0$. Then since $(p\psi_i')' + rp\psi_i + \lambda_i pq\psi_i = 0$ we have

$$\begin{aligned} &Q_0 + \int_0^1 [-p(w')^2 + prw^2 + \mu pqw^2]\,dt + \int_0^1 [p(u')^2 - pru^2 - \mu pqu^2]\,dt \\ &= \sum_{i=m}^{\infty} (\mu - \lambda_i) c_i^2 \int_0^1 pq\psi_i^2\,dt + \sum_{i=0}^{m-1} (\lambda_i - \mu) c_i^2 \int_0^1 pq\psi_i^2\,dt \\ &\le (\mu - \lambda_m) \int_0^1 pqw^2\,dt + (\lambda_{m-1} - \mu) \int_0^1 pqu^2\,dt. \end{aligned}$$

Put this into (2.9) to obtain

$$\lambda\int_0^1 (w-u)pg(t,y,py')\,dt + (1-\lambda)(\lambda_m-\mu)\int_0^1 pqw^2\,dt$$
$$+ (\mu-\lambda_{m-1})\int_0^1 pqu^2\,dt + \lambda(\lambda_m-\mu)\int_0^1 pqu^2\,dt$$
$$\le -\lambda\int_0^1 (w-u)ph(t,y,py')\,dt.$$

Consequently

$$\begin{aligned}\int_0^1 pyg(t,y,py')\,dt + (\lambda_m-\mu)\int_0^1 pqu^2\,dt &\le 2\int_0^1 pug(t,y,py')\,dt \\ &+ \int_0^1 p|y||h(t,y,py')|\,dt + 2\int_0^1 p|u||h(t,y,py')|\,dt.\end{aligned} \tag{2.10}$$

Assumption (2.3) yields

$$\begin{aligned}\int_0^1 pyg(t,y,py')\,dt &\ge A\int_0^1 p\phi H_{\alpha_0,\theta}(y)\,dt \\ &= A\int_0^1 p\phi|y|^{\alpha_0+1}\,dt + A\int_{\{t:|y(t)|\le 1\}} p\phi[|y|^{\theta+1}-|y|^{\alpha_0+1}]\,dt \\ &\ge A\int_0^1 p\phi|y|^{\alpha_0+1}\,dt - A\int_0^1 p\phi\,dt\end{aligned}$$

and put this into (2.10), and use (2.4) and (2.5), to obtain

$$\begin{aligned}A\int_0^1 p\phi|y|^{\alpha_0+1}\,dt + (\lambda_m-\mu)\int_0^1 pqu^2\,dt &\le A\int_0^1 p\phi\,dt + 2\int_0^1 p\phi_4|u|\,dt \\ &+ 2\int_0^1 p\phi_5|u||y|^{\gamma}\,dt + \int_0^1 p\phi_1|y|\,dt \\ &+ \int_0^1 p\phi_2|y|^{\beta_0+1}\,dt + \int_0^1 p\phi_3|y||py'|^{\sigma}\,dt \\ &+ 2\int_0^1 p\phi_1|u|\,dt + 2\int_0^1 p\phi_2|u||y|^{\beta_0}\,dt \\ &+ 2\int_0^1 p\phi_3|u||py'|^{\sigma}\,dt.\end{aligned} \tag{2.11}$$

For the remainder of the proof we assume without loss of generality that $\sigma > 0$ and $\phi_3 \not\equiv 0$ on $[0,1]$. Let $\epsilon > 0$ be given. Hölder's inequality together with assumption (2.6) immediately yields the following inequalities:

$$2\int_0^1 p\phi_4|u|\,dt \le 2Q_1\left(\int_0^1 pqu^2\,dt\right)^{\frac{1}{2}} \le \epsilon\int_0^1 pqu^2\,dt + \frac{Q_1}{\epsilon}\,;$$

$$
\begin{aligned}
2\int_0^1 p\phi_1|u|\,dt &\le \epsilon\int_0^1 pqu^2\,dt + \frac{Q_2}{\epsilon};\\
2\int_0^1 p\phi_5|u||y|^{\gamma}\,dt &\le 2Q_3\left(\int_0^1 pqu^2\,dt\right)^{\frac12}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\gamma}{\alpha_0+1}}\\
&\le \epsilon Q_3\int_0^1 pqu^2\,dt + \frac{Q_3}{\epsilon}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{2\gamma}{\alpha_0+1}};\\
2\int_0^1 p\phi_2|u||y|^{\beta_0}\,dt &\le \epsilon Q_4\int_0^1 pqu^2\,dt + \frac{Q_4}{\epsilon}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{2\beta_0}{\alpha_0+1}};\\
\int_0^1 p\phi_1|y|\,dt &\le Q_5\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}};\\
\int_0^1 p\phi_2|y|^{\beta_0+1}\,dt &\le Q_6\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\beta_0+1}{\alpha_0+1}};\\
\int_0^1 p\phi_3|y||py'|^{\sigma}\,dt &\le \left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}}\\
&\quad\times\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}}|py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{\alpha_0}{\alpha_0+1}};\\
2\int_0^1 p\phi_3|u||py'|^{\sigma}\,dt &\le 2Q_7\left(\int_0^1 pqu^2\,dt\right)^{\frac12}\\
&\quad\times\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}}|py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{\alpha_0}{\alpha_0+1}}\\
&\le \epsilon Q_7\int_0^1 pqu^2\,dt\\
&\quad+\frac{Q_7}{\epsilon}\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}}|py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{2\alpha_0}{\alpha_0+1}}
\end{aligned}
$$

