Forcing countable networks for spaces satisfying $\mathrm{R}(X^\omega) = \omega$

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Abstract. We show that all finite powers of a Hausdorff space X do not contain uncountable weakly separated subspaces iff there is a c.c.c poset P such that in V^P X is a countable union of 0-dimensional subspaces of countable weight. We also show that this theorem is sharp in two different senses: (i) we cannot get rid of using generic extensions, (ii) we have to consider all finite powers of X.

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1. Introduction

We use standard topological notation and terminology throughout, cf. [4]. The following definitions are less well-known.

Definition 1.1. Given a topological space $\langle X, \tau\rangle$ and a subspace $Y \subset X$ a function f is called a *neighbourhood assignment on* Y iff $f : Y \to \tau$ and $y \in f(y)$ for each $y \in Y$.

Definition 1.2. A space Y is *weakly separated* if there is a neighbourhood assignment f on Y such that

$$\forall\, y \neq z \in Y \ (y \notin f(z) \vee z \notin f(y)),$$

moreover

$$\mathrm{R}(X) = \sup\{|Y| : Y \subset X \text{ is weakly separated}\}.$$

The notion of weakly separated spaces and the cardinal function R were introduced by Tkačenko in [7], where the following question was also raised: does $\mathrm{R}(X^\omega) = \omega$ (or even $\mathrm{R}(X) = \omega$) imply that X has a countable network (i.e. $\mathrm{nw}(X) = \omega$)? (Note that $\mathrm{R}(X^\omega) = \omega$ is equivalent to $\mathrm{R}(X^n) = \omega$ for all $n \in \omega$, moreover $\mathrm{nw}(X) = \omega$ implies $\mathrm{R}(X^\omega) = \mathrm{nw}(X^\omega) = \omega$.) Several consistent counterexamples to this were given, e.g. in [1], [2], [5] and [8, p. 43], but no ZFC

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counterexample is known. (In [8] it is stated that under PFA the implication is valid, but no proof is given.) The counterexamples given in [5] and [8] from CH are also first countable.

Our main result here says that, at least for T_2 spaces, a weaker version of Tkačenko's conjecture is valid, namely $\mathrm{R}(X^\omega)=\omega$ implies that $\mathrm{nw}(X)=\omega$ holds in a suitable c.c.c and hence cardinal preserving generic extension! In fact, in this extension X becomes σ-second countable, i.e. X is the union of countably many subspaces of countable weight.

In Section 3 we show that the main result is sharp in different senses. Firstly, we force, for every natural number n, a 0-dimensional, first countable space X such that $\mathrm{R}(X^n)=\omega$, but $\mathrm{nw}(X)>\omega$ in any cardinal preserving extension of the ground model. Secondly, we construct in ZFC a 0-dimensional T_2 space X such that $\chi(X)=\mathrm{nw}(X)=\omega$ but X is not σ-second countable.

It easily follows from the proof of our main result that if MA_κ holds and X is a Hausdorff space with $|X|+\mathrm{w}(X)\le\kappa$, then $\mathrm{R}(X^\omega)=\omega$ if and only if X is σ-second countable. We also prove that $\mathrm{MA}(\mathit{Cohen})$ is not enough to yield this equivalence. To do this we use a result of Shelah (the proof of which presented here with his kind permission) saying that in any generic extension by Cohen reals the ideal of the first category subspaces of a space from the ground model is generated by the subspaces of first category from the ground model.

2. The main result

Theorem 2.1. *Given a Hausdorff topological space X the following are equivalent:*

(1) $\mathrm{R}(X^\omega)=\omega$,
(2) *there is a c.c.c poset P such that* $V^P\models$ "$\mathrm{nw}(X)=\omega$",
(3) *there is a c.c.c poset P such that* $V^P\models$ *"X is a countable union of 0-dimensional subspaces of countable weight."*

Proof: Since the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are clear, it remains to prove only that (1) implies (3). So assume that X is Hausdorff with $\mathrm{R}(X^\omega)=\omega$, and fix a base $\mathcal{B}$ of X and a well-ordering $\prec$ on $X\cup\mathcal{B}$.

The space X contains at most countably many isolated points by $\mathrm{R}(X)=\omega$, so it is enough to force an appropriate partition of X', the set of non-isolated points of X. We say that a 4-tuple $\langle A,\mathcal{U},f,g\rangle$ is in P provided (i)–(v) below hold:

(i) $A\in\left[X'\right]^{<\omega}$ and $\mathcal{U}\in\left[\mathcal{B}\right]^{<\omega}$,
(ii) f and g are functions,
(iii) $f:A\to\omega$, $g:\mathcal{U}\to\omega\times\omega$,
(iv) if $g(U)=\langle n,i\rangle$ and $f(x)=n$ then $x\notin(\overline{U}\setminus U)$ whenever $x\in A$ and $U\in\mathcal{U}$,
(v) if $g(U)=g(V)=\langle n,i\rangle$ and $f(x)=n$ then $x\in U$ iff $x\in V$ whenever $x\in A$ and $U,V\in\mathcal{U}$.

Our idea is that f will guess the partition of X' into countably many pieces, $\{F_n : n < \omega\}$; if $g(U) = \langle n, i\rangle$ then $U \cap F_n$ will be clopen in the subspace F_n, and $g(U) = g(V) = \langle n, i\rangle$ implies $U \cap F_n = V \cap F_n$. Consequently, each F_n will have a countable clopen base.

For $p \in P$ we write $p = \langle A^p, \mathcal{U}^p, f^p, g^p\rangle$. If $p, q \in P$ we set $p \leq q$ iff $f^p \supset f^q$ and $g^p \supset g^q$.

