# Applications of the spectral radius to some integral equations

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*Abstract.* In the paper [13] we proved a fixed point theorem for an operator $\mathcal{A}$, which satisfies a generalized Lipschitz condition with respect to a linear bounded operator $A$, that is:

$$m(\mathcal{A}x - \mathcal{A}y) \prec Am(x-y).$$

The purpose of this paper is to show that the results obtained in [13], [14] can be extended to a nonlinear operator $A$.

*Keywords:* fixed point theorem, spectral radius, integral-functional equation

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## 1. Fixed point theorem

Let $X$ be a Banach space. An operator $A : X \to X$ is said to be linearly bounded if (analogously to a linear operator)

$$\exists_{M>0}\ \forall_{x\in X}\ \|Ax\| \le M\|x\|.$$

This definition implies that $A$ vanishes at zero. The number

$$\|A\| = \inf\{M > 0 : \|Ax\| \le M\|x\|,\ x \in X\}$$

we call the norm of $A$. Since, as in the case of linear operator,

$$\|A^{n+m}\| \le \|A^n\|\ \|A^m\|,$$

there exists the limit

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}. \tag{1}$$

We call $r(A)$ the generalized spectral radius of $A$. If we assume additionally that $A$ is a positively homogeneous operator then the following formula holds:

$$\|A\| = \sup_{\|x\|=1} \|Ax\|. \tag{2}$$

Let $(X, \|\cdot\|, \prec, m)$ denote a Banach space of elements $x \in X$, with a binary relation $\prec$ and a mapping $m : X \to X$. We shall assume that:

1° the relation $\prec$ is transitive,

2° $\theta \prec m(x)$ and $\|m(x)\| = \|x\|$ for all $x \in X$,

3° the norm $\|\cdot\|$ is monotonic, that is, if $\theta \prec x \prec y$ then $\|x\| \le \|y\|$.

Now we can formulate a variant of Banach's contraction principle.

**Theorem 1.** *In the Banach space considered above, let the operators $\mathcal{A} : X \to X$, $A : X \to X$ be given with the following properties:*

$4^\circ$ *$A$ is linearly bounded and $r(A) < 1$,*

$5^\circ$ *$A$ is positively increasing, that is, if $\theta \prec x \prec y$ then $Ax \prec Ay$,*

$6^\circ$ *$m(\mathcal{A}x - \mathcal{A}y) \prec Am(x - y)$ for all $x, y \in X$.*

*Then the equation*

$$\mathcal{A}x = x$$

*has a unique solution in the set $X$.*

The proof of Theorem 1 is analogous to that of Theorem 1 [13], so it can be omitted. Similar theorems can be found in [5], [8], [9], [11].

## 2. An integral-functional equation

In this section we shall show an application of Theorem 1 to an integral-functional equation. Consider the equation

$$x(t) = \int_0^t f\Big(s, \max_{[0,\sqrt{s}]}\{x(\tau)\}\Big)\, ds, \; t \in [0, T], \; T \geq 1. \tag{3}$$

We show that under suitable assumptions the equation (3) has exactly one solution in the set of continuous functions on the interval $[0, T]$.

**Remark.** *The equation* (3) *can be considered with connection to the Cauchy problem*

$$x'(t) = f\Big(t, \max_{[0,\sqrt{t}]}\{x(\tau)\}\Big), \; t \in [0, T], \; T \geq 1,$$
$$x(0) = 0.$$

*Differential equations with maxima or suprema were studied for example in the papers* [3], [6] *and in the monograph* [1].

**Theorem 2.** *Suppose that*

$7^\circ$ *$f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies the Lipschitz condition*

$$|f(t, x) - f(t, y)| \leq L(t)|x - y|,$$

*where $L$ is continuous and non-negative function on the interval $[0, T]$,*

$8^\circ$ $\max_{[0,T]} L(t) < 2$.