for some constants $Q_1,\ldots,Q_7$. Put these into (2.11) to obtain

$$
\begin{aligned}
&A\int_0^1 p\phi|y|^{\alpha_0+1}\,dt + (\lambda_m-\mu-2\epsilon-\epsilon Q_3-\epsilon Q_4-\epsilon Q_7)\int_0^1 pqu^2\,dt\\
&\le Q_8 + \frac{Q_3}{\epsilon}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{2\gamma}{\alpha_0+1}} + \frac{Q_4}{\epsilon}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{2\beta_0}{\alpha_0+1}}\\
&+ Q_5\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}} + Q_6\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\beta_0+1}{\alpha_0+1}}\\
&+\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}}\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}}|py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{\alpha_0}{\alpha_0+1}}
\end{aligned}
$$

$$+\frac{Q_7}{\epsilon}\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}}|py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{2\alpha_0}{\alpha_0+1}}$$

for some constant Q_8. We may choose ϵ so that $\lambda_m-\mu-2\epsilon-\epsilon Q_3-\epsilon Q_4-\epsilon Q_7>0$ and we have

$$\begin{aligned}
A\int_0^1 p\phi|y|^{\alpha_0+1}\,dt &\le Q_8+\frac{Q_3}{\epsilon}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{2\gamma}{\alpha_0+1}}\\
&+\frac{Q_4}{\epsilon}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{2\beta_0}{\alpha_0+1}}+Q_5\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}}\\
&+Q_6\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\beta_0+1}{\alpha_0+1}}\\
&+\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}}\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}}|py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{\alpha_0}{\alpha_0+1}}\\
&+\frac{Q_7}{\epsilon}\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}}|py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{2\alpha_0}{\alpha_0+1}}.
\end{aligned}\tag{2.12}$$

We now consider two cases $\int_0^1 p\phi|y|^{\alpha_0+1}\,dt>1$ and $\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\le 1$ separately.

Case (i). $\int_0^1 p\phi|y|^{\alpha_0+1}\,dt>1$.

Divide (2.12) by $\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}}$ and use $\int_0^1 p\phi|y|^{\alpha_0+1}\,dt>1$ to obtain

$$\begin{aligned}
A\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\alpha_0}{\alpha_0+1}} &\le Q_8+\frac{Q_3}{\epsilon}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{2\gamma-1}{\alpha_0+1}}\\
&+\frac{Q_4}{\epsilon}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{2\beta_0-1}{\alpha_0+1}}\\
&+Q_5+Q_6\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\beta_0}{\alpha_0+1}}\\
&+\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}}|py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{\alpha_0}{\alpha_0+1}}\\
&+\frac{Q_7}{\epsilon}\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}}|py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{2\alpha_0}{\alpha_0+1}}.
\end{aligned}$$

Now since $\max\{2\gamma-1,2\beta_0-1,\beta_0\}<\alpha_0$ there exist constants Q_9, Q_{10} and Q_{11}

with

$$\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\alpha_0}{\alpha_0+1}} \le Q_9 + Q_{10}\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}} |py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{\alpha_0}{\alpha_0+1}}$$
$$+ Q_{11}\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}} |py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{2\alpha_0}{\alpha_0+1}}.$$

Using the inequality $(a+b)^c \le 2^c(a^c+b^c)$ for $a \ge 0$, $b \ge 0$, $c \ge 0$ we see that there exist constants Q_{12} and Q_{13} with

$$\int_0^1 p\phi|y|^{\alpha_0+1}\,dt \le Q_{12} + Q_{13}\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}} |py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^2. \tag{2.13}$$

Case (ii). $\int_0^1 p\phi|y|^{\alpha_0+1}\,dt \le 1$.

In this case (2.13) is clearly true with $Q_{12} = 1$.

Consequently in all cases (2.13) is true. Returning to $(2.8)_\lambda$ we have

$$y(t) = \lambda\int_0^1 G(t,s)[f(s,y(s),p(s)y'(s)) + (\mu-\lambda_m)q(s)y(s)]\,ds \tag{2.14}$$

and

$$p(t)y'(t) = \lambda\int_0^1 p(t)G_t(t,s)[f(s,y(s),p(s)y'(s)) + (\mu-\lambda_m)q(s)y(s)]\,ds \tag{2.15}$$

where $G(t,s)$ is the Green's function associated with $\frac{1}{p}(pv')' + rv + \mu qv = 0$ a.e. on $[0,1]$, $v \in$ (SL) or (P).