Two conditions p and q from P are called *twins* provided $|A^p| = |A^q|$, $|\mathcal{U}^p| = |\mathcal{U}^q|$ and denoting by η and by ϱ the unique $\prec$-preserving bijections between A^p and A^q, and between $\mathcal{U}^p$ and $\mathcal{U}^q$, respectively, we have

(1) $\eta\lceil A^p \cap A^q = \mathrm{id}$, $\varrho\lceil\mathcal{U}^p \cap \mathcal{U}^q = \mathrm{id}$,
(2) $\forall\, x \in A^p\ f^p(x) = f^q(\eta(x))$,
(3) $\forall\, U \in \mathcal{U}^p\ g^p(U) = g^q(\varrho(U))$,
(4) $\forall\, x \in A^p\ \forall\, U \in \mathcal{U}^p$ ($x \in U$ iff $\eta(x) \in \varrho(U)$, and $x \in \overline{U}$ iff $\eta(x) \in \overline{\varrho(U)}$).

Lemma 2.2. *$\mathcal{P} = \langle P, \leq\rangle$ satisfies c.c.c.*

Proof of Lemma 2.2: Let $\{p_\alpha : \alpha < \omega_1\} \subset P$. Write $p_\alpha = \langle A^\alpha, \mathcal{U}^\alpha, f^\alpha, g^\alpha\rangle$. Using standard Δ-system and counting arguments we can assume that these conditions are pairwise twins. Let $k = |A^\alpha|$ and $\{a_{\alpha,i} : i < k\}$ be the $\prec$-increasing enumeration of A^α. For each $\alpha < \omega_1$ and $i < k$ put

$$\mathcal{U}^0_{\alpha,i} = \{U \in \mathcal{U}^\alpha : \exists\, j\ g^\alpha(U) = \big\langle f^\alpha(a_{\alpha,i}), j\big\rangle \wedge a_{\alpha,i} \in U\},$$
$$\mathcal{U}^1_{\alpha,i} = \{U \in \mathcal{U}^\alpha : \exists\, j\ g^\alpha(U) = \big\langle f^\alpha(a_{\alpha,i}), j\big\rangle \wedge a_{\alpha,i} \notin U\},$$

and finally

$$V_{\alpha,i} = \bigcap\{U : U \in \mathcal{U}^0_{\alpha,i}\} \cap \bigcap\{X \setminus \overline{U} : U \in \mathcal{U}^1_{\alpha,i}\}.$$

By (iv) we have $a_{\alpha,i} \notin \overline{U}$ for $U \in \mathcal{U}^1_{\alpha,i}$, so $a_{\alpha,i} \in V_{\alpha,i}$. Since $\mathrm{R}(X^k) = \omega$ there are $\alpha < \beta < \omega_1$ such that

$$(\dagger) \qquad a_{\alpha,i} \in V_{\beta,i} \text{ and } a_{\beta,i} \in V_{\alpha,i} \text{ for each } i < k.$$

We claim that p_α and p_β are compatible in P. Let η and ϱ be the functions witnessing that p_α and p_β are twins. Put $A = A^\alpha \cup A^\beta$, $\mathcal{U} = \mathcal{U}^\alpha \cup \mathcal{U}^\beta$, $f = f^\alpha \cup f^\beta$, $g = g^\alpha \cup g^\beta$ and $p = \langle A, \mathcal{U}, f, g\rangle$. Since p_α and p_β are twins, p satisfies (i)–(iii) and $p \leq p_\alpha, p_\beta$. So all we have to do is to show that p satisfies (iv) and (v).

Claim. *If $U \in \mathcal{U}^0_{\alpha,i} \cup \mathcal{U}^1_{\alpha,i}$, then $a_{\alpha,i} \in U$ iff $a_{\alpha,i} \in \varrho(U)$.*

Proof of the claim: We know $a_{\alpha,i} \in U$ iff $a_{\beta,i} \in \varrho(U)$. Thus $a_{\alpha,i} \in U$ implies that $\varrho(U) \in \mathcal{U}^0_{\beta,i}$ and so $a_{\alpha,i} \in V_{\beta,i} \subset \varrho(U)$. On the other hand, if $a_{\alpha,i} \notin U$, then $a_{\beta,i} \notin \varrho(U)$, hence $\varrho(U) \in \mathcal{U}^1_{\beta,i}$, and so $a_{\alpha,i} \in V_{\beta,i} \subset X \setminus \overline{\varrho(U)}$, i.e. $a_{\alpha,i} \notin \overline{\varrho(U)}$. $\square$

Now we check (iv). Assume $a_{\alpha,i} \in A^\alpha$, $V \in \mathcal{U}^\beta$ with $g^\beta(V) = \langle f^\alpha(a_{\alpha,i}), l \rangle$ and $a_{\alpha,i} \in \overline{V}$. We have to show that $a_{\alpha,i} \in V$. Since $a_{\alpha,i} \in V_{\beta,i}$ by (†), we have $V \notin \mathcal{U}^1_{\beta,i}$. But $f^\alpha(a_{\alpha,i}) = f^\beta(a_{\beta,i})$ since p_α and p_β are twins, hence $g^\beta(V) = \left\langle f^\beta(a_{\beta,i}), l \right\rangle$ implies $V \in \mathcal{U}^0_{\beta,i} \cup \mathcal{U}^1_{\beta,i}$, so $V \in \mathcal{U}^0_{\beta,i}$. Hence $a_{\alpha,i} \in V_{\beta,i} \subset V$ by (†), which was to be proved.

Finally we check (v). Assume that $a_{\alpha,i} \in A^\alpha$ and $U, V \in \mathcal{U}^\alpha \cup \mathcal{U}^\beta$ are such that $g(U) = g(V) = \langle f(a_{\alpha,i}), l \rangle$. Define the function $\varrho^* : \mathcal{U} \to \mathcal{U}^\alpha$ by the formula $\varrho^* = \mathrm{id}\lceil \mathcal{U}^\alpha \cup (\varrho)^{-1}$. Then $a_{\alpha,i} \in U$ iff $a_{\alpha,i} \in \varrho^*(U)$ and $a_{\alpha,i} \in V$ iff $a_{\alpha,i} \in \varrho^*(V)$ by the previous claim.

