

On the asymmetric divisor problem with congruence conditions

MANFRED KÜHLEITNER

Abstract. A certain generalized divisor function $d^*(n)$ is studied which counts the number of factorizations of a natural number n into integer powers with prescribed exponents under certain congruence restrictions. An Ω -estimate is established for the remainder term in the asymptotic for its Dirichlet summatory function.

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Introduction

For $N = p + q \geq 2$ (where p and q are positive integers), and fixed natural numbers $a_1, \dots, a_p, a_{p+1} = b_1, \dots, a_{p+q} = b_q$, let $d^*(n)$ denote the number of ways to write the positive integer n as a product of different powers of N factors, of which p satisfy certain congruence conditions,

$$d^*(n) = d(a_1, \dots, a_N; m_1, \dots, m_p; n) = \#\{(u_1, \dots, u_N) \in \mathbb{N}^N : u_1^{a_1} \dots u_N^{a_N} = n, u_j \equiv l_j \pmod{m_j} \quad (j = 1, \dots, p)\},$$

where l_j and m_j are given natural numbers, with $l_j < m_j$.

For a large real variable x , we consider the remainder term $E(x)$ in the asymptotic formula

$$D^*(x) = \sum_{n \leq x} d^*(n) = H(x) + E(x)$$

where

$$H(x) = \sum_{s_0=0, \frac{1}{b_1}, \dots, \frac{1}{b_q}} \operatorname{Res}_{s=s_0} \left(F(s) \frac{x^s}{M^s s} \right)$$

where $M = m_1^{a_1} \dots m_p^{a_p}$ and $F(s)$ is the generating function

$$F(s) = M^s \sum_{n=1}^{\infty} d^*(n) n^{-s} = \prod_{j=1}^p \zeta(a_j s, \lambda_j) \prod_{i=1}^q \zeta(b_i s) \quad (\operatorname{Re} s > 1),$$

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$\lambda_j = \frac{l_j}{m_j}$ for $j = 1, \dots, p$ and $\zeta(s), \zeta(s, \cdot)$ denote the Riemann and Hurwitz zeta-functions, respectively.

Upper bounds for the error term $E(x)$ can be readily established as a trivial generalization of the corresponding results for the asymmetric divisor problem. For a historical survey see e.g. the textbooks of Ivić [7], Krätzel [8], Titchmarsh [16].

As in Nowak [10], [11] we generalize the asymmetric divisor problem with respect to arithmetic progressions. In the present paper, we shall be concerned with a lower bound for this remainder term. We therefore use a classical method of Szegő and Walfisz [14] with a more recent technique due to Hafner [5].

Remark. Throughout the paper we denote by $C(\lambda, \mu)$, λ, μ real numbers, the oriented polygonal line which joins the points $\lambda - i\infty, \lambda - i, \mu - i, \mu + i, \lambda + i\infty$ in this order.

Statement of results

Theorem 1. For each integer $m > \frac{1}{2}(N - 1)$, the Liouville-Riemann integral of order m of the error term $E(x)$ possesses an absolutely convergent series representation

$$\begin{aligned}
 E_m(x) &\stackrel{\text{def}}{=} \frac{1}{\Gamma(m)} \int_0^x (x - u)^{m-1} E(u) du = \\
 (1) \quad &= \pi^{\frac{N}{2} - \Sigma(1+m)} M^m \sum_{h=1}^{\infty} h^{-m} \sum_{\substack{(l_1, \dots, l_p) \\ (l_i=0,1)}} \beta(l_1, \dots, l_p; h) I_{l_1, \dots, l_p; m}^* \left(\frac{x}{M} \pi^\Sigma h \right)
 \end{aligned}$$

where $\Sigma = a_1 + \dots + a_N$ for short, and

$$\begin{aligned}
 (2) \quad &\beta(l_1, \dots, l_p; h) = \\
 &= \sum_{\substack{j_1, \dots, j_p, i_1, \dots, i_q \\ j_1^{a_1} \dots j_p^{a_p} i_1^{b_1} \dots i_q^{a_q} = h}} \frac{1}{j_1 \dots j_p i_1 \dots i_q} \prod_{k=1}^p (\sin(2\pi j_k \lambda_k))^{l_k} (\cos(2\pi j_k \lambda_k))^{1-l_k}.
 \end{aligned}$$

The functions $I_{l_1, \dots, l_p; m}^*(y)$ are defined, for every integer $m \geq 0$, by

$$\begin{aligned}
 I_{l_1, \dots, l_p; m}^*(y) &= \\
 &= \sum_{k=-1, \dots, -m} \text{Res}_{s=k} \left(G_{l_1, \dots, l_p}(s) \frac{\Gamma(s)}{\Gamma(s+m+1)} y^{s+m} \right) + I_{l_1, \dots, l_p; m}(y)
 \end{aligned}$$

where $I_{l_1, \dots, l_p; m}(y)$ is given by an absolutely convergent integral representation

$$I_{l_1, \dots, l_p; m}(y) = \frac{1}{2\pi i} \int_{C(\lambda, \mu)} G_{l_1, \dots, l_p}(s) \frac{\Gamma(s)}{\Gamma(s+m+1)} y^{s+m} ds.$$

Here λ, μ , are real numbers satisfying

$$\lambda > \frac{N}{2\Sigma}, \quad \mu < -m,$$

and

$$(3) \quad G_{l_1, \dots, l_p}(s) = \prod_{i=1}^q \frac{\Gamma(\frac{1}{2} - \frac{b_i s}{2})}{\Gamma(\frac{b_i s}{2})} \prod_{k=1}^p \left(\frac{\Gamma(\frac{1}{2} - \frac{a_k s}{2})}{\Gamma(\frac{a_k s}{2})} \right)^{1-l_k} \left(\frac{\Gamma(1 - \frac{a_k s}{2})}{\Gamma(\frac{1}{2} + \frac{a_k s}{2})} \right)^{l_k}.$$

The functions $I_{l_1, \dots, l_p; m}(y)$ possess an asymptotic expansion

$$(4) \quad \begin{aligned} I_{l_1, \dots, l_p; m}(y) &= \\ &= \sum_{j=0}^L C_{m,j} y^{m + \frac{1}{\Sigma}(-\frac{1}{2} + \frac{N}{2} + m - j)} \\ &\cos(e^{\frac{K}{\Sigma}} y^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N - 3) - \frac{\pi}{2}(l_1 + \dots + l_p) + \frac{\pi}{2}j - \pi m) + \\ &+ O(y^{m + \frac{N}{2\Sigma} - \frac{M+m+\frac{3}{2}}{\Sigma}}) \end{aligned}$$

where L is an arbitrary positive integer and the coefficients $C_{m,j}$ are computable. In particular, the leading coefficient is given by

$$C_{0,0} = \pi \sqrt{\frac{\pi}{2}} \Sigma^{1 - \frac{N}{2}} \prod_{i=1}^N \sqrt{a_i}.$$

Theorem 2. Let a^* be the minimum value of the numbers a_1, \dots, a_N and $\theta = \frac{1}{\Sigma}(-\frac{1}{2} + \frac{N}{2})$.