Notice ([16], [17]) that $\sup_{t\in[0,1]} |p(t)G_t(t,s)| \le Q_{14}p(s)$ for some constant Q_{14}. Now (2.15) together with (2.4) and (2.5) imply for $t \in (0,1)$ that

$$|p(t)y'(t)| \le Q_{15}\int_0^1 p\phi_1\,ds + Q_{15}\int_0^1 p\phi_2|y|^{\beta_0}\,ds + Q_{15}\int_0^1 p\phi_3|py'|^{\sigma}\,ds$$
$$+ Q_{15}\int_0^1 p\phi_4\,ds + Q_{15}\int_0^1 p\phi_5|y|^{\gamma}\,ds + Q_{16}\int_0^1 pq|y|\,ds$$

for some constants Q_{15} and Q_{16}. Hölder's inequality together with (2.6) implies

$$|p(t)y'(t)| \le Q_{17} + Q_{18}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\beta_0}{\alpha_0+1}}$$
$$+ Q_{19}\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}} |py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{\alpha_0}{\alpha_0+1}}$$
$$+ Q_{20}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\gamma}{\alpha_0+1}} + Q_{21}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}}$$

for some constants $Q_{17}, \dots, Q_{21}$. Thus for $t \in (0,1)$ we have

$$
\begin{aligned}
|p(t)y'(t)|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}} &\le Q_{22} + Q_{23} \left(\int_0^1 p\phi|y|^{\alpha_0+1}\, dt \right)^{\frac{\sigma\beta_0}{\alpha_0}} \\
&\quad + Q_{24} \left(\int_0^1 p \left(\phi_3^{\alpha_0+1}\phi^{-1} \right)^{\frac{1}{\alpha_0}} |py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\, dt \right)^{\sigma} \\
&\quad + Q_{25} \left(\int_0^1 p\phi|y|^{\alpha_0+1}\, dt \right)^{\frac{\sigma\gamma}{\alpha_0}} \\
&\quad + Q_{26} \left(\int_0^1 p\phi|y|^{\alpha_0+1}\, dt \right)^{\frac{\sigma}{\alpha_0}}
\end{aligned}
\tag{2.16}
$$

for some constants $Q_{22}, \dots, Q_{26}$. This together with (2.13) implies

$$
\begin{aligned}
&\int_0^1 p \left(\phi_3^{\alpha_0+1}\phi^{-1} \right)^{\frac{1}{\alpha_0}} |py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\, dt \\
&\le Q_{27} + Q_{28} \left(\int_0^1 p \left(\phi_3^{\alpha_0+1}\phi^{-1} \right)^{\frac{1}{\alpha_0}} |py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\, dt \right)^{\frac{2\sigma\beta_0}{\alpha_0}} \\
&+ Q_{29} \left(\int_0^1 p \left(\phi_3^{\alpha_0+1}\phi^{-1} \right)^{\frac{1}{\alpha_0}} |py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\, dt \right)^{\sigma} \\
&+ Q_{30} \left(\int_0^1 p \left(\phi_3^{\alpha_0+1}\phi^{-1} \right)^{\frac{1}{\alpha_0}} |py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\, dt \right)^{\frac{2\sigma\gamma}{\alpha_0}} \\
&+ Q_{31} \left(\int_0^1 p \left(\phi_3^{\alpha_0+1}\phi^{-1} \right)^{\frac{1}{\alpha_0}} |py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\, dt \right)^{\frac{2\sigma}{\alpha_0}}
\end{aligned}
$$

for some constants $Q_{27}, \dots, Q_{31}$. Finally since $\max\{\frac{2\sigma\beta_0}{\alpha_0}, \sigma, \frac{2\sigma\gamma}{\alpha_0}, \frac{2\sigma}{\alpha_0}\} < 1$ there exists a constant Q_{32} with

$$
\int_0^1 p \left(\phi_3^{\alpha_0+1}\phi^{-1} \right)^{\frac{1}{\alpha_0}} |py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\, dt \le Q_{32} \tag{2.17}
$$

and this together with (2.13) implies that there exists a constant Q_{33} with

$$
\int_0^1 p\phi|y|^{\alpha_0+1}\, dt \le Q_{33}. \tag{2.18}
$$

Putting these inequalities into (2.16) establishes the existence of a constant Q_{34} with

$$
\sup_{t\in(0,1)} |p(t)y'(t)| \le Q_{34}. \tag{2.19}
$$

Now (2.14) together ([16], [17]) with $\sup_{t\in[0,1]}|G(t,s)| \le Q_{35}p(s)$, for some constant Q_{35}, and Hölder's inequality implies for $t \in [0,1]$ that

$$\begin{aligned}|y(t)| \le Q_{36} + Q_{37}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\beta_0}{\alpha_0+1}} \\ + Q_{38}\left(\int_0^1 p\left(\phi_3^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}}|py'|^{\frac{\sigma(\alpha_0+1)}{\alpha_0}}\,dt\right)^{\frac{\alpha_0}{\alpha_0+1}} \\ + Q_{39}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\gamma}{\alpha_0+1}} + Q_{40}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}}\end{aligned}$$

for some constants $Q_{36},\dots,Q_{40}$. This together with (2.17) and (2.18) implies that there is a constant Q_{41} with

$$\sup_{t\in[0,1]}|y(t)| \le Q_{41}. \tag{2.20}$$

Now (2.19), (2.20) together with Theorem 1.2 establish the result. □

Example. Theorem 2.1 (here $H_{\alpha_0,\theta}(u) = H_{\frac{1}{3},\frac{1}{3}}(u)$) immediately guarantees that

$$\begin{cases} y'' + n^2\pi^2\, y = y^{\frac{1}{3}} + [y']^{\frac{1}{7}} + 1 \ \text{ a.e. on } [0,1] \\ y(0) = y(1) = 0,\ n \in \{1,2,\dots\} \end{cases}$$

has a solution.