But $g(\varrho^*(U)) = g(U) = g(V) = g(\varrho^*(V)) = \langle f(a_{\alpha,i}), l \rangle$, so $a_{\alpha,i} \in \varrho^*(U)$ iff $a_{\alpha,i} \in \varrho^*(V)$ for p_α satisfies (v). Thus $a_{\alpha,i} \in U$ iff $a_{\alpha,i} \in \varrho^*(U)$ iff $a_{\alpha,i} \in \varrho^*(V)$ iff $a_{\alpha,i} \in V$, which proves (v).

Now let $\mathcal{G}$ be a P-generic filter and let $F = \bigcup\{f^p : p \in \mathcal{G}\}$ and $G = \bigcup\{g^p : p \in \mathcal{G}\}$. For $n < \omega$ let $F_n = F^{-1}\{n\}$.

Lemma 2.3. $\mathrm{dom}(F) = X'$ *and* $\mathrm{dom}(G) = \mathcal{B}$.

Proof: Let $p = \langle A, \mathcal{U}, f, g \rangle \in P$, $x \in X' \setminus A$ and $U \in \mathcal{B} \setminus \mathcal{U}$. Let $A^* = A \cup \{x\}$, $\mathcal{U}^* = \mathcal{U} \cup \{U\}$ and $n = \max \mathrm{ran}(f) + 1$. Let $\mathrm{dom}(f^*) = A^*$, $f^* \supset f$ and $f^*(x) = n$, $\mathrm{dom}(g^*) = \mathcal{U}^*$, $g^* \supset g$ and $g^*(U) = \langle n+1, 0 \rangle$. Then it is easy to check $p^* = \langle A^*, \mathcal{U}^*, f^*, g^* \rangle \in P$ and obviously $p^* \leq p$. So the lemma holds because a generic filter intersects every dense set. $\square$

For $m \in \omega$ let

$$\mathcal{B}_m = \{U \cap F_m : U \in \mathcal{B} \ \text{ and } \ G(U) = \langle m, i \rangle \ \text{ for some } \ i \in \omega\}.$$

Lemma 2.4. *$\mathcal{B}_m$ is a countable, clopen base of the subspace F_m of X'.*

Proof: If $U \in \mathcal{B}$, $G(U) = \langle m, i \rangle$ then $U \cap F_m$ is clopen in F_m by (iv). If $U, V \in \mathcal{B}$, $G(U) = G(V) = \langle m, i \rangle$ then $U \cap F_m = V \cap F_m$ by (v). So $\mathcal{B}_m$ is countable.

Finally we show that it is a base of F_m. So fix $x \in F_m$ and $V \in \mathcal{B}$ with $x \in V$. Let $p = \langle A, \mathcal{U}, f, g \rangle \in P$ such that $f(x) = m$. Since X is Hausdorff and x is non-isolated in X, we can choose $U \in \mathcal{B} \setminus \mathcal{U}$ such that $x \in U \subset V$ and $\overline{U} \cap A = \{x\}$.

Let $\mathcal{U}^* = \mathcal{U} \cup \{U\}$. Define the function $g^* : \mathcal{U}^* \to \omega \times \omega$ such that $g^* \supset g$ and $g^*(U) = \langle m, k \rangle$ where $k = \min\{l : \mathrm{ran}\, g \subset l \times l\}$. Then $p^* = \langle A, \mathcal{U}^*, f, g^* \rangle$ is an extension of p in $\mathcal{P}$ and $p^* \Vdash x \in U \cap F_m \subset V \cap F_m \wedge U \cap F_m \in \mathcal{B}_m$. Consequently, if $p \Vdash$ "$x \in F_m \cap V$" then we also have $p \Vdash$ "$\exists\, U \in \mathcal{B}_m (x \in U \subset F_m \cap V)$", which completes the proof. $\square$

Thus Theorem 2.1 is proved.

It is easy to check that the above proof needs the genericity of $\mathcal{G}$ over $|X| + |\mathcal{B}|$ many dense sets only, and this immediately yields the following result.

Corollary 2.5. *If MA_κ holds then for a Hausdorff space X with $|X| + \mathrm{w}(X) \le \kappa$ the following are equivalent:*

(1) $\mathrm{R}(X^\omega) = \omega$,
(2) $\mathrm{nw}(X) = \omega$,
(3) *X is the union of countably many 0-dimensional subspaces of countable weight,*
(4) *X is the union of countably many separable metrizable subspaces.*

3. Sharpness of the main result

Our aim in this section is to examine how sharp the above main result is. The co-finite topology on any uncountable set X clearly satisfies $\mathrm{R}(X^\omega) = \omega$, while, in any extension, $\mathrm{nw}(X) = |X|$. This show that in the proof of (1) $\to$ (2) of 2.1 the Hausdorffness of X cannot be replaced by T_1.

The next result in this section implies that, at least in ZFC, the exponent ω in proving (1) $\to$ (2) in Theorem 2.1 cannot be lowered.

Theorem 3.1. *For each uncountable cardinal κ and natural number m there is a c.c.c poset $\mathcal{P}$ of cardinality κ such that in $V^{\mathcal{P}}$ there is a 0-dimensional first countable topological space $X = \langle \kappa, \tau \rangle$ such that $\mathrm{R}(X^m) = \omega$ but $\mathrm{R}(X^{m+1}) = \kappa$, hence $\mathrm{nw}(X) = \kappa$ in any cardinal preserving extension.*

In [5, Theorem 3.5] we constructed a c.c.c poset $\langle P^\kappa, \le \rangle$ which adds to the ground model a 0-dimensional, first countable topology τ on κ such that $\mathrm{R}(X^\omega) = \omega$ and $\mathrm{w}(X) = \kappa$ for $X = \langle \kappa, \tau \rangle$. The conditions in P^κ are finite approximations of the space X and the property $\mathrm{R}(X^\omega) = \omega$ is guaranteed by some Δ-system and amalgamation arguments. Here we will use a subset P of P^κ with the inherited order. To ensure $\mathrm{R}(X^{m+1}) = \kappa$ we thin out P^κ in the following way. We fix a family $\mathcal{D} = \{d_\alpha : \alpha < \kappa\}$ of pairwise disjoint elements of κ^{m+1} with the intention to make $\mathcal{D}$ discrete in X^{m+1}. A condition $p \in P^\kappa$ is put into P if and only if every neighbourhood given by p witnesses that $\mathcal{D}$ is discrete. The main step of the proof is to show that P is large enough to allow the Δ-system and amalgamation arguments to work in showing $\mathrm{R}(X^m) = \omega$.