For $N \geq 4$, and $x \rightarrow \infty$,

$$E(a_1, \dots, a_N; m_1, \dots, m_p; x) = \Omega_{\pm}(x^{\theta}(\log x)^{a^* \theta}(\log \log x)^{q-1}(\log \log \log x)^{-\left(\frac{\Sigma}{2} + a^*\right)\theta}).$$

For $N \geq 2$ and $x \rightarrow \infty$,

$$E(a_1, \dots, a_N; m_1, \dots, m_p; x) = \Omega(x^{\theta}(\log x)^{a^* \theta}(\log \log x)^{q-1}(\log \log \log x)^{-\left(\frac{\Sigma}{2} + a^*\right)\theta}).$$

For the case of $N = 2$, this can be refined to

$$E(x) = \Omega_{\pm}((x(\log x)^{a^*})^{\theta}(\log \log \log x)^{-\left(\frac{\Sigma}{2} + a^*\right)\theta})$$

if

$$0 < \frac{l}{m} < \frac{1}{6} \quad \text{or} \quad \frac{1}{2} < \frac{l}{m} < \frac{5}{6}.$$

For the case of $N = 3$, the remainder term $E(x)$ satisfies

$$E(x) = \Omega_{\pm}((x(\log x)^{a^*})^{\theta}(\log \log \log x)^{-(\frac{\Sigma}{2}+a^*)\theta}),$$

if we induce only on one factor a congruence condition, and this satisfies

$$\frac{l}{m} \neq \frac{1}{2},$$

whereas if we induce congruence conditions on two factors, the remainder term $E(x)$ satisfies

$$E(x) = \Omega_{\pm}((x(\log x)^{a^*})^{\theta}(\log \log \log x)^{-(\frac{\Sigma}{2}+a^*)\theta}),$$

if

$$\log(2 \sin(\pi \frac{l_1}{m_1}))(\frac{1}{2} - \frac{l_2}{m_2}) + \log(2 \sin(\pi \frac{l_2}{m_2}))(\frac{1}{2} - \frac{l_1}{m_1}) \neq 0.$$

Proof of Theorem 1

A version of Perron's formula yields

$$\begin{aligned} (5) \quad D_m^*(x) &\stackrel{\text{def}}{=} \frac{1}{\Gamma(m)} \int_0^{\infty} (x-u)^{m-1} D^*(u) du = \\ &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{F(s)}{M^s} \frac{\Gamma(s)}{\Gamma(s+m+1)} x^{s+m} ds \end{aligned}$$

where m is an integer greater than $\frac{N}{2}$. Now we shift the line of integration left to zero, observing that for δ be a suitable small positive constant, then for each $\varepsilon > 0$

$$\zeta(\sigma + it, \lambda) \ll (1 + |t|)^{\frac{1}{2} + \varepsilon}$$

in $|t| \geq 1$, $\sigma \geq -\delta$ (this is a consequence of the Phragmén-Lindelöf principle). For the Gamma-functions involved, we recall Stirling's formula in the weak form

$$|\Gamma(\sigma + it)| \asymp |t|^{\sigma - \frac{1}{2}} \exp(-\frac{\pi}{2}|t|)$$

uniformly in $|t| \geq 1$, $\sigma_1 \leq \sigma \leq \sigma_2$, (σ_1, σ_2 arbitrary). From this it is an immediate consequence that the integrand in (5) is $\ll |t|^{-m-1+\frac{N}{2}+\varepsilon'}$ where ε' can be made arbitrarily small by the choice of δ . The sum of the residues at $s = 0, \frac{1}{b_1}, \dots, \frac{1}{b_q}$ is obviously just the order term $H(x)$, thus we obtain

$$E_m(x) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} F(s) \frac{\Gamma(s)}{\Gamma(s+m+1)} \frac{x^{m+s}}{M^s} ds$$

for the new integral is absolutely convergent, since $m > \frac{N}{2}$.

By the functional equations of the Riemann and the Hurwitz zeta-function (see e.g. [1], pp. 257–259)

$$\zeta(s) = \frac{1}{(2\pi)^{1-s}} 2\Gamma(1-s)\zeta(1-s)\sin\left(\frac{\pi}{2}s\right),$$

$$\zeta(s, \lambda) = \frac{1}{(2\pi)^{1-s}} 2\Gamma(1-s) \sum_{h=1}^{\infty} \frac{1}{h^{1-s}} \sin\left(2\pi h\lambda + \frac{\pi}{2}s\right) \quad (\operatorname{Re} s < 0),$$

we conclude that, for $\operatorname{Re} s < 0$,

$$F(s) = \frac{2^\Sigma}{(2\pi)^{\Sigma(1-s)}} \prod_{i=1}^N \Gamma(1-a_i s) \prod_{i=1}^q \zeta(1-b_i s) \sin\left(\frac{\pi}{2}b_i s\right) \times$$

$$\times \prod_{j=1}^p \sum_{h=1}^{\infty} \frac{1}{h^{1-a_j s}} \sin\left(2\pi h\lambda_j + \frac{\pi}{2}a_j s\right).$$

Inserting the Dirichlet series for all of the factors $\zeta(1-a_i s)$ gives,

$$F(s) = \frac{2^{\Sigma s}}{\pi^{\Sigma(1-s)}} \prod_{i=1}^N \Gamma(1-a_i s) \sum_{h=1}^{\infty} h^s \sum_{\substack{(l_1, \dots, l_p) \\ (l_i=0,1)}} \beta(l_1, \dots, l_p; h) \times$$

$$\times \underbrace{\prod_{i=1}^q \sin\left(\frac{\pi}{2}b_i s\right) \prod_{k=1}^p \left(\cos\left(\frac{\pi}{2}a_k s\right)\right)^{l_k} \left(\sin\left(\frac{\pi}{2}a_k s\right)\right)^{1-l_k}}_{G_{l_1, \dots, l_p}(s)},$$

with $\beta(l_1, \dots, l_p; h)$ defined in (2).