*Under the assumptions $7^\circ$–$8^\circ$ the equation* (3) *has a unique solution in the set of continuous functions on the interval* $[0, T]$.

Proof: We set the Banach space $(X, \|\cdot\|, \prec, m)$ from Theorem 1 as follows: let $X$ be a set of continuous functions on $[0, T]$, $\|x\| = \max_{[0,T]} |x(t)|$ and $(m(x))(t) =$

$|x(t)|$ for $t \in [0,T]$. Moreover, we say that $x \prec y$ if and only if $x(t) \le y(t)$ for all $t \in [0,T]$. Obviously, the conditions 1°–3° are satisfied in this case. Consider the operator

$$(\mathcal{A}x)(t) = \int_0^t f\Big(s, \max_{[0,\sqrt{s}]}\{x(\tau)\}\Big)\,ds,\ t \in [0,T],\ T \ge 1. \tag{4}$$

To prove Theorem 2 we shall show that $\mathcal{A}$ has a unique fixed point in $X$. From 7° it follows that

$$\begin{aligned} |(\mathcal{A}x)(t) - (\mathcal{A}y)(t)| &\le \int_0^t L(s)|\max_{[0,\sqrt{s}]}\{x(\tau)\} - \max_{[0,\sqrt{s}]}\{y(\tau)\}|\,ds \\ &\le \int_0^t L \max_{[0,\sqrt{s}]} |x(\tau) - y(\tau)|\,ds, \end{aligned} \tag{5}$$

where $L = \max_{[0,T]} |L(t)|$. Let

$$(Ax)(t) = \int_0^t L \max_{[0,\sqrt{s}]} |x(\tau)|\,ds,\ t \in [0,T]. \tag{6}$$

The operator (6) maps $X$ into $X$ and it is linearly bounded. Moreover, in view of (5), the condition 6° of Theorem 1 is fulfilled. It remains to show that the spectral radius of the operator (6) is less than 1. Observe that

$$\begin{aligned} (A^2x)(t) &= \int_0^t L \max_{[0,\sqrt{s}]} \Big| \int_0^\tau L \max_{[0,\sqrt{s_1}]} |x(\tau_1)|\,ds_1 \Big|\,ds \\ &= L^2 \int_0^t \int_0^{\sqrt{s}} \max_{[0,\sqrt{s_1}]} |x(\tau_1)|\,ds_1\,ds. \end{aligned}$$

Continuing this process, we get

$$(A^n x)(t) = L^n \int_0^t \int_0^{\sqrt{s_1}} \cdots \int_0^{\sqrt{s_{n-1}}} \max_{[0,\sqrt{s_n}]} |x(\tau)|\,ds_n\,ds_{n-1}\ldots ds_1.$$

Thus

$$\|A^n x\| \le L^n \frac{2}{3}\cdot\frac{4}{7}\cdot\ldots\cdot\frac{2^{n-1}}{2^n-1} T^{\frac{2^n-1}{2^{n-1}}} \|x\|$$

and

$$\|A^n\|^{1/n} \le L\Big(\frac{2}{3}\cdot\frac{4}{7}\cdot\ldots\cdot\frac{2^{n-1}}{2^n-1} T^{\frac{2^n-1}{2^{n-1}}}\Big)^{1/n}.$$

Therefore $r(A) \le \frac{L}{2}$. By the assumption 8°, $r(A) < 1$. Hence, in virtue of Theorem 1, the operator (4) has a unique fixed point in $X$. This completes the proof of Theorem 2. □

## 3. A method of evaluation of the generalized spectral radius

Evaluation of the spectral radius of a linearly bounded operator by definition (1) is not easy. It is known that if $A$ is a linear bounded operator then we can use the formula

$$r(A) = \lim_{n\to\infty} \|A^n x_0\|^{1/n}, \tag{7}$$

where $x_0$ is a suitably chosen element of a Banach space (see [2], [4]). We shall show that (7) holds also for some nonlinear operators.