One can improve considerably the above theorem if $m = 0$ (at the first eigenvalue). In particular the condition $0 < \alpha_0 < 1$ is replaced by $\alpha_0 > 0$ in this case; also condition (2.5) can be improved and the condition $\sigma < \frac{\alpha_0}{2}$ can be relaxed. We present two existence results.

Consider

$$\begin{cases} \frac{1}{p}(py')' + ry + \lambda_0 qy = f(t,y,py') \ \text{ a.e. on } [0,1] \\ y \in \text{(SL) or (P)} \end{cases} \tag{2.21}$$

where λ_0 is the first eigenvalue of (2.2).

Theorem 2.2. *Let $pf : [0,1]\times\mathbf{R}^2 \to \mathbf{R}$ be an L^1-Carathéodory function with (1.3) and (1.9) satisfied. Suppose f has the decomposition $f(t,u_1,u_2) = g(t,u_1,u_2) + h(t,u_1,u_2)$ with $pg, ph : [0,1]\times\mathbf{R}^2 \to \mathbf{R}$ L^1-Carathéodory functions and*

$$\begin{cases} \text{there exist constants } A > 0, \alpha_0 > 0 \text{ and a function } \phi \in L^1_p[0,1], \\ \phi > 0 \text{ a.e. on } [0,1] \text{ with } u_1 g(t,u_1,u_2) \ge A\phi(t)H_{\alpha_0,\theta}(u_1) \\ \text{for a.e. } t\in[0,1];\ \text{here } \alpha_0 \le \theta \end{cases} \tag{2.22}$$

$$(2.23)\quad \begin{cases} \textit{there exist } \phi_i \in L^1_p[0,1],\ i=1,2,3 \textit{ and constants } \beta_0 \textit{ and} \\ \sigma \textit{ with } |h(t,u_1,u_2)| \le \phi_1(t)+\phi_2(t)|u_1|^{\beta_0}+\phi_3(t)|u_2|^{\sigma} \textit{ for} \\ \textit{a.e. } t\in[0,1];\ \textit{here } \beta_0<\alpha_0 \textit{ and } \phi_3>0 \\ \textit{a.e. on } [0,1] \textit{ or } \phi_3\equiv 0 \textit{ on } [0,1] \end{cases}$$

$$(2.24)\quad \begin{cases} \textit{there exist } \phi_i \in L^1_p[0,1],\ i=4,5,6 \textit{ and constants } \gamma\le\alpha_0,\ \tau>\sigma \\ \textit{with } |g(t,u_1,u_2)| \le \phi_4(t)+\phi_5(t)|u_1|^{\gamma}+\phi_6(t)|u_2|^{\tau} \\ \textit{for a.e. } t\in[0,1]; \\ \textit{here } \phi_6>0 \textit{ a.e. on } [0,1] \textit{ or } \phi_6\equiv 0 \textit{ on } [0,1] \end{cases}$$

$$(2.25)\quad \sigma<\min\{1,\frac{\alpha_0}{\gamma},\alpha_0\} \textit{ and } \ \tau<1$$

$$(2.26)\quad \begin{cases} \left(\phi_1^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}} \in L^1_p[0,1],\ \left(\phi_2^{\alpha_0+1}\phi^{-(\beta_0+1)}\right)^{\frac{1}{\alpha_0-\beta_0}} \in L^1_p[0,1], \\ \left(\phi_5^{\alpha_0+1}\phi^{-\gamma}\right)^{\frac{1}{\alpha_0+1-\gamma}} \in L^1_p[0,1] \textit{ and } \ \left(q^{\alpha_0+1}\phi^{-1}\right)^{\frac{1}{\alpha_0}} \in L^1_p[0,1] \end{cases}$$

and

$$(2.27)\quad \begin{cases} \textit{with } \kappa=\max\{\frac{\alpha_0+1}{\alpha_0},2\},\ \left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^{\kappa} \in L^1_p[0,1].\ \textit{Also need} \\ \phi_6^{\kappa}\in L^1_p[0,1] \textit{ and } \left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^{\frac{\kappa\tau}{\tau-\sigma}} (\phi_6)^{-\kappa\left(\frac{\sigma}{\tau-\sigma}\right)} \in L^1_p[0,1] \\ \textit{if } \phi_6>0 \textit{ a.e. on } [0,1] \end{cases}$$

holding. Then (2.21) *has at least one solution* $y\in C[0,1]\cap C^1(0,1)$ *with* $py'\in AC[0,1]$.