By well known properties of the Gamma function,

$$\Gamma(1-us)\sin\left(\frac{\pi}{2}us\right) = \sqrt{\pi}2^{-us} \frac{\Gamma\left(\frac{1}{2} - \frac{us}{2}\right)}{\Gamma\left(\frac{us}{2}\right)}$$

$$\Gamma(1-us)\left(\cos\left(\frac{\pi}{2}us\right)\right)^l \left(\sin\left(\frac{\pi}{2}us\right)\right)^{1-l} = \begin{cases} \sqrt{\pi}2^{-us} \frac{\Gamma\left(\frac{1}{2} - \frac{us}{2}\right)}{\Gamma\left(\frac{us}{2}\right)}, & \text{for } l = 0 \\ \sqrt{\pi}2^{-us} \frac{\Gamma\left(1 - \frac{us}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{us}{2}\right)}, & \text{for } l = 1 \end{cases}$$

we obtain

$$E_m(x) = \pi^{\frac{N}{2} - \Sigma(1+m)} M^m \sum_{h=1}^{\infty} h^{-m} \sum_{\substack{(l_1, \dots, l_p) \\ (l_i=0,1)}} \beta(l_1, \dots, l_p; h) I_{l_1, \dots, l_p; m}^{**} \left(\frac{x}{M} \pi^\Sigma h\right)$$

with

$$I_{l_1, \dots, l_p}^{**}(y) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} G_{l_1, \dots, l_p}(s) \frac{\Gamma(s)}{\Gamma(s+m+1)} y^{s+m} ds.$$

It is evident from the functional equation that all the singularities of $G_{l_1, \dots, l_p}(s)$ are on the positive real axis. Observing this, we can deform the line of integration such that $I_{l_1, \dots, l_p; m}^{**}(y) = I_{l_1, \dots, l_p; m}^*(y)$, provided that $\lambda \geq 0$ and $\mu < -m$. In order to get absolutely convergent integrals $I_{l_1, \dots, l_p; m}(y)$ for $m \geq 0$ we choose λ greater than $\frac{N}{2\Sigma}$. Therefore

$$(6) \quad \frac{d}{dy}(I_{l_1, \dots, l_p; m}^*(y)) = I_{l_1, \dots, l_p; m}^*(y).$$

(Notice that this is also valid for $I_{l_1, \dots, l_p; m}(y)$ for this differs from $I_{l_1, \dots, l_p}^*(y)$ only by a finite sum of differentiable functions.)

To complete the proof of Theorem 1, it remains to establish the asymptotic expansion of

$$(7) \quad G_{l_1, \dots, l_p}(s) \frac{\Gamma(s)}{\Gamma(s+m+1)}.$$

In what follows we write $R_k(s)$ for expressions of the form

$$R_k(s) = \sum_{j=1}^{L+1} c_{k,j} s^{-j}$$

where $c_{k,j}$ are any complex coefficients. We use Stirling's formula in the form

$$\log \Gamma(s+c) = (s+c - \frac{1}{2}) \log s - s + \frac{1}{2} \log 2\pi + R_1(s) + O(|s|^{-L-2})$$

with $c \in \mathbb{C}$ arbitrary, which holds uniformly for $|\arg(s+c)| \leq \beta_0 < \pi$. (The coefficients $c_{1,j}$ and the O -constant may depend on c .) Employing this we compute an asymptotic expansion for the logarithm of (7) and compare it with the asymptotic expansion of the logarithm of

$$(8) \quad \frac{\Gamma(-a's+b')}{\Gamma(\frac{1}{2} - \frac{a'}{2} + c')\Gamma(\frac{1}{2} + \frac{a'}{2}s - c')} e^{Ks+c}.$$

This yields that the logarithm of (7)

$$F_0(s) = C_m^* e^{Ks} \Gamma(-\Sigma s + \frac{N}{2} - m - \frac{1}{2}) \cos \pi(\frac{\Sigma s}{2} + 1 + m - \frac{N}{2} + \frac{1}{2}(l_1 + \dots + l_p))$$

has the same asymptotic expansion as the logarithm of (8), where

$$K = \frac{\Sigma}{2} \log\left(\frac{\Sigma}{2}\right) - \frac{\Sigma}{2} + \sum_{i=1}^N a_i \left(1 - \log\left(\frac{a_i}{2}\right)\right),$$

$$C_m^* = \exp\left(\frac{1}{2} \log(2\pi) + \log \pi + \left(1 + m - \frac{N}{2}\right) \log\left(\frac{\Sigma}{2}\right) + \sum_{i=1}^N \frac{1}{2} \log\left(\frac{a_i}{2}\right)\right).$$

Thus, on any set avoiding the poles of the terms involved,

$$\begin{aligned} G_{l_1, \dots, l_p}(s) \frac{\Gamma(s)}{\Gamma(s+m+1)} &= F_0(s)(1 + R_2(s) + O(|s|^{-L-2})) = \\ &= F_0(s) \left(1 + \sum_{j=1}^{L+1} c_j^* \prod_{i=1}^j \left(-\Sigma s + \frac{N}{2} - m - \frac{1}{2} - i\right) + O((1 + |s|^{-L-2}))\right) = \\ &= F_0(s) + \sum_{j=1}^{L+1} c_j^* F_j(s) + \Delta(s) \end{aligned}$$

with

$$F_j(s) = C_m^* e^{Ks} \Gamma\left(-\Sigma s + \frac{N}{2} - m - \frac{1}{2} - j\right) \cos \pi \left(\frac{\Sigma s}{2} + 1 + m - \frac{N}{2} + \frac{1}{2}(l_1 + \dots + l_p)\right)$$

by the functional equation for the Γ -function, and

$$\Delta(s) \ll |t|^{-L-2} |F_0(s)| \ll |t|^{-L-m-3+\frac{N}{2}-\Sigma\sigma}$$

uniformly in $|t| \geq 1$, $\sigma_1 \leq \sigma \leq \sigma_2$ (σ_1, σ_2 arbitrary). We can therefore bound the contribution of $\Delta(s)$ to the integral $I_{l_1, \dots, l_p; m}(y)$,

$$\int_{C(\Lambda, \mu)} \Delta(s) y^{s+m} ds \ll y^{\mu+m} + y^{\Lambda+m} \ll y^{m - \frac{L+m+\frac{3}{2}}{\Sigma} + \frac{N}{2}}$$

by the choice of $\Lambda = -\frac{L+m+\frac{3}{2}}{\Sigma} + \frac{N}{2}$ (notice that μ is only restricted by $\mu \leq -m$ and may therefore be assumed to be less than Λ). Consequently,

$$I_{l_1, \dots, l_p; m}(y) = J_{l_1, \dots, l_p; 0}(y) + \sum_{j=1}^{L+1} c_j^* J_{l_1, \dots, l_p; j}(y) + O\left(y^{m+\frac{N}{2\Sigma} - \frac{L+m+\frac{3}{2}}{\Sigma}}\right)$$

where, for $j = 0, 1, \dots, L+1$,

$$J_{l_1, \dots, l_p; j}(y) = \frac{1}{2\pi i} \int_{C(\lambda, \mu)} F_j(s) y^{s+m} ds.$$