Proof of Theorem 3.1: First we recall some definitions and lemmas from the proof of [5, Theorem 3.5]. A quadruple $\langle A, n, f, g \rangle$ is said to be in P_0^κ provided (a)–(b) below hold:

(a) $A \in \left[\kappa\right]^{<\omega}$, $n \in \omega$, f and g are functions,
(b) $f : A \times A \times n \to 2$, $g : A \times n \times A \times n \to 3$,

For $p \in P_0^\kappa$ we write $p = \langle A^p, n^p, f^p, g^p \rangle$. If $p, q \in P_0^\kappa$ we set $p \le q$ iff $f^p \supseteq f^q$ and $g^p \supseteq g^q$. If $p \in P_0^\kappa$, $\alpha \in A^p$, $i < n^p$ we defined $U(\alpha, i) = U^p(\alpha, i) = \{\beta \in A^p : f^p(\beta, \alpha, i) = 1\}$.

A quadruple $\langle A, n, f, g \rangle \in P_0^\kappa$ is put in P^κ iff (i)–(ii) below are also satisfied:

(i) $\forall\, \alpha \in A\ \forall\, i < j < n\ \alpha \in U(\alpha, j) \subset U(\alpha, i)$,

(ii) $\forall\,\alpha \neq \beta \in A\ \forall\, i, j < n$
$g(\alpha, i, \beta, j) = 0$ if and only if $U(\alpha, i) \subset U(\beta, j)$,
$g(\alpha, i, \beta, j) = 1$ if and only if $U(\alpha, i) \cap U(\beta, j) = \emptyset$,
$g(\alpha, i, \beta, j) = 2$ if $\alpha \in U(\beta, j)$ and $\beta \in U(\alpha, i)$.

Definition 3.2 ([5, Definition 3.6])**.** Assume that $p_i = \langle A^i, n^i, f^i, g^i \rangle \in P_0^\kappa$ for $i \in 2$. We say that p_0 and p_1 are *twins* iff $n_0 = n_1$, $|A_0| = |A_1|$ and taking $n = n_0$ and denoting by σ the unique $<_{\mathrm{On}}$-preserving bijection between A_0 and A_1 we have

(1) $\sigma\lceil A_0 \cap A_1 = \mathrm{id}_{A_0 \cap A_1}$.
(2) σ is an isomorphism between p_0 and p_1, i.e. $\forall\,\alpha, \beta \in A_0$, $\forall\, i, j < n$
$$f_0(\alpha, \beta, i) = f_1(\sigma(\alpha), \sigma(\beta), i),$$
$$g_0(\alpha, i, \beta, j) = g_1(\sigma(\alpha), i, \sigma(\beta), j),$$

We say that σ is the *twin function* of p_0 and p_1. Define the *smashing function* $\overline{\sigma}$ of p_0 and p_1 as follows: $\overline{\sigma} = \sigma \cup \mathrm{id}_{A_1}$. The function σ^* defined by the formula $\sigma^* = \sigma \cup \sigma^{-1}\lceil A_1$ is called the *exchange function* of p_0 and p_1.

Definition 3.3 ([5, Definition 3.7])**.** Assume that p_0 and p_1 are twins and $\varepsilon : A^{p_1} \setminus A^{p_0} \to 2$. A common extension $q \in P^\kappa$ of p_0 and p_1 is called an ε-*amalgamation* of the twins p_0 and p_1 provided
$$\forall\,\alpha \in A^{p_0} \triangle A^{p_1}\ \ f^q(\alpha, \sigma^*(\alpha), i) = \varepsilon(\overline{\sigma}(\alpha)).$$

Lemma 3.4 ([5, Lemma 3.8])**.** *If p_0, $p_1 \in \mathcal{P}^\kappa$ are twins and $\varepsilon : A^{p_1} \setminus A^{p_0} \to 2$, then p_0 and p_1 have an ε-amalgamation in P^κ.*

In [5] we used the poset $\mathcal{P}^\kappa = \langle P^\kappa, \leq \rangle$. Here we will apply a subset P of P^κ. To define it let $\{d_\alpha : \alpha < \kappa\}$ be a family of pairwise disjoint elements of $\left[\kappa\right]^{m+1}$ such that $\kappa \setminus \bigcup\{d_\alpha : \alpha < \kappa\}$ is still infinite. Write $d_\alpha = \{d_{\alpha,i} : i \leq m\}$.

Definition 3.5. A condition $p = \langle A, n, f, g \rangle \in P^\kappa$ is in P iff it satisfies (1) and (2) below:

(1) $d_\alpha \subset A$ or $d_\alpha \cap A = \emptyset$ for each $\alpha < \kappa$;
(2) if $\alpha < \beta < \kappa$, $d_\alpha \cup d_\beta \subset A$, then there is an $i = i_{\alpha,\beta} \leq m$ such that
$$d_{\alpha,i} \notin U^p(d_{\beta,i}, 0) \ \text{ and } \ d_{\beta,i} \notin U^p(d_{\alpha,i}, 0).$$

Let $\mathcal{P} = \langle P, \leq \rangle$.

We define X as expected. Let $\mathcal{G}$ be a $\mathcal{P}$-generic filter and let $F = \bigcup\{f^p : p \in \mathcal{G}\}$. For each $\alpha < \kappa$ and $n \in \omega$ let $V(\alpha, i) = \{\beta < \kappa : F(\beta, \alpha, i) = 1\}$. Put $\mathcal{B}_\alpha = \{V(\alpha, i) : i < \kappa\}$ and $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha : \alpha < \kappa\}$. We choose $\mathcal{B}$ as the base of $X = \langle \kappa, \tau \rangle$. By standard density arguments we can see that X is first countable and 0-dimensional.