Proof: Let y be a solution of $(2.8)_\lambda$ with $m=0$. Following the ideas of Theorem 2.1 with $u=0$ and $y=w$ we obtain the analogue of (2.11), namely

$$(2.28)\quad \begin{aligned} A\int_0^1 p\phi|y|^{\alpha_0+1}\,dt \le A\int_0^1 p\phi\,dt+\int_0^1 p\phi_1|y|\,dt+\int_0^1 p\phi_2|y|^{\beta_0+1}\,dt \\ +\int_0^1 p\phi_3|y||py'|^{\sigma}\,dt. \end{aligned}$$

Hölder's inequality implies

$$(2.29)\quad \begin{aligned} A\int_0^1 p\phi|y|^{\alpha_0+1}\,dt \le N_0+N_1\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}} \\ +N_2\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\beta_0+1}{\alpha_0+1}}+\int_0^1 p\phi_3|y||py'|^{\sigma}\,dt \end{aligned}$$

for some constants N_0, N_1 and N_2. Let $\kappa = \max\{2, \frac{\alpha_0+1}{\alpha_0}\}$. Hölder's inequality together with assumption (2.27) implies

$$\int_0^1 p\phi_3|y||py'|^\sigma\,dt \le \left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}} \left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^\kappa |py'|^{\sigma\kappa}\,dt\right)^{\frac{1}{\kappa}}$$

if $\kappa = \frac{\alpha_0+1}{\alpha_0}$ whereas

$$\int_0^1 p\phi_3|y||py'|^\sigma\,dt \le \left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}} \times \left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^\kappa |py'|^{\sigma\kappa}\,dt\right)^{\frac{1}{\kappa}} \left(\int_0^1 p(t)\,dt\right)^{\frac{\alpha_0-1}{2(\alpha_0+1)}}$$

if $\kappa = 2$.Put this into (2.29) and essentially the same reasoning as in Theorem 2.1 establishes the existence of constants N_3 and N_4 with

$$\int_0^1 p\phi|y|^{\alpha_0+1}\,dt \le N_3 + N_4\left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^\kappa |py'|^{\sigma\kappa}\,dt\right)^{\frac{\alpha_0+1}{\kappa\alpha_0}}. \tag{2.30}$$

Also (2.15) implies (as in Theorem 2.1) for $t \in (0,1)$ that

$$\begin{aligned} |p(t)y'(t)| \le{}& N_5 + N_6\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\beta_0}{\alpha_0+1}} + N_7\int_0^1 p\phi_3|py'|^\sigma\,dt \\ &+ N_8\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\gamma}{\alpha_0+1}} + N_9\int_0^1 p\phi_6|py'|^\tau\,dt \\ &+ N_{10}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}} \end{aligned} \tag{2.31}$$

for some constants $N_5, \ldots, N_{10}$. Again with $\kappa = \max\{2, \frac{\alpha_0+1}{\alpha_0}\}$ we have

$$\int_0^1 p\phi_3|py'|^\sigma\,dt \le \left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^\kappa |py'|^{\sigma\kappa}\,dt\right)^{\frac{1}{\kappa}} \left(\int_0^1 p\phi\,dt\right)^{\frac{1}{\alpha_0+1}} \quad \text{if } \kappa = \frac{\alpha_0+1}{\alpha_0}$$

$$\int_0^1 p\phi_3|py'|^\sigma\,dt \le \left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^\kappa |py'|^{\sigma\kappa}\,dt\right)^{\frac{1}{\kappa}} \left(\int_0^1 p\phi^{\frac{2}{\alpha_0+1}}\,dt\right)^{\frac{1}{2}} \quad \text{if } \kappa = 2$$

$$\int_0^1 p\phi_6|py'|^\tau\,dt \le \left(\int_0^1 p\phi_6^\kappa|py'|^{\tau\kappa}\,dt\right)^{\frac{1}{\kappa}} \left(\int_0^1 p(t)\,dt\right)^{1-\frac{1}{\kappa}}.$$

There are two cases to consider, namely $\phi_6 > 0$ a.e. on $[0,1]$ or $\phi_6 \equiv 0$ on $[0,1]$.
Case (i). $\phi_6 > 0$ a.e. on $[0,1]$.

Putting the above into (2.31) and using (2.30) leads to

$$\begin{aligned}\int_0^1 p\phi_6^\kappa |py'|^{\tau\kappa}\,dt \le N_{11} &+ N_{12}\left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^\kappa |py'|^{\sigma\kappa}\,dt\right)^{\frac{\beta_0\tau}{\alpha_0}}\\ &+ N_{13}\left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^\kappa |py'|^{\sigma\kappa}\,dt\right)^{\tau}\\ &+ N_{14}\left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^\kappa |py'|^{\sigma\kappa}\,dt\right)^{\frac{\gamma\tau}{\alpha_0}}\\ &+ N_{15}\left(\int_0^1 p\phi_6^\kappa |py'|^{\tau\kappa}\,dt\right)^{\tau}\\ &+ N_{16}\left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^\kappa |py'|^{\sigma\kappa}\,dt\right)^{\frac{\tau}{\alpha_0}}\end{aligned}\tag{2.32}$$

for some constants $N_{11},\ldots,N_{16}$. Also Hölder's inequality implies

$$\begin{aligned}&\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^\kappa |py'|^{\sigma\kappa}\,dt\\ &\quad\le \left(\int_0^1 p\phi_6^\kappa |py'|^{\tau\kappa}\,dt\right)^{\frac{\sigma}{\tau}}\left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^{\frac{\kappa\tau}{\tau-\sigma}}(\phi_6)^{-\kappa\left(\frac{\sigma}{\tau-\sigma}\right)}\,dt\right)^{\frac{\tau-\sigma}{\tau}}\end{aligned}$$