To evaluate the remaining integrals, we use the following identity (valid for $\lambda_1 > \frac{1}{2}$, $\mu_1 < 0$, $z \in \mathbb{R}^+$),

$$\frac{1}{2\pi i} \int_{C(\lambda_1, \mu_1)} \Gamma(-s_1) \cos\left(\frac{\pi}{2}s_1 + \gamma\right) z^{s_1} ds_1 = \cos(z - \gamma)$$

(see e.g. [12]). Recalling the definition of $F_j(s)$, we substitute

$$s_1 = \Sigma * s - \frac{N}{2} + m + \frac{1}{2} + j, \quad \gamma = \frac{\pi}{2} * \left(\frac{3}{2} + m - N - j + (l_1 + \dots + l_p)\right), \quad z = (e^K * y)^{\frac{1}{\Sigma}}$$

in this last identity. After a few simple calculations the assertion of Theorem 1 follows, at least for $m \geq \frac{1}{2}Np$. But since $\sum_{h=1}^{\infty} \beta(l_1, \dots, l_p; h)h^{-\varepsilon} < \infty$ for each $\varepsilon > 0$, it is evident from (4) that the series in (1) converges absolutely for every $m > \frac{1}{2}(Np - 1)$. Appealing to (6), we complete the proof for this slightly larger range of m .

Proof of Theorem 2

We employ a classic method of Szegö and Walfisz [14] involving the Borel mean-value with more recent technique due to Hafner [5]. For a large real parameter t , we put

$$(9) \quad X = X(t) = K_1(\log t)^{-a^*} (\log \log \log t)^{\frac{\Sigma}{2} + a^*}$$

and

$$(10) \quad k = k(t) = K_2(\zeta + tX^{-\frac{1}{\Sigma}})^2,$$

with positive constants K_1, K_2 and real ζ to be specified later. We consider

$$B(t) = \frac{1}{\Gamma(k+1)} \int_0^\infty e^{-u} u^k E(u^{\frac{\Sigma}{2}} X) du.$$

We substitute $v = u^{\frac{\Sigma}{2}}$ and put $h(v) = \frac{2}{\Sigma} \exp(-v^{\frac{2}{\Sigma}}) v^{\frac{2(k+1)}{\Sigma} - 1}$.

We choose $m = \lfloor \frac{1}{2}N \rfloor + 1$ and observe that $h(v)$ and its first m derivatives vanish at $v = 0$ and at $v = \infty$ if t and thus k is sufficiently large. Therefore, an iterated integration by parts gives

$$B(t) = \frac{1}{\Gamma(k+1)} \int_0^\infty h(v) E(Xv) dv = \frac{(-1)^m X^{-m}}{\Gamma(k+1)} \int_0^\infty h^{(m)}(v) E_m(Xv) dv.$$

We insert the series representation (1), interchange the order of summation and integration and apply iterated integration by parts one more time, keeping (6) in mind. This leads to

$$B(t) = \pi^{\frac{N}{2} - \Sigma} \sum_{h=1}^{\infty} \sum_{\substack{(l_1, \dots, l_p) \\ (l_i=0,1)}} \beta(l_1, \dots, l_p; h) \frac{1}{\Gamma(k+1)} \times \\ \times \int_0^\infty e^{-u} u^k I_{l_1, \dots, l_p; 0}^* \left(u^{\frac{\Sigma}{2}} \frac{X}{M} \pi^\Sigma h\right) du.$$

Now we insert the asymptotic expansion (4) for the integrals $I_{l_1, \dots, l_p; 0}^*(y) = I_{l_1, \dots, l_p; 0}(y)$ and remark that $\beta(l_1, \dots, l_p; h) \ll h^\varepsilon$ for each $\varepsilon > 0$. We choose L so that the exponent of n in the error term of (4) be less than -1 . This is achieved for

$$L = \left[\frac{1}{2}(N - 3) + \Sigma \right] + 1.$$

The contribution of the O -term to the asymptotic expansion of $B(t)$ is then bounded by

$$\begin{aligned} &\ll \frac{k^\varepsilon}{\Gamma(k+1)} \int_0^\infty e^{-u} u^{k + \frac{N}{4} - \frac{1}{2}(L + \frac{3}{2})} du \ll \\ &\ll k^{\varepsilon + \frac{N}{4} - \frac{1}{2}(L + \frac{3}{2})} \ll k^{\varepsilon - \frac{1}{2}\Sigma} \ll k^{-\frac{1}{2}}, \end{aligned}$$

in view of Stirling's formula.

To deal with the main terms of (4), we make use of a result from classic analysis going back to Szegö [14], and Szegö and Walfisz [15].

Lemma 1. *Let α, c, c' , be real constants. Then for $k \rightarrow \infty$,*

$$\begin{aligned} J(k, T) &= \frac{1}{\Gamma(k+1)} \int_0^\infty e^{-u} u^{k+\alpha} \exp(iT\sqrt{u}) du = \\ &= \begin{cases} k^\alpha \exp(-\frac{1}{8}T^2) \exp(iT\sqrt{k}) + O(k^{\alpha - \frac{1}{2} + \varepsilon}) & \text{if } ck^{-\varepsilon} \leq T \leq ck^\varepsilon \\ \ll T^{-C} & \text{for every real constant } C, \text{ if } T \geq c'k^\varepsilon \end{cases} \end{aligned}$$

PROOF: This is an immediate consequence of a result of Szegö [14, pp. 100–102], and Szegö-Walfisz [15]. Applying this Lemma to the integrals which arise if we insert the significant terms of (4), we conclude that the main term, with $j = 0$ is of the form

$$\begin{aligned} &c^* \frac{(hX)^\theta}{\Gamma(k+1)} \int_0^\infty e^{-u} u^{k + \frac{\Sigma}{2}\theta} \cos(c_1(hX)^{\frac{1}{\Sigma}}\sqrt{u} + \frac{\pi}{4}(N-3) - \frac{\pi}{2}(l_1 + \dots + l_p)) du = \\ &= c^*(hX)^\theta k^{\frac{N-1}{4}} e^{-c_2(hX)^{\frac{2}{\Sigma}}} \cos(c_1(hX)^{\frac{1}{\Sigma}}\sqrt{k} + \frac{\pi}{4}(N-3) + \frac{\pi}{2}(l_1 + \dots + l_p)) + \\ &+ \begin{cases} O((hX)^\theta k^{\frac{N-3}{4} + \varepsilon}), & \text{for } ck^{-\varepsilon} \leq c_1(hX)^{\frac{1}{\Sigma}} \leq c'k^\varepsilon \\ \ll (hX)^{-C}, & \text{for every real constant } C, \text{ if } c_1(hX)^{\frac{1}{\Sigma}} > c'k^\varepsilon \end{cases} \end{aligned}$$

where $c^* = c_{0,0}(M^{-1}\pi^\Sigma)^\theta$ and $c_1 = (e^K\pi^\Sigma M^{-1})^{\frac{1}{\Sigma}}$.