Let $S(X)$ denote a class of linearly bounded operators $A : X \to X$ satisfying the following implication

$$\Big(\limsup_{n\to\infty} \|A^n x\|^{1/n} \le a\Big) \Longrightarrow (r(A) \le a),\ x \in X. \tag{8}$$

Particularly, the linear bounded operators belong to $S(X)$ (see [10]). It is easy to show that the linearly bounded and positively homogeneous operators for which there exists $\overline{x} \in X$, $\|\overline{x}\| = 1$ such that for $n \in \mathbb{N}$ $\|A^n\| = \|A^n\overline{x}\|$, belong to $S(X)$, too. Indeed, if $A$ is linearly bounded and positively homogeneous then (2) holds. Suppose, on the contrary, that $\limsup_{n\to\infty} \|A^n x\|^{1/n} \le a$ and $r(A) > a$, that is, there exists $\delta > 0$ such that $r(A) \ge a + \delta$. Then there exists $N_1 \in \mathbb{N}$ such that for $n > N_1$

$$\|A^n\| \ge \Big(a + \frac{\delta}{2}\Big)^n.$$

On the other hand, it follows from $\limsup_{n\to\infty} \|A^n x\|^{1/n} \le a$ that for $\overline{x}$ there exists $N_2 \in \mathbb{N}$ such that for $n > N_2$

$$\|A^n\overline{x}\| \le \Big(a + \frac{\delta}{4}\Big)^n.$$

Put $n_0 = \max(N_1, N_2) + 1$. Then

$$\|A^{n_0}\| = \sup_{\|x\|=1} \|A^{n_0}x\| = \|A^{n_0}\overline{x}\| \ge \Big(a + \frac{\delta}{2}\Big)^{n_0}. \tag{9}$$

and

$$\|A^{n_0}\overline{x}\| \le \Big(a + \frac{\delta}{4}\Big)^{n_0},$$

contrary to (9).

Let $K$ be a solid and normal cone in a Banach space $X$. For $x_0 \in \operatorname{int} K$ we define $\|\cdot\|_{x_0}$-norm of an element $x \in X$ as follows (see [4], [12])

$$\|x\|_{x_0} = \inf\{t > 0 : -tx_0 \prec_K x \prec_K tx_0\}, \tag{10}$$

where the relation $\prec_K$ is generated by $K$.

**Lemma.** *Suppose that the operator $A : X \to X$ belongs to $S(X)$. Suppose further that $A$ is positive, subadditive, positively increasing (with respect to the relation $\prec_K$) and positively homogeneous. Then $r(A) \le \|Ax_0\|_{x_0}$.*

PROOF: In view of (10) we get

$$Ax_0 \prec_K \|Ax_0\|_{x_0} x_0.$$

Let $x \in K$. Then $Ax \in K$ and, by (10),

$$Ax \prec_K \|Ax\|_{x_0} x_0.$$

Put $u(x) = \|Ax\|_{x_0}$. Since $A$ is positively increasing and positively homogeneous, we get for $x \in K$ and $n \in \mathbb{N}$:

$$A^n x \prec_K u(x) A^{n-1} x_0 \prec_K u(x) \|Ax_0\|_{x_0}^{n-1} x_0. \tag{11}$$

The cone $K$ is normal, so there exists $M > 0$ such that

$$\|A^n x\| \le M u(x) \|Ax_0\|_{x_0}^{n-1} \|x_0\|.$$

Moreover, $K$ is generating (since $\operatorname{int} K \ne \emptyset$). Therefore for every $x \in X$ there exist $x_1, x_2 \in K$ such that $x = x_1 - x_2$. Thus, by positive homogeneity and subadditivity of $A$ we have

$$\|A^n x\| \le \|A^n x_1\| + \|A^n x_2\| \le 2 \max\{\|A^n x_1\|, \|A^n x_2\|\}.$$