and putting this into (2.32) yields

$$\begin{aligned}\int_0^1 p\phi_6^\kappa |py'|^{\tau\kappa}\,dt \le N_{17} &+ N_{18}\left(\int_0^1 p\phi_6^\kappa |py'|^{\tau\kappa}\,dt\right)^{\frac{\beta_0\sigma}{\alpha_0}}\\ &+ N_{19}\left(\int_0^1 p\phi_6^\kappa |py'|^{\tau\kappa}\,dt\right)^{\sigma} + N_{20}\left(\int_0^1 p\phi_6^\kappa |py'|^{\tau\kappa}\,dt\right)^{\frac{\gamma\sigma}{\alpha_0}}\\ &+ N_{21}\left(\int_0^1 p\phi_6^\kappa |py'|^{\tau\kappa}\,dt\right)^{\tau} + N_{22}\left(\int_0^1 p\phi_6^\kappa |py'|^{\tau\kappa}\,dt\right)^{\frac{\sigma}{\alpha_0}}\end{aligned}$$

for some constants $N_{17},\ldots,N_{22}$. Now since $\max\{\frac{\sigma\beta_0}{\alpha_0},\sigma,\frac{\sigma\gamma}{\alpha_0},\tau,\frac{\sigma}{\alpha_0}\} < 1$ then there exists a constant N_{23} with

$$\int_0^1 p\phi_6^\kappa |py'|^{\tau\kappa}\,dt \le N_{23}$$

and essentially the same reasoning as in Theorem 2.1 establishes the result.

Case (ii). $\phi_6 \equiv 0$ on $[0,1]$.

We may assume without loss of generality that $\sigma > 0$ and $\phi_3 > 0$ a.e. on $[0,1]$; otherwise the result is easy. Then (2.31) for $t \in (0,1)$ becomes

$$|p(t)y'(t)| \le N_{23} + N_{24}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\beta_0}{\alpha_0+1}} + N_{25}\left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^{\kappa}|py'|^{\sigma\kappa}\,dt\right)^{\frac{1}{\kappa}} + N_{26}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{\gamma}{\alpha_0+1}} + N_{27}\left(\int_0^1 p\phi|y|^{\alpha_0+1}\,dt\right)^{\frac{1}{\alpha_0+1}}$$

for some constants $N_{23},\ldots,N_{27}$. This together with (2.30) leads to

$$\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^{\kappa}|py'|^{\sigma\kappa}\,dt \le N_{28} + N_{29}\left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^{\kappa}|py'|^{\sigma\kappa}\,dt\right)^{\frac{\beta_0\sigma}{\alpha_0}} + N_{30}\left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^{\kappa}|py'|^{\sigma\kappa}\,dt\right)^{\sigma} + N_{31}\left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^{\kappa}|py'|^{\sigma\kappa}\,dt\right)^{\frac{\gamma\sigma}{\alpha_0}} + N_{32}\left(\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^{\kappa}|py'|^{\sigma\kappa}\,dt\right)^{\frac{\sigma}{\alpha_0}}$$

for some constants $N_{28},\ldots,N_{32}$. Thus there exists a constant N_{33} with

$$\int_0^1 p\left(\phi_3\phi^{-\frac{1}{\alpha_0+1}}\right)^{\kappa}|py'|^{\sigma\kappa}\,dt \le N_{33}$$

and the result follows as in Theorem 2.1. □

The next theorem establishes the existence of a nonnegative solution to

$$\left\{\begin{array}{l}\frac{1}{p}(py')' + \lambda_0 qy = \psi(t)f(t,y,py'),\ 0<t<1\\ y \in (\mathrm{SL}) \text{ or } (\mathrm{P})\end{array}\right. \tag{2.33}$$

where λ_0 is the first eigenvalue of (2.2) with $r \equiv 0$ and q, ψ satisfies

$$q,\ \psi \in L^1_p[0,1] \text{ with } q,\ \psi > 0 \text{ on } (0,1). \tag{2.34}$$

Let

$$H^*_{\alpha_0,\theta}(u_1) = \left\{\begin{array}{l} u_1^{\theta+1},\ 0 \le u_1 \le 1\\ u_1^{\alpha_0+1},\ 1 < u_1 < \infty.\end{array}\right.$$

Theorem 2.3. *Let $f : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ be continuous with* (1.3), (2.34) *and*

$$f(t,0,0) \le 0 \tag{2.35}$$

holding. Suppose ψf has the decomposition $\psi(t)f(t,u_1,u_2) = g(t,u_1,u_2) + h(t,u_1,u_2)$ with $pg, ph : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ L^1-Carathéodory functions and

$$(2.36) \qquad \begin{cases} \text{there exist constants } A > 0, \alpha_0 > 0 \text{ and a function } \phi \in L^1_p[0,1], \\ \phi > 0 \text{ on } (0,1) \text{ with } u_1 g(t,u_1,u_2) \ge A\phi(t) H^*_{\alpha_0,\theta}(u_1) \\ \text{for } t \in (0,1), u_1 \ge 0 \text{ and } u_2 \in \mathbf{R};\ \text{here } \alpha_0 \le \theta \end{cases}$$