The contribution of the other terms is

$$\ll (hX)^{\theta - \frac{1}{2}} k^{\frac{\Sigma}{2}(\theta - \frac{1}{2})} e^{-c_2(hX)^{\frac{2}{\Sigma}}} \ll (hX)^\theta k^{\frac{N-2}{4}},$$

for $c_1 k^{-\varepsilon} \leq c_1(hX)^{\frac{1}{\Sigma}} \leq c' k^\varepsilon$ and $j = 1, \dots, L$.

We estimate the contribution of the error term to the asymptotic expansion of $B(t)$. The terms corresponding to h which satisfy $c_1(hX)^{\frac{1}{\Sigma}} \leq c' k^\varepsilon$, contribute

$$\begin{aligned} &\ll \sum_{h \leq c_3 X^{-1} k^{\varepsilon \Sigma}} \sum_{\substack{(l_1, \dots, l_p) \\ (l_i=0,1)}} \beta(l_1, \dots, l_p; h) (hX)^\theta k^{\frac{N-2}{4}} \ll \\ &\ll X^\theta k^{\frac{N-2}{2}} (X^{-1} k^{\varepsilon \Sigma})^{1+\varepsilon+\theta} \ll k^{\frac{N-2}{4}+\varepsilon'} \ll k^{\frac{N}{4}-\frac{3}{8}}, \end{aligned}$$

whereas the terms corresponding to h which satisfy $c_1(hX)^{\frac{1}{\Sigma}} \geq c' k^\varepsilon$, contribute only

$$\ll \sum_{h \geq c_3 X^{-1} k^{\varepsilon \Sigma}} \sum_{\substack{(l_1, \dots, l_p) \\ (l_i=0,1)}} \beta(l_1, \dots, l_p; h) (hX)^{-C} \ll X^{-C} (X^{-1} k^{\varepsilon \Sigma})^{-1} = o(1)$$

as $t \rightarrow \infty$ by the choice of $C = 1 + \theta + 2$.

Altogether, we deduce that

$$\begin{aligned} B(t) &= C^{**} X^\theta k^{\frac{N-1}{4}} \sum_{h \leq c_3 X^{-1} k^{\varepsilon \Sigma}} \sum_{\substack{(l_1, \dots, l_p) \\ (l_i=0,1)}} \beta(l_1, \dots, l_p; h) \times \\ &\times h^\theta e^{-c_2(hX)^{\frac{2}{\Sigma}}} \cos\left(c_1(hX)^{\frac{1}{\Sigma}} \sqrt{k} + \frac{\pi}{4}(N-3) - \frac{\pi}{2}(l_1 + \dots + l_p)\right) + O\left(k^{\frac{N}{4}-\frac{3}{8}}\right) \end{aligned}$$

where

$$C^{**} = \pi^{\frac{N}{2} + \Sigma(\theta-1) + 1} M^{-\theta} \sqrt{\frac{\pi}{2}} \Sigma^{1-\frac{N}{2}} \prod_{i=1}^N \sqrt{a_i}.$$

In order to extend the range of summation in this series to $1 \leq h < \infty$, it suffices to observe that

$$\begin{aligned} &X^\theta k^{\frac{N-1}{4}} \sum_{h > c_3 X^{-1} k^{\varepsilon \Sigma}} \sum_{\substack{(l_1, \dots, l_p) \\ (l_i=0,1)}} \beta(l_1, \dots, l_p; h) h^\theta \exp(-c_2(hX)^{\frac{2}{\Sigma}}) \ll \\ &\ll k^{\frac{N-1}{4}} \sum_{h > c_3 X^{-1} k^{\varepsilon \Sigma}} \exp(-c_4(hX)^{\frac{2}{\Sigma}}) \ll \\ &\ll k^{\frac{N-1}{4}} \left(\exp(-c_5 k^{2\varepsilon}) + \int_{c_3 X^{-1} k^{\varepsilon \Sigma}}^\infty \exp(-c_4(uX)^{\frac{2}{\Sigma}}) du \right) \ll \\ &\ll \exp(-c_6 k^{2\varepsilon}) \ll k^{-1} \end{aligned}$$

Consequently,

$$(11) \quad B(t) = C^{**} X^\theta k^{\frac{N-1}{4}} \sum_{h=1}^{\infty} \sum_{\substack{(l_1, \dots, l_p) \\ (l_i=0,1)}} \beta(l_1, \dots, l_p; h) h^\theta \exp(-c_2(hX)^{\frac{2}{5}}) \times \\ \times (\cos(c_1(hX)^{\frac{1}{5}} \sqrt{k} + \frac{\pi}{4}(N-3) - \frac{\pi}{2}(l_1 + \dots + l_p))) + O(k^{\frac{N}{4} - \frac{3}{8}}).$$

We recall the definition of $\beta(l_1, \dots, l_p; h)$, keep $h = j_1^{a_1} \dots j_p^{a_p} i_1^{b_1} \dots i_q^{b_q}$ fixed for the moment and compute (with $Z = c_1(hX)^{\frac{1}{5}} \sqrt{k} + \frac{\pi}{4}(N-3)$ for short)

$$\begin{aligned} & \sum_{\substack{(l_1, \dots, l_p) \\ (l_i=0,1)}} \beta(l_1, \dots, l_p; h) \cos(Z - \frac{\pi}{4}(l_1 + \dots + l_p)) = \\ & = \sum_{\substack{(j_1, \dots, j_p, i_1, \dots, i_q) \\ j_1^{a_1} \dots j_p^{a_p} i_1^{b_1} \dots i_q^{b_q} = h}} \frac{1}{j_1 \dots j_p i_1 \dots i_q} \times \\ & \times \sum_{\substack{(l_1, \dots, l_p) \\ (l_i=0,1)}} \prod_{k=1}^p (\sin(2\pi j_k \lambda_k))^{l_k} (\cos(2\pi j_k \lambda_k))^{1-l_k} \cos(Z - \frac{\pi}{2}(l_1 + \dots + l_p)) = \\ & = \sum_{\substack{(j_1, \dots, j_p, i_1, \dots, i_q) \\ j_1^{a_1} \dots j_p^{a_p} i_1^{b_1} \dots i_q^{b_q} = h}} \frac{1}{j_1 \dots j_p i_1 \dots i_q} \cos(Z - 2\pi \sum_{j=1}^p h_j \lambda_j) \end{aligned}$$

by the general addition theorems for the cosine and sine functions.