Hence

$$\|A^n x\|^{1/n} \le \big(2 \max\{\|A^n x_1\|, \|A^n x_2\|\}\big)^{1/n}.$$

But, in view of (11), for $x_1, x_2 \in K$ there exist the constants $u(x_1)$, $u(x_2)$ such that

$$\|A^n x_1\| \le M u(x_1) \|Ax_0\|_{x_0}^{n-1} \|x_0\|$$

and

$$\|A^n x_2\| \le M u(x_2) \|Ax_0\|_{x_0}^{n-1} \|x_0\|.$$

Thus

$$\|A^n x\|^{1/n} \le \big(2 \max\{M u(x_1) \|Ax_0\|_{x_0}^{n-1} \|x_0\|, M u(x_2) \|Ax_0\|_{x_0}^{n-1} \|x_0\|\}\big)^{1/n}$$

and consequently

$$\limsup_{n \to \infty} \|A^n x\|^{1/n} \le \|Ax_0\|_{x_0}. \tag{12}$$

Since the operator $A$ belongs to $S(X)$, we conclude from (12) that $r(A) \le \|Ax_0\|_{x_0}$, which ends the proof of the lemma. □

**Theorem 3.** *Let $K$ be a normal and solid cone in a Banach space $X$ and let $x_0 \in \operatorname{int} K$. If the assumptions of the lemma are satisfied then* (7) *holds.*

Proof: It is easily seen that

$$A^n x_0 \prec_K \|A^n x_0\|_{x_0} x_0.$$

Hence, in virtue of the lemma, we get

$$r(A^n) \le \|A^n x_0\|_{x_0},$$

but

$$r(A^n) = [r(A)]^n.$$

Thus

$$r(A) \le \liminf_{n\to\infty} \|A^n x_0\|_{x_0}^{1/n}. \tag{13}$$

On the other hand, since the norms $\|\cdot\|$, $\|\cdot\|_{x_0}$ are equivalent (see for example [12]), there exists a constant $m > 0$ such that

$$\|A^n x_0\|_{x_0} \le m\|A^n x_0\| \le m\|A^n\|\,\|x_0\|.$$

Hence

$$\limsup_{n\to\infty} \|A^n x_0\|_{x_0}^{1/n} \le r(A). \tag{14}$$

Combining (13) with (14) we obtain

$$r(A) = \lim_{n\to\infty} \|A^n x_0\|_{x_0}^{1/n}.$$

Finally, we apply equivalence of the norms $\|\cdot\|$, $\|\cdot\|_{x_0}$ again, which gives (7). This ends the proof of Theorem 3. □

**Remark.** *The proof of Theorem* 3 *is similar to that of Theorem* 9.1 [4].

## 4. The generalized spectral radius of the sum of two operators

In applications of Theorem 1 it may occur that the operator $A$ has the form $A = A_1 + A_2$. It is known that if $A_1$ and $A_2$ are linear, bounded and commutative then ([4], [7])

$$r(A_1 + A_2) \le r(A_1) + r(A_2). \tag{15}$$

In this section we give a sufficient condition for linearly bounded operators, different from the global commutativity, under which the inequality (15) holds.

Consider a Banach space $(X, \|\cdot\|, \prec)$ assuming that the conditions 1° and 3° are satisfied and moreover:

9° the relation $\prec$ is reflexive,
10° if $x \prec y$ then $x + z \prec y + z$.

**Theorem 4.** *In the Banach space considered above, let the linearly bounded operators $A_1 : X \to X$, $A_2 : X \to X$ be given. Suppose that if $\theta \prec x$ then $\theta \prec A_1 x$ and $\theta \prec A_2 x$. Moreover, we assume that there exists an element $x_0 \in X$, $\theta \prec x_0$ such that:*

$11^\circ$ $r(A_1 + A_2) = \lim_{n\to\infty} \|(A_1 + A_2)^n x_0\|^{1/n}$,

$12^\circ$ $A_2 A_1^j A_2^k x_0 \prec A_1^j A_2^{k+1} x_0$ *for* $j = 1, 2, \dots,$ $k = 0, 1, \dots$ .