$$(2.37) \qquad \begin{cases} \text{there exist } \phi_i \in L^1_p[0,1],\ i = 1,2,3 \text{ and constants } \beta_0 \text{ and} \\ \sigma \text{ with } |h(t,u_1,u_2)| \le \phi_1(t) + \phi_2(t) u_1^{\beta_0} + \phi_3(t)|u_2|^\sigma \text{ for} \\ t \in (0,1), u_1 \ge 0 \text{ and } u_2 \in \mathbf{R};\ \text{here } \beta_0 < \alpha_0 \\ \text{and } \phi_3 > 0 \text{ on } (0,1) \text{ or } \phi_3 \equiv 0 \end{cases}$$

and

$$(2.38) \qquad \begin{cases} \text{there exist } \phi_i \in L^1_p[0,1],\ i = 4,5,6 \text{ and constants } \gamma \le \alpha_0,\ \tau > \sigma \\ \text{with } |g(t,u_1,u_2)| \le \phi_4(t) + \phi_5(t) u_1^{\gamma} + \phi_6(t)|u_2|^\tau \text{ for } t \in (0,1), u_1 \ge 0 \\ \text{and } u_2 \in \mathbf{R} \text{ and } \phi_6 > 0 \text{ on } (0,1) \text{ or } \phi_6 \equiv 0 \end{cases}$$

hold. Finally suppose (2.25) *and* (2.26) *are satisfied. Then* (2.33) *has at least one nonnegative solution $y \in C[0,1] \cap C^1(0,1)$ with $py' \in AC[0,1]$.*

Proof: Consider the family of problems

$$(2.39)_\lambda \qquad \begin{cases} \frac{1}{p}(py')' + \mu q y = \lambda f^*(t,y,py'),\ 0 < t < 1 \\ y \in (\mathrm{SL}) \text{ or } (\mathrm{P}) \end{cases}$$

where $0 < \lambda < 1$ and

$$\mu = \begin{cases} 0 & \text{if } y \in (SL) \text{ and } \alpha^2 + a^2 > 0 \\ -1 & \text{if } y \in (\mathrm{P}) \text{ or } y \in (\mathrm{SL}) \text{ with } \alpha = a = 0. \end{cases}$$

Also

$$f^*(t,u_1,u_2) = \begin{cases} \psi(t)f(t,u_1,u_2) + (\mu - \lambda_0) q u_1,\ u_1 \ge 0 \\ \psi(t)f(t,0,u_2) + (\mu+1) q u_1,\ u_1 < 0. \end{cases}$$

Notice $pf^* : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is an L^1-Carathéodory function.

Let y be a solution to $(2.39)_\lambda$ for some $0 < \lambda < 1$. We **claim** that $y \ge 0$ on $[0,1]$. If not then y would have a negative absolute minimum somewhere on $[0,1]$, say at t_0. If $t_0 \in (0,1)$ then $y'(t_0) = 0$ and this together with the differential equation and (2.35) yields

$$\begin{aligned} y''(t_0) &= \frac{1}{p(t_0)}(p(t_0)y'(t_0))' \\ &= \lambda\left(\psi(t_0)f(t_0,0,0) + q(t_0)y(t_0)\right) + (\lambda-1)\mu q(t_0)y(t_0) < 0, \end{aligned}$$

a contradiction. Next suppose the negative absolute minimum were to occur at $t_0 = 0$. Consider first the Sturm Liouville boundary condition. Of course we need only consider $\beta \neq 0$. If $\alpha \neq 0$ as well then

$$y(0)\lim_{t\to 0^+} p(t)y'(t) = \frac{\alpha}{\beta}y^2(0) > 0,$$

which implies $y^2(t)$ is an increasing function near 0, a contradiction. So it remains to consider the case $\alpha = 0$ and $\beta \neq 0$. The boundary condition is $\lim_{t\to 0^+} p(t)y'(t) = 0$. Now $f(0,0,0) \leq 0$ and this together with the differential equation and (2.34) implies there exists $\delta > 0$ with $(p(t)y'(t))' < 0$ for $t \in (0,\delta)$. Thus the boundary condition implies $p(t)y'(t) < 0$ for $t \in (0,\delta)$, a contradiction. Consequently $t_0 \neq 0$. A similar argument shows $t_0 \neq 1$. Thus our claim is established for Sturm Liouville boundary data.

Consider now Periodic boundary data. If the absolute minimum of y occurs at $t_0 = 0$ then, since $y(0) = y(1)$, it must also occur at 1. Thus $\lim_{t\to 0^+} p(t)y'(t) \geq 0$ and $\lim_{t\to 1^-} p(t)y'(t) \leq 0$. Consequently

$$\lim_{t\to 0^+} p(t)y'(t) = \lim_{t\to 1^-} p(t)y'(t) = 0$$

because of the second boundary condition. As above there exists $\delta > 0$ with $(p(t)y'(t))' < 0$ for $t \in (0,\delta)$ and so $p(t)y'(t) < 0$ for $t \in (0,\delta)$, a contradiction.