We conclude that

$$\begin{aligned} B(t) = C^{**} k^{\frac{N-1}{4}} \{ & X^\theta \sum_{h=1}^{\infty} h^\theta \exp(-c_2(hX)^{\frac{2}{5}}) \times \\ & \times (a_h \cos(c_1(hX)^{\frac{1}{5}} \sqrt{k} + \frac{\pi}{4}(N-3)) + b_h \sin(c_1(hX)^{\frac{1}{5}} \sqrt{k} + \frac{\pi}{4}(N-3))) + \\ & + O(k^{-\frac{1}{8}}) \} \end{aligned}$$

where

$$\begin{aligned} a_h & = \sum_{\substack{(j_1, \dots, j_p, i_1, \dots, i_p) \\ j_1^{a_1} \dots j_p^{a_p} i_1^{b_1} \dots i_q^{b_q} = h}} \frac{\cos(2\pi \sum_{k=1}^p j_k \lambda_k)}{j_1 \dots j_p i_1 \dots i_q}, \\ b_h & = \sum_{\substack{(j_1, \dots, j_p, i_1, \dots, i_p) \\ j_1^{a_1} \dots j_p^{a_p} i_1^{b_1} \dots i_q^{b_q} = h}} \frac{\sin(2\pi \sum_{k=1}^p j_k \lambda_k)}{j_1 \dots j_p i_1 \dots i_q}. \end{aligned}$$

The next step is to approximate a finite partial sum of the infinite series in (11) by an expression of the form

$$f(X, \zeta) = \sum_{h \leq B_0} (a_h g_1(X, h, \zeta) + b_h g_2(X, h, \zeta)),$$

where, for short,

$$\begin{aligned} g_1(X, u, \zeta) &= \exp(-c_2(Xu)^{\frac{2}{\Xi}}) u^\theta \cos(\zeta(Xu)^{\frac{1}{N}} + \frac{\pi}{4}(N-3)), \\ g_2(X, u, \zeta) &= \exp(-c_2(Xu)^{\frac{2}{\Xi}}) u^\theta \sin(\zeta(Xu)^{\frac{1}{N}} + \frac{\pi}{4}(N-3)). \end{aligned}$$

Let a^* be the minimum value of $a_1, \dots, a_p, b_1, \dots, b_q$, then it is clear that if either of a_h, b_h is $\neq 0$, then h must be a^* -full. It is known that the number of a^* -full numbers $h \leq B_1$ is $\leq c_8 B_1^{\frac{1}{a^*}}$ (see e.g. Krätzel [8]). We now apply Dirichlet's approximation principle (see e.g. [8]): Let B_1 be a large positive integer and $q = \lceil (\log B_1)^N \rceil$. Then there exists a value of t in the interval

$$(12) \quad B_1 \leq t \leq B_1 q^{c_8 B_1^{\frac{1}{a^*}}}$$

such that $\| \frac{1}{2\pi} h^{\frac{1}{\Xi}} t \| \leq \frac{1}{q}$ for the a^* -full $h \leq B_1$, where $\| \cdot \|$ denotes the distance from the nearest integer. It is an easy consequence of (12) that

$$B_1 \gg (\log t)^{a^*} (\log q)^{-a^*}.$$

Let us define

$$B_0 = c_8 (\log t)^{a^*} (\log q)^{-a^*}$$

with c_8 so small that $B_0 \leq B_1$ for $q \geq 2$ and sufficiently large t .

Choosing in (10) $K_2 = c_1^{-2}$, we thus may conclude that

$$\begin{aligned} \left| \cos(c_1(hX)^{\frac{1}{\Xi}} \sqrt{k} + \frac{\pi}{4}(N-3)) - \cos(\zeta(hX)^{\frac{1}{\Xi}} + \frac{\pi}{4}(N-3)) \right| &\leq \\ &\leq \left\| \frac{1}{2\pi} h^{\frac{1}{\Xi}} t \right\| \leq \frac{1}{q}, \end{aligned}$$

for all $h \in \mathbb{N}$ with $h \leq B_0$ and $\beta(l_1, \dots, l_p; h) \neq 0$.

We consider the contribution from those h with $h \leq B_0$. This is bounded by

$$\begin{aligned}
 X^\theta \left| \sum_{h \leq B_0} h^\theta \exp(-c_2(hX)^{\frac{2}{\Sigma}}) \times \left((a_h \cos(c_1(hX)^{\frac{1}{\Sigma}} \sqrt{k} + \frac{\pi}{4}(N-3))) + \right. \right. \\
 \left. \left. + b_h \sin(c_1(hX)^{\frac{1}{\Sigma}} \sqrt{k} + \frac{\pi}{4}(N-3)) \right) - f(X, \zeta) \right| \ll \\
 \ll \frac{1}{q} X^\theta \sum_{h \leq B_0} h^\theta \exp(-c_2(hX)^{\frac{2}{\Sigma}}) \sum_{\substack{h_1, \dots, h_N \\ h_1^{a_1} \dots h_N^{a_N} = h}} \frac{1}{h_1 \dots h_N} \ll \\
 \ll \frac{1}{q} X^\theta \int_{1^-}^{B_0} \exp(-c_2(hX)^{\frac{2}{\Sigma}}) dS(u) \ll \\
 \ll \frac{1}{q} X^\theta B_0^\theta (\log B_0)^{N-1}
 \end{aligned}$$

where

$$S(u) = \sum_{h \leq u} h^\theta \sum_{\substack{h_1, \dots, h_N \\ h_1^{a_1} \dots h_N^{a_N} = h}} \frac{1}{h_1 \dots h_N} \asymp u^\theta (\log u)^{N-1}.$$

Those h with $h \geq B_0$ contribute,

$$\begin{aligned}
 & \ll X^\theta \sum_{h \geq B_0} h^\theta \exp(-c_2(hX)^{\frac{2}{\Sigma}}) \sum_{\substack{h_1, \dots, h_N \\ h_1^{a_1} \dots h_N^{a_N} = h}} \frac{1}{h_1 \dots h_N} \ll \\
 & \ll X^\theta \left\{ B_0^\theta (\log B_0)^{N-1} e^{(-c_2(B_0 X)^{\frac{2}{\Sigma}})} + \int_{B_0}^{\infty} e^{(-c_2(uX)^{\frac{2}{\Sigma}})} (uX)^{\frac{2}{\Sigma}-1} X S(u) du \right\}.
 \end{aligned}$$

We split up the last integral in $\int_{B_0}^{B_0^2} + \int_{B_0^2}^{\infty}$. The first integral contributes,

$$\begin{aligned}
 & \ll (\log B_0)^{N-1} \int_{B_0}^{B_0^2} \exp(-c_2(uX)^{\frac{2}{\Sigma}}) (uX)^{\frac{2}{\Sigma}} u^{\theta-1} du \ll \\
 & \ll B_0^\theta (\log B_0)^{N-1} \exp(-c_2(B_0 X)^{\frac{2}{\Sigma}}).
 \end{aligned}$$

In a similar way one verifies that the contribution of the second integral is $o(1)$ (as $t \rightarrow \infty$). In exactly the same way the infinite ‘‘tail’’ of the series in (11) can be estimated.