It is easy to see that $\mathrm{R}(X^{m+1}) = \kappa$, in fact $\mathrm{s}(X^{m+1}) = \kappa$. Indeed, by 3.5 (2), $\{d_\alpha : \alpha < \kappa\}$ is discrete in X^{m+1}, as witnessed by the open neighborhoods $V(d_{\alpha,0}, 0) \times V(d_{\alpha,1}, 0) \cdots \times V(d_{\alpha,m}, 0)$.

Finally we need to show that $\mathcal{P}$ satisfies c.c.c and $V^{\mathcal{P}} \models \mathrm{R}(X^m) = \omega$. Clearly, both of these statements follow from the next lemma.

Lemma 3.6. *If $\{p_\gamma : \gamma < \omega_1\} \subset \mathcal{P}$, $\{c_\gamma : \gamma < \omega_1\} \subset \kappa^m$ and $j_0, \dots, j_{m-1}$ are natural numbers, then there are ordinals $\alpha < \beta < \omega_1$ and a condition $p \in P$ such that*

(+) $p \le p_\alpha, p_\beta$ *and* $p \Vdash c_\alpha \in \prod_{i<m} V(c_\beta(i), j_i) \wedge c_\beta \in \prod_{i<m} V(c_\alpha(i), j_i)$.

Proof: We can assume that $c_\gamma \subset A^{p_\gamma}$ holds for each $\gamma < \omega_1$.

Pick α and β such that p_α and p_β are twins, and denoting by ϱ their twin function we have $\varrho'' c_\alpha = c_\beta$ and $\{\varrho'' d_\xi : d_\xi \subset A^{p_\alpha}\} = \{d_\zeta : d_\zeta \subset A^{p_\beta}\}$.

Let $x = \{\xi < \kappa : d_\xi \subset A^{p_\alpha}\}$ and $y = \{\xi < \kappa : d_\xi \subset A^{p_\beta}\}$. Define the function $\varepsilon : A^{p_\alpha} \setminus A^{p_\beta} \to 2$ by the stipulations $\varepsilon(\nu) = 1$ iff $\nu \in c_\alpha$. By Lemma 3.4 p_α and p_β have an ε-amalgamation p in P^κ. Since $C = \kappa \setminus \bigcup\{d_\xi : \xi < \kappa\}$ is infinite, we can assume that $A^p \setminus (A^{p_\alpha} \cup A^{p_\beta}) \subset C$.

First we show that $p \in P$. Observe that for any $\xi < \kappa$ we have $d_\xi \cap A^p \ne \emptyset$ if and only if $d_\xi \cap (A^{p_\alpha} \cup A^{p_\beta}) \ne \emptyset$ if and only if $(d_\xi \subset A^{p_\alpha} \vee d_\xi \subset A^{p_\beta})$ by $A^p \setminus (A^{p_\alpha} \cup A^{p_\beta}) \subset C$. Thus 3.5 (1) holds. To check 3.5 (2) assume that $\xi \ne \zeta < \kappa$ and $d_\xi \cup d_\zeta \subset A^p$. Then $d_\xi \cup d_\zeta \subset A^{p_\alpha} \cup A^{p_\beta}$ and since p_α and p_β are in P we can assume that $d_\xi \subset A^{p_\alpha}$ and $d_\zeta \subset A^{p_\beta}$. If $d_\xi \cap A^{p_\beta} \ne \emptyset$ then $d_\xi \subset A^{p_\beta}$ as well. Therefore $d_\xi \cup d_\zeta \subset A^{p_\beta}$, and so 3.5 (2) holds for ξ and ζ because $p_\beta \in P$. Thus we can assume that $d_\xi \subset A^{p_\alpha} \setminus A^{p_\beta}$, and similarly that $d_\zeta \subset A^{p_\beta} \setminus A^{p_\alpha}$. If $\varrho'' d_\xi \ne d_\zeta$, then $\varrho'' d_\xi = d_\mu$ for some $\mu \in y \setminus \{\zeta\}$. Since $p_\beta \in P$, there is $i \le m$ such that $d_{\mu,i} \notin U^{p_\beta}(d_{\zeta,i}, 0)$ and $d_{\zeta,i} \notin U^{p_\beta}(d_{\mu,i}, 0)$. So, by the definition of ε-amalgamation, $d_{\xi,i} \notin U^p(d_{\zeta,i}, 0)$ and $d_{\zeta,i} \notin U^p(d_{\xi,i}, 0)$. On the other hand, if $\varrho'' d_\xi = d_\zeta$, then there is $i \le m$ such that $\varepsilon(d_{\xi,i}) = 0$, for $|d_\xi| = m + 1 > m = |\varepsilon^{-1}\{1\}|$. So, by the definition of ε-amalgamation, $d_{\xi,i} \notin U^p(d_{\zeta,i}, 0)$ and $d_{\zeta,i} \notin U^p(d_{\xi,i}, 0)$. Thus $p \in P$.

Finally we show that $p \Vdash$ "$c_\alpha \in \prod_{i<m} V(c_\beta(i), j_i) \wedge c_\beta \in \prod_{i<m} V(c_\alpha(i), j_i)$". Indeed, $c_\beta(i) \in U^p(c_\alpha(i), j_i)$ and $c_\alpha(i) \in U^p(c_\beta(i), j_i)$ for each $i < m$ because $\varrho(c_{\alpha,i}) = c_{\beta,i}$ and either $c_{\alpha,i} = c_{\beta,i}$ or $\varepsilon(c_{\alpha,i}) = 1$. □

Theorem 3.1 is proved.

Next we show that the use of forcing in the implications (1) → (3) and (2) → (3) from Theorem 2.1 is essential because in 3.8 we shall produce a ZFC example of a 0-dimensional, first countable space X that satisfies $\mathrm{nw}(X) = \omega$ (hence $\mathrm{R}(X^\omega) = \omega$) but still X is not σ-second countable. To achieve this we need the following lemma. If $X = \langle X, \tau\rangle$ is a topological space, $\mathrm{D}(X)$ denotes the discrete topology on X. If A and B are sets, let $\mathrm{Fin}(A, B)$ be the family of functions mapping a finite subset of A into B.