*Then* (15) *holds.*

The proof of Theorem 4 is analogous to that of Theorem 1 [14], so it can be omitted.

Finally we shall show an application of Theorems 1, 3 and 4. Consider the integral-functional equation

$$x(t) = \int_0^t f\Big(s, \max_{[0,s^a]}\{x(\tau)\}, x(s^a)\Big)\, ds, \tag{16}$$

where $t \in [0, T]$, $T \geq 1$, $0 < a < 1$.

**Theorem 5.** *Assume that:*

$13^\circ$ $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ *is continuous and satisfies the Lipschitz condition*

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq L_1(t)|x_1 - y_1| + L_2(t)|x_2 - y_2|,$$

*where the functions $L_1$, $L_2$ are continuous and non-negative on the interval* $[0, T]$,

$14^\circ$ $\max_{[0,T]}\{L_1(t)\} + \max_{[0,T]}\{L_2(t)\} < \frac{1}{1-a}$.

*Then the equation* (16) *has a unique solution in the set of continuous functions on the interval* $[0, T]$.

Proof: Let $(X, \|\cdot\|, \prec, m)$ be the Banach space from the proof of Theorem 2. We shall show that the operator

$$(\mathcal{A}x)(t) = \int_0^t f\Big(s, \max_{[0,s^a]}\{x(\tau)\}, x(s^a)\Big)\, ds,\ t \in [0, T],\ T \geq 1,$$

has exactly one fixed point in $X$. In view of our assumptions we have

$$|(\mathcal{A}x)(t) - (\mathcal{A}y)(t)| \leq \int_0^t L_1 \max_{[0,s^a]} |x(\tau) - y(\tau)|\, ds + \int_0^t L_2 |x(s^a) - y(s^a)|\, ds,$$

where $L_i = \max_{[0,T]}\{L_i(t)\}$, $i = 1, 2$. Let

$$(Ax)(t) = \int_0^t L_1 \max_{[0,s^a]} |x(\tau)|\, ds + \int_0^t L_2 |x(s^a)|\, ds.$$

Obviously, $A$ is linearly bounded and positively increasing. To prove our theorem it is sufficient to show that $r(A) < 1$. Observe that $A = A_1 + A_2$, where

$$(A_1x)(t) = \int_0^t L_1 \max_{[0,s^a]} |x(\tau)|\, ds$$

and

$$(A_2x)(t) = \int_0^t L_2 |x(s^a)|\, ds.$$

It is easy to check that $A$, $A_1$ and $A_2$ belong to $S(X)$. In the space of continuous functions on the interval $[0,T]$ we choose the cone $K$ of non-negative functions. Such a cone is solid and normal and $x_0(t) \equiv 1$ for $t \in [0,T]$ is its interior element. Clearly, $A$, $A_1$ and $A_2$ satisfy the remaining assumptions of Theorem 3. Thus the condition 11° of Theorem 4 is fulfilled. Moreover, for $j = 1, 2, \dots$, $k = 0, 1, \dots$ we have

$$(A_2 A_1^j A_2^k x_0)(t) = L_1^j L_2^{k+1} \frac{1}{a_1 a_2 \dots a_{k+j+1}} t^{a_{k+j+1}} = (A_1^j A_2^{k+1} x_0)(t),$$

where $a_1 = a + 1$, $a_n = a \cdot a_{n-1} + 1$. Hence

$$A_2 A_1^j A_2^k x_0 \prec A_1^j A_2^{k+1} x_0, \quad j = 1, 2, \dots, \quad k = 0, 1, \dots .$$

Therefore, in virtue of Theorem 4

$$r(A) \le r(A_1) + r(A_2).$$

Using (7), we obtain

$$r(A_1) = (1-a)L_1$$

and

$$r(A_2) = (1-a)L_2.$$

Thus, by 14°, $r(A) < 1$. This ends the proof of Theorem 4. □

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