Combining this, we arrive at

$$\begin{aligned}
 B(t) = C^{**} k^{\frac{N-1}{4}} \left\{ X^\theta \sum_{h=1}^{\infty} h^\theta \exp(-c_2(hX)^{\frac{2}{\Sigma}}) \times \right. \\
 \times (a_h \cos(\zeta(hX)^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3)) + b_h \sin(\zeta(hX)^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3))) + \\
 \left. + O\left(\frac{1}{q} X^\theta B_0^\theta (\log B_0)^{N-1}\right) + O\left(X^\theta B_0^\theta (\log B_0)^{N-1} \exp(-c_2(B_0 X)^{\frac{2}{\Sigma}})\right) + o(1) \right\}.
 \end{aligned}$$

We conclude that

$$B(t) = C^{**} k^{\frac{N-1}{4}} \{X^\theta \sum_{h=1}^\infty h^\theta \exp(-c_2(hX)^{\frac{2}{\Xi}}) \times \\ \times (a_h \cos(\zeta(hX)^{\frac{1}{\Xi}} + \frac{\pi}{4}(N-3)) + b_h \sin(\zeta(hX)^{\frac{1}{\Xi}} + \frac{\pi}{4}(N-3))) + o(1)\}.$$

Our next step is an asymptotic formula for this last series, as $X \rightarrow 0^+$, ζ some real constant, in the spirit of Berndt [2]. To this end, we need an asymptotic formula for $S_1(u) = \sum_{h \leq u} h^\theta a_h$, $S_2(u) = \sum_{h \leq u} h^\theta b_h$. This can be done in one step.

For $\text{Re } s > 1$, consider the generating function of $a_h + ib_h$,

$$Z(s) \stackrel{\text{def}}{=} \sum_{h=1}^\infty \frac{a_h + ib_h}{h^s} = \prod_{i=1}^q \zeta(b_i s + 1 - b_i \theta) \prod_{k=1}^p \sum_{n=1}^\infty \frac{\exp(2\pi i n \lambda_k)}{n^{a_k s + 1 - a_k \theta}}, \quad \text{Re } s > 0.$$

By standard techniques it follows that

$$S_1(u) + iS_2(u) = \text{Res}_{s=\theta}(Z(S) \frac{u^s}{s}) + o(u^\rho) = B_q u^\theta (\log u)^{q-1} + O(u^\theta (\log u)^{q-2})$$

where $\rho < 1$ and

$$B_q = C(q) \prod_{k=1}^p \sum_{n=1}^\infty \frac{\exp(2\pi i n \lambda_k)}{n} = C(q) \prod_{k=1}^p (-\log(2 \sin(\pi \lambda_k)) + i(\frac{\pi}{2} - \pi \lambda_k)).$$

Let $B_q = |B_q| e^{2\pi i \beta}$ with $0 \leq \beta \leq 1$, then

$$(13) \quad S_1(u) + iS_2(u) = (|B_q| \cos(2\pi\beta) + i|B_q| \sin(2\pi\beta)) u^\theta (\log u)^{q-1} + \\ + O(u^\theta (\log u)^{q-2}).$$

□

Lemma 2. For $X \rightarrow 0^+$,

$$F(X, \zeta) \stackrel{\text{def}}{=} \sum_{h=1}^\infty h^\theta \exp(-c_2(hX)^{\frac{2}{\Xi}}) \times \\ \times ((a_h \cos(\zeta(hX)^{\frac{1}{\Xi}} + \frac{\pi}{4}(N-3)) + (b_h \sin(\zeta(hX)^{\frac{1}{\Xi}} + \frac{\pi}{4}(N-3)))) = \\ = c_6 |B_q| X^{-\theta} |\log X|^{q-1} (G(\zeta) + o(1)),$$

where

$$G(\zeta) = \int_0^\infty e^{-v^2} v^{-\frac{N-3}{4}} \cos(c_2^{-\frac{1}{2}} \zeta v - \frac{\pi}{4}(N-3) - 2\pi\beta) dv.$$

PROOF: With our previous notation, put $S(u) = S_1(u) + iS_2(u)$ and write $H_1(u) + iH_2(u)$ for the main term on the right hand side of the assertion of Lemma 2. Using Stieltjes integral notation

$$F(X, \zeta) = \int_0^\infty \exp(-c_2(uX)^{\frac{2}{\Sigma}}) \cos(\zeta(uX)^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3)) dS_1(u) + \int_0^\infty \exp(-c_2(uX)^{\frac{2}{\Sigma}}) \sin(\zeta(uX)^{\frac{1}{\Sigma}} + \frac{\pi}{4}(N-3)) dS_2(u).$$

Integration by parts and inserting the asymptotic expansion given in (1), we estimate the contribution of the error to be less than

$$\begin{aligned} &\ll \exp(-c_2(uX)^{\frac{2}{\Sigma}}) u^\theta (1 + \log u)^{q-1} |_{u=0}^\infty + \\ &+ \int_0^\infty \exp(-c_2(uX)^{\frac{2}{\Sigma}}) ((uX)^{\frac{2}{\Sigma}-1} X + (uX)^{\frac{1}{\Sigma}-1} X) u^\theta (1 + \log u)^{q-1} \ll \\ &\ll X^{-\theta} |\log X|^{q-2} \int_0^\infty (v^{\frac{2}{\Sigma}} + v^{\frac{1}{\Sigma}}) v^{\theta-1} |\log v|^{q-2} dv \ll X^{-\theta} |\log X|^{q-2}. \end{aligned}$$

We obtain the order term by a quite similar reasoning and a change of variable $v = \sqrt{c_2(nX)^{\frac{1}{\Sigma}}}$.

Using this Lemma, we arrive at our desired asymptotic expansion,

$$B(t) = c_{10} k^{\frac{N-1}{4}} |\log X|^{q-1} (G(\zeta) + o(1)) + o(k^{\frac{N-1}{4}})$$

with a positive constant c_{10} . □

We now make use of a deep result due to Steinig [13] which provides necessary and sufficient conditions for functions like our $G(\zeta)$ to have a change of sign.