Lemma 3.7. *If $Z \subset X^\omega$ and Z is somewhere dense in the space* $\mathrm{D}(X)^\omega$, *then* $\mathrm{w}(Z) = \mathrm{w}(X)$.

Proof: Fix a natural number $n \in \omega$ and a function $f : n \to X$ such that Z is dense in the basic open set $U_f = \{g \in X^\omega : f \subset g\}$ of $\mathrm{D}(X)^\omega$. This means that

$$(\dagger) \qquad \forall f' \in \mathrm{Fin}(\omega \setminus n, X)\, \exists g \in Z \; f \cup f' \subset g.$$

From now on we forget about the $\mathrm{D}(X)^\omega$ topology, we will use only (†). Without loss of generality we can assume that $Z \subset U_f$. Let $\mathcal{Z}$ be a base of Z in the subspace topology of X^ω.

Let $\pi_{\mathrm{m}} : X^\omega \to X$ be the projection to the m^{th} factor, i.e. $\pi_{\mathrm{m}}(g) = g(m)$. Set $\mathcal{X} = \{\pi_{\mathrm{n}}(U) : U \in \mathcal{Z}\}$. Since $\mathrm{w}(Z) \leq \mathrm{w}(X)$, it is enough to show that $\mathcal{X}$ is a base of X.

Claim. *If U is open in Z then $\pi_{\mathrm{n}}(U)$ is open in X.*

Proof of the claim: Let $x \in \pi_{\mathrm{n}}(U)$. We need to show that $\pi_{\mathrm{n}}(U)$ contains a neighbourhood of x. Pick $g \in U$ with $x = g(n)$. Then, by the definition of the product topology on X^ω, there is a function σ which maps a finite subset of $\omega \setminus n$ into the family of non-empty open subsets of X such that

$$(\star) \qquad g \in Z \cap \bigcap_{m \in \mathrm{dom}(\sigma)} {\pi_{\mathrm{m}}}^{-1} \sigma(m) \subset U.$$

We can assume that $n \in \mathrm{dom}(\sigma)$. Let $g' = g \lceil (\mathrm{dom}\, \sigma \setminus \{n\})$. By (†), for each $x' \in \sigma(n)$ there is $h_{x'} \in Z$ such that

$$(\star\star) \qquad f \cup \{\langle n, x' \rangle\} \cup f' \subset h_{x'}.$$

Now $(\star)$ implies $h_{x'} \in U$ and so $x' \in \pi_{\mathrm{n}}(U)$. Thus $x \in \sigma(n) \subset \pi_{\mathrm{n}}(U)$, which was to be proved. □

To show that $\mathcal{X}$ is a base let $x \in V \subset X$, V open. By (†) we can find a point $g \in Z$ with $f \subset g$ and $g(n) = x$. The family $\mathcal{Z}$ is a base of Z in the subspace topology of X^ω, so there is $U \in \mathcal{Z}$ such that $g \in U \subset Z \cap {\pi_{\mathrm{n}}}^{-1} V$. Thus $x \in \pi_{\mathrm{n}}(U) \subset V$ and $\pi_{\mathrm{n}}(U)$ is open by the previous claim.

Thus $\mathcal{X}$ is a base of X, and so $\mathrm{w}(X) \leq |\mathcal{X}| \leq |\mathcal{Z}|$. Since $\mathrm{w}(Z) \leq \omega\, \mathrm{w}(X) = \mathrm{w}(X)$, we are done.

After this preparation we can give the ZFC example promised above.

Theorem 3.8. *There is 0-dimensional Hausdorff space Y such that $\chi(Y)\,\mathrm{nw}(Y) = \omega$, but Y is not σ-second countable.*

Proof: By [5, Theorem 3.1] there is a 0-dimensional Hausdorff space X of size 2^ω such that $\chi(X)\,\mathrm{nw}(X) = \omega$, but $\mathrm{w}(X) = 2^\omega$. We show that $Y = X^\omega$ is as required. Clearly $\chi(Y) = \chi(X) = \omega$ and $\mathrm{nw}(Y) = \mathrm{nw}(X) = \omega$.

Assume that $Y = \bigcup_{k<\omega} Z_k$. The Baire category theorem implies that some Z_k is somewhere dense in $\mathrm{D}(X)^\omega$. Then $\mathrm{w}(Z_k) = \mathrm{w}(X) = 2^\omega$ by Lemma 3.7. □

By Corollary 2.5, if Martin's Axiom holds, then every Hausdorff space X of size and weight $< 2^\omega$ is σ-second countable if and only if $\mathrm{nw}(X) = \omega$. The next theorem shows that MA(*Cohen*) is not enough to get this equivalence. Note that for a first countable space X we have $\mathrm{w}(X) \leq |X|$.

Theorem 3.9. *If ZFC is consistent, then so is ZFC* + MA(*Cohen*) + "there is a first countable 0-dimensional Hausdorff space Y such that $\mathrm{nw}(Y) = \omega$ and $|Y| < 2^\omega$ but Y is not σ-second countable".

The proof is based on Theorem 3.9 above and Theorem 3.11 below.

Definition 3.10. Given a topological space Z, let $\mathcal{I}(Z)$ be the σ-ideal generated by the nowhere dense subsets of Z. The elements of $\mathcal{I}(Z)$ are called *first category in* Z.

The next result is due to Saharon Shelah [6] and it is included here with his kind permission.

Theorem 3.11. *If Z is a topological space and the forcing notion $P = \mathrm{Fn}(\kappa, 2, \omega)$ adds κ Cohen reals to the ground model, then the ideal $\mathcal{I}^{V^P}(Z)$ is generated by $\mathcal{I}^V(Z)$, that is, for each $T \in \mathcal{I}^{V^P}(Z)$ there is $T' \in \mathcal{I}^V(Z)$ with $T \subset T'$.*

Proof: Since $\mathcal{I}^{V^P}(Z)$ is σ-generated by the nowhere dense subsets of Z in V^P, we can assume that T is nowhere dense. Let $\dot{T}$ be a P-name of T such that $1_P \Vdash$"$\dot{T}$ *is nowhere dense*".