Thus $y \geq 0$ on $[0,1]$ for any solution y to $(2.39)_\lambda$. Consequently y satisfies

$$\frac{1}{p}(py')' + \mu qy = \lambda\left(\psi(t)f(t,y,py') + (\mu-\lambda_0)qy\right),\ 0 < t < 1.$$

Essentially the same reasoning as in Theorem 2.2 (in this case we look at $\int_0^1 p\phi y^{\alpha_0+1}\,dt$) guarantees the existence of a solution y to $(2.39)_1$. Of course y is automatically a solution of (2.33) since $y \geq 0$ on $[0,1]$. $\square$

Existence theory II.

In this subsection we examine the resonant problem (2.1) on the "right" of the eigenvalue.

Theorem 2.4. *Let $pf : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ be an L^1-Carathéodory function with* (1.3) *and* (1.9) *holding. Suppose f has the decomposition $f(t,u_1,u_2) =$*

$g(t,u_1,u_2)+h(t,u_1,u_2)$ *with* $pg, ph : [0,1]\times \mathbf{R}^2 \to \mathbf{R}$ L^1*-Carathéodory functions and assume*

$$(2.40)\qquad \begin{cases} \text{there exist constants } A>0,\, 0<\alpha_0<1 \text{ and a function} \\ \phi\in L^1_p[0,1],\ \phi>0 \text{ a.e. on } [0,1] \text{ with} \\ u_1 g(t,u_1,u_2)\le -A\phi(t)H_{\alpha_0,\theta}(u_1) \text{ for a.e. } t\in[0,1];\ \text{here } \alpha_0\le\theta \end{cases}$$

holds. In addition assume (2.4), (2.5), (2.6) *and* (2.7) *are satisfied. Then* (2.1) *has at least one solution* $y\in C[0,1]\cap C^1(0,1)$ *with* $py'\in AC[0,1]$.

Proof: Consider the family of problems

$$(2.41)_\lambda\qquad \begin{cases} \frac{1}{p}(py')'+ry+\mu qy=\lambda[f(t,y,py')+(\mu-\lambda_m)qy] \ \text{ a.e. on } [0,1] \\ y\in(\mathrm{SL}) \text{ or } (\mathrm{P}) \end{cases}$$

where $0<\lambda<1$ and $\lambda_m<\mu<\lambda_{m+1}$.

Notice $L^2_{pq}[0,1]=\Gamma\bigoplus\Gamma^\perp$ where $\Gamma = span\,\{\psi_0,\psi_1,\dots,\psi_m\}$. Multiply $(2.41)_\lambda$ by $w-u$ and integrate from 0 to 1 to obtain as in Theorem 2.1 (Q_0 is as in Theorem 2.1)

$$(2.42)\qquad \begin{aligned} Q_0 &+ \int_0^1[-p(w')^2+prw^2+\mu pqw^2]\,dt+\int_0^1[p(u')^2-pru^2-\mu pqu^2]\,dt \\ &=\lambda\int_0^1(w-u)pf(t,y,py')\,dt+\lambda(\mu-\lambda_m)\int_0^1 pqw^2\,dt \\ &-\lambda(\mu-\lambda_m)\int_0^1 pqu^2\,dt. \end{aligned}$$

Now since $u\in\Gamma$, $w\in\Gamma^\perp$ and $y=u+w$ we have

$$u=\sum_{i=0}^m c_i\psi_i \quad\text{and}\quad w=\sum_{i=m+1}^\infty c_i\psi_i \quad\text{where}\quad c_i=\langle y,\psi_i\rangle.$$

Also as before

$$\begin{aligned} Q_0 &+ \int_0^1[-p(w')^2+prw^2+\mu pqw^2]\,dt+\int_0^1[p(u')^2-pru^2-\mu pqu^2]\,dt \\ &\le(\mu-\lambda_{m+1})\int_0^1 pqw^2\,dt+(\lambda_m-\mu)\int_0^1 pqu^2\,dt \end{aligned}$$

so putting this into (2.42) yields

$$\begin{aligned} \lambda\int_0^1(w-u)pg(t,y,py')\,dt &+(1-\lambda)(\mu-\lambda_m)\int_0^1 pqu^2\,dt \\ &+(\lambda_{m+1}-\mu)\int_0^1 pqw^2\,dt+\lambda(\mu-\lambda_m)\int_0^1 pqw^2\,dt \\ &\le-\lambda\int_0^1(w-u)ph(t,y,py')\,dt. \end{aligned}$$

Now $w-u=-y+2w$ and $-yg(t,y,py') \geq A\phi(t)H_{\alpha_0,\theta}(y)$ for a.e. $t \in [0,1]$ so with the above we have

$$\begin{aligned} A\int_0^1 p\phi H_{\alpha_0,\theta}(y)\,dt + (\mu-\lambda_m)\int_0^1 pqw^2\,dt \leq -2\int_0^1 pwg(t,y,py')\,dt \\ + \int_0^1 p|y||h(t,y,py')|\,dt \\ + 2\int_0^1 p|w||h(t,y,py')|\,dt. \end{aligned}$$

Essentially the same reasoning as in Theorem 2.1 (the only difference is that we use $\int_0^1 pqw^2\,dt$ in place of $\int_0^1 pqu^2\,dt$) establishes the result. □

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