Lemma 3. For $\zeta, B, \gamma \in \mathbb{R}, \gamma > -1$, let

$$G_{\gamma, B}(\zeta) \stackrel{\text{def}}{=} \int_0^\infty e^{-u^2} u^\gamma \cos(au + B\gamma) du.$$

Then $G_{\gamma, B}(\zeta)$ as a function of ζ has a sign change if and only if

$$(14) \quad \gamma > -2|B - [B + \frac{1}{2}]|.$$

Otherwise, $G_{\gamma, B}(\zeta) \neq 0$ for all real values of ζ .

For $N \geq 4$, (14) is satisfied for any choice of the λ_j . Thus there exist real numbers ζ_1 and ζ_2 and a positive constant c_{11} such that $G(\zeta_1) \leq -c_{11}$, $G(\zeta_2) \geq c_{11}$. We take once $\zeta = \zeta_1$, then $\zeta = \zeta_2$ in the definition (9), i.e. we put

$$k_i = k_i(t) = K_2(\zeta_i + tX(t)^{-\frac{1}{\Sigma}})^2 \quad (i = 1, 2),$$

define $B_i(t)$ like $B(t)$ before, with k replaced by k_i , and infer from the above argument that there exists an unbounded sequence of reals t with

$$B_1(t) \leq -c_{12}k_1^{\frac{N-1}{4}}(\log \log t)^{q-1}$$

$$B_2(t) \leq -c_{12}k_2^{\frac{N-1}{4}}(\log \log t)^{q-1}.$$

To complete the proof, let us suppose that, for some small positive constant K_3 ,

$$\pm E(x) \leq K_3(x(\log x)^{a^*})^\theta(\log \log x)^{q-1}(\log \log \log x)^{-\left(\frac{\Sigma}{2}+a^*\right)\theta}$$

for all sufficiently large x . By the definition of $B_i(t)$, this would imply that, for every large real t ,

$$(-1)^i B_i(t) \leq \frac{K_3}{\Gamma(k_i(t) + 1)} \int_0^\infty e^{-u} u^{k_i(t)} (X(t)u^{\frac{\Sigma}{2}})^\theta L(X(t)u^{\frac{\Sigma}{2}}) du$$

where $L(w) = (\log w)^{a^*\theta}(\log \log w)^{q-1}(\log \log \log w)^{-\left(\frac{\Sigma}{2}+a^*\right)\theta}$ for $w \geq 10$ and $L(w) = L(10)$ else. Estimating this integral by Hafner's Lemma 2.3.6 in [5, p. 51], we obtain

$$(-1)^i B_i(t) \leq c_{13}(k_i(t))^{\frac{N-1}{4}}(\log \log t)^{q-1}.$$

Together this yields a positive lower bound for K_3 (for both $i = 1, 2$) and thus completes the proof of Theorem 2. □

It remains to deal with the case that $N = 2, 3$.

Case $N = 2$. We have to check under which conditions (14) is satisfied. Comparing our asymptotic expansion with the Lemma of Steinig we have

$$\gamma = \frac{1}{2}(N - 3), \quad B = \frac{1}{4}(N - 3) - 2\beta.$$

Here $\gamma = \frac{1}{2}, B = -\frac{1}{4} - 2\beta$. Hence (14) becomes

$$\frac{1}{4} < \left| \frac{1}{4} + 2\beta + \left[\frac{1}{4} - 2\beta \right] \right|,$$

which is easily seen to be satisfied if and only if

$$0 < \beta < \frac{1}{4} \quad \text{or} \quad \frac{1}{2} < \beta < \frac{3}{4}.$$

Now β depends on $\lambda = \frac{l}{m}$ by the equation

$$B_1 = C_1(-\log(2 \sin(\pi\lambda)) + i\left(\frac{\pi}{2} - \pi\lambda\right)) \quad (\beta \in \mathbb{R}, 0 \leq \beta \leq 1).$$

This implies for the values λ ,

$$0 < \lambda < \frac{1}{6} \quad \text{or} \quad \frac{1}{2} < \lambda < \frac{5}{6},$$

which completes the proof of Theorem 2.

Case $N = 3$. Here we have $\gamma = 0$ and $B = -2\beta$, hence (14) is true if and only if $B \notin \mathbb{Z}$ or equivalently $\beta \notin \{0, \frac{1}{2}, 1\}$. Now there are two possibilities. In the case where $p = 1, q = 2$ the above arguments holds, and we simply get $\lambda \neq \frac{1}{2}$. In the second case, where $p = 2, q = 1$, we have that $\beta \notin \{0, \frac{1}{2}, 1\}$ is equivalent that

$$B_2 = C_2 \prod_{j=1,2} (-\log(2 \sin(\pi \lambda_j)) + i(\frac{\pi}{2} - \pi \lambda_j)).$$

We simplify this equation by writing u, v for $\frac{l_1}{m_1}, \frac{l_2}{m_2}$, which yields

$$(15) \quad \log(2 \sin(\pi u))(v - \frac{1}{2}) + \log(2 \sin(\pi v))(u - \frac{1}{2}) = 0.$$

Writing

$$w = 1 - v,$$

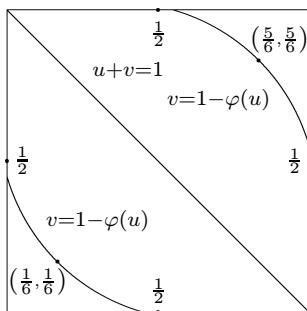
this equation simplifies to

$$(*) \quad \Phi(u) = \Phi(w),$$

with

$$\Phi(t) = \frac{\log(2 \sin(\pi t))}{t - \frac{1}{2}}.$$

Now $\Phi(t)$ decreases monotonically from $+\infty$ to $-\infty$ on each of the subintervals $]0, \frac{1}{2}[$ and $] \frac{1}{2}, 1[$. It follows that $(*)$ possesses two solutions for w : the trivial one $w = u$ and a second function $w = \phi(u)$ which is smooth on $]0, \frac{1}{2}[$ and on $] \frac{1}{2}, 1[$. Consequently, (15) is satisfied for $u + v = 1$ and for $v = 1 - \phi(u)$. Both curves are shown in the picture below. Note that the second one contains the rational points $(\frac{1}{6}, \frac{1}{6})$ and $(\frac{5}{6}, \frac{5}{6})$.



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INSTITUT FÜR MATHEMATIK, UNIVERSITÄT FÜR BODENKULTUR, GREGOR MENDEL-STR. 33,
A-1180 WIEN, AUSTRIA

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