For each $m \in \omega$ define, in V, the subset B_m of Z as follows:

$$(*) \qquad B_m = \{x \in Z : \exists\, p \in P \ |p| = m \wedge p \Vdash x \in \dot{T}\}.$$

Clearly $1_P \Vdash \dot{T} \subset \bigcup_{m\in\omega} B_m$, so it is enough to show that every B_m is nowhere dense. Assume on the contrary that there are an open set $U \subset Z$ and $m \in \omega$ such that B_m is dense in U.

Now, by finite induction, we can define open sets $U \supset U_0 \supset U_1 \cdots \supset U_m$ and conditions $q_0, \dots, q_m$ with pairwise disjoint domains such that for each $j \le m$ and for each $f \in P$ if $\mathrm{dom}(f) = \bigcup_{i<j} \mathrm{dom}(q_i)$ then

$$(\dagger) \qquad f \cup q_j \Vdash \dot{T} \cap U_j = \emptyset.$$

Since B_m is dense in U and $U_m \subset U$ there is $x \in B_m \cap U_m$. Then, by the definition of B_m, we have a condition $p \in P$ with $|p| = m$ such that $p \Vdash x \in \dot{T}$. But the domains of the q_j are pairwise disjoint, so there is $j \le m$ with $\mathrm{dom}(p) \cap \mathrm{dom}(q_j) = \emptyset$. Thus $p \cup q_j \in P$. Let $f = p \lceil \bigcup_{i<j} \mathrm{dom}(q_i)$. By ($\dagger$) $f \cup q_j \Vdash \dot{T} \cap U_j = \emptyset$, so $p \cup q_j \Vdash \dot{T} \cap U_m = \emptyset$ as well. But this contradicts $x \in U_m$ and $p \Vdash x \in \dot{T}$, and thus the theorem is proved. □

Proof of Theorem 3.9: Using again [5, Theorem 3.1] we have a 0-dimensional Hausdorff space X such that $\chi(X)\,\mathrm{nw}(X) = \omega$, but $\mathrm{w}(X) = 2^\omega$. Let $Y = X^\omega$ and $Z = \mathrm{D}(X)^\omega$. Then add $\kappa > 2^\omega$ Cohen reals to the ground model. In the generic extension clearly $\chi(Y) = \mathrm{nw}(Y) = \omega$ will remain valid.

Since $Z \notin \mathcal{I}^V(Z)$ by the Baire Category theorem, we have $Z \notin \mathcal{I}^{V^P}(Z)$ as well by Theorem 3.11.

Therefore, if $Y = \bigcup\{Y_k : k < \omega\}$ holds in V^P, then $Y_k \notin \mathcal{I}^{V^P}(Z)$ for some $k \in \omega$, i.e. Y_k is somewhere dense in Z. Thus there is a natural number $n \in \omega$ and a function $f : n \to X$ such that

$$(\dagger) \qquad \forall\, f' \in \mathrm{Fin}(\omega \setminus n, X)\ \exists\, g \in Y_k\ \ f \cup f' \subset g.$$

But ($\dagger$) implies that Y_k is also dense in the basic open set $(V_f)^{V^P} = \{g \in X^\omega \cap V^P : f \subset g\}$ of $(\mathrm{D}(X)^\omega)^{V^P}$. Thus, applying Lemma 3.7 in V^P we have $\mathrm{w}(Y_k) = \mathrm{w}(X) > \omega$. Thus, in V^P, MA($Cohen$) holds, and still the first countable, 0-dimensional T_2 space Y is not σ-second countable, though $\mathrm{w}(Y) = (2^\omega)^V < \kappa = (2^\omega)^{V^P}$. □

4. Examples for higher cardinals

In this section we generalize the constructions of [5, Theorem 3.1] and 3.8 for cardinals greater than ω.

Theorem 4.1. *Let κ and λ be cardinals, $\mathrm{cf}(\kappa) = \kappa$. Then there is a 0-dimensional Hausdorff space X^κ_λ such that $\chi(X^\kappa_\lambda) = \kappa$, $\mathrm{nw}(X^\kappa_\lambda) = \lambda^{<\kappa}$ and $\mathrm{w}(X^\kappa_\lambda) = \lambda^\kappa$.*

Proof: For each $f \in \mathrm{Fn}(\kappa, \lambda, \kappa)$ put $U_f = \{g \in {}^\kappa\lambda : f \subset g\}$. Write $U(g, \alpha) = U_{g\lceil\alpha}$ for $g \in {}^\kappa\lambda$ and $\alpha < \kappa$. For $g \neq h \in {}^\kappa\lambda$ define $\Delta(g, h) = \min\{\alpha : g(\alpha) \neq h(\alpha)\}$. Consider the topological space $C^\kappa_\lambda = \langle {}^\kappa\lambda, \tau\rangle$ that has as a base $\{U_f : f \in {}^{<\kappa}\lambda\}$. Let

$$Y = \{g \in {}^\kappa\lambda : \exists\, \alpha < \kappa\ (g(\beta) = 0 \text{ iff } \ \beta \geq \alpha)\}$$

and

$$Z = \{g \in {}^\kappa\lambda : 0 \notin \mathrm{ran}(g)\}.$$

Clearly Y and Z are disjoint, $|Y| = \lambda^{<\kappa}$, $|Z| = \lambda^\kappa$, Z is closed and nowhere dense in C^κ_λ. Let $X = Y \cup Z$. Our required space will be $X^\kappa_\lambda = \langle X, \varrho\rangle$, where ϱ refines the topology τ_X. To define ϱ put

$$X_g = \bigcup\{U_f : \exists\, \alpha < \kappa\ f = g\lceil\alpha \cup \{\langle\alpha, 0\rangle\}\}$$

for $g \in Z$ and $X_g = \emptyset$ for $g \in Y$. For $g \in X$ and $\alpha < \kappa$ let $V(g, \alpha) = (U(g, \alpha) \setminus X_g) \cap X$. Let the neighbourhood base of $g \in X$ in ϱ be

$$\mathcal{B}_g = \{V(g, \alpha) : \alpha < \kappa\}.$$

First we note that $\mathcal{B}=\bigcup\{\mathcal{B}_x : x\in X\}$ is a base of a topology because

$$(\dagger)\qquad \forall\,\alpha<\kappa\;\forall\,h\in V(g,\alpha)\setminus\{g\}\;U(h,\Delta(g,h)+1)\cap X\subset V(g,\alpha).$$

Since $V(g,\alpha)\cap Y=U(g,\alpha)\cap Y$ and $V(h,\alpha)\cap Z=U(h,\alpha)\cap Z$ for each $g\in Y$, $h\in Z$ and $\alpha<\kappa$ we have that $\langle Y,\tau\rangle=\langle Y,\varrho\rangle$ and $\langle Z,\tau\rangle=\langle Z,\varrho\rangle$. Thus

$$\mathrm{nw}(\langle X,\varrho\rangle)=\mathrm{nw}(\langle Y,\varrho\rangle)+\mathrm{nw}(\langle Z,\varrho\rangle)\le \mathrm{w}(\langle Y,\tau\rangle)+\mathrm{w}(\langle Z,\tau\rangle)\le \mathrm{w}(\langle C^\kappa_\lambda\rangle)=\lambda^{<\kappa}.$$

Obviously $\chi(\langle X,\varrho\rangle)=\kappa$.

Finally we show that $\mathrm{w}(\langle X,\varrho\rangle)=\lambda^\kappa$. This will follow if we show that $\mathcal{B}$ is an irreducible base for X, by [5, Lemma 2.6]. We claim that $\{\mathcal{B}_x : x\in X\}$ is an irreducible decomposition of the base $\mathcal{B}$ (see [5, Definition 2.3]). Since Y is discrete in ϱ it is enough to show that if $g\ne h$ are from Z with $g\in V(h,\alpha)$ for some $\alpha<\kappa$ then $V(h,\alpha)\not\subset V(g,0)$. Let $\delta=\Delta(g,h)$. Then $U(g,\delta+1)\subset V(h,\alpha)$ by ($\dagger$). Consider the element y of Y defined by the formulas $y\lceil\delta+1=g\lceil\delta+1$ and $y(\delta+1)=0$. Then $y\in X_g$ and so $y\notin V(g,0)$. On the other hand $y\in U(g,\delta+1)$, so $V(h,\alpha)\not\subset V(g,0)$. $\square$

Lemma 4.2. *If X is any topological space and $Z\subset X^\kappa$, $\alpha<\kappa$, $f:\alpha\to X$ are such that*

$$(\ddagger)\qquad \forall\,f'\in\mathrm{Fin}(\kappa\setminus\alpha,X)\;\exists\,g\in Z\;f\cup f'\subset g,$$

then $\mathrm{w}(Z)\ge\mathrm{w}(X)$.

The proof is similar to that of Lemma 3.7, so we omit it.

Let us recall that given a cardinal μ the *Singular Cardinal Hypothesis* (*SCH*) is said to hold below μ provided $\nu^{\mathrm{cf}(\nu)}=2^{\mathrm{cf}(\nu)}\nu^+$ for each singular cardinal $\nu\le\mu$. By [3, Lemma 1.8.1], if μ is regular and SCH holds below μ, then $\log(\mu^+)=\min\{\nu : 2^\nu\ge\mu^+\}$ is also regular. It is well-known that the failure of SCH requires the consistency of large cardinals, therefore the assumption of our next lemma is quite reasonable. Also note that *SCH* trivially holds below $\aleph_\omega$.

Theorem 4.3. *If μ and $\log(\mu^+)$ are both regular cardinals then there is a 0-dimensional T_2 space Y such that $\chi(Y)\,\mathrm{nw}(Y)\le\mu$, but Y is not the union of μ subspaces of weight μ.*

Proof: Let $\varrho=\log(\mu^+)$. Applying Theorem 4.1 for $\kappa=\varrho$ and $\lambda=2$ we get a space X with $\chi(X)\,\mathrm{nw}(X)=2^{<\varrho}\le\mu<\mathrm{w}(X)$. Let $Y=X^\varrho$ and consider a partition $Y=\bigcup_{\alpha<\varrho}Y_\alpha$. Since $\kappa=\varrho$ is regular, applying the technique of the standard proof of the Baire Category theorem we can see that the space $C^\varrho_{|X|}$ is not the union of ϱ nowhere dense subspaces. Therefore there are ordinals $\alpha,\xi<\varrho$ and a function $f:\alpha\to X$ such that

$$(\ddagger')\qquad \forall\,f'\in\mathrm{Fn}(\varrho\setminus\alpha,X,\varrho)\;\exists\,g\in Y_\xi\;f\cup f'\subset g.$$

But ($\ddagger'$) clearly implies ($\ddagger$). So, by Lemma 4.2, $\mathrm{w}(Y_\xi)\ge\mathrm{w}(X)>\mu$. Thus Y satisfies the requirements. The theorem is proved. $\square$

Problem 1 (ZFC). *If μ is a cardinal, is there a space X such that $\chi(X)\,\mathrm{nw}(X) = \mu$ (or just $\mathrm{nw}(X) = \mu$), but X is not the union of $< 2^\mu$ many subspaces of weight $< 2^\mu$ (or just $\mathrm{w}(X) = 2^\mu$)?*

Remark. The simplest case of Problem 1 left open by Theorem 4.1 is that $2^\omega = \omega_2$, $2^{\omega_1} = \omega_3$ and $\mu = \omega_1$. Indeed, if X^κ_λ is the space constructed in Theorem 4.1 and $\mathrm{nw}(X^\kappa_\lambda) = \lambda^{<\kappa} \le \omega_1$, then $\lambda \le \omega_1$ and $\kappa \le \omega$. So $\mathrm{w}(X^\kappa_\lambda) = \lambda^\kappa \le \omega_1^\omega = \omega_2 < 2^{\omega_1}$.

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