Local cardinal functions of H-closed spaces

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Abstract. The cardinal functions of pseudocharacter, closed pseudocharacter, and character are used to examine H-closed spaces and to contrast the differences between H-closed and minimal Hausdorff spaces. An H-closed space X is produced with the properties that $|X| > 2^{2^{\psi(X)}}$ and $\overline{\psi}(X) > 2^{\psi(X)}$.

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For a compact Hausdorff space X it is well known (see e.g. [H]) that $\psi(X) = \overline{\psi}(X) = \chi(X)$, where ψ, $\overline{\psi}$ and χ are the local cardinal functions pseudocharacter, closed pseudocharacter and character respectively, and for any space X $\psi(X) \leq \overline{\psi}(X) \leq \chi(X)$. In [DP] the authors extend one of Arhangel'skii's cardinal inequalities for compact Hausdorff spaces, namely $|X| \leq 2^{\overline{\psi}(X)}$, to H-closed spaces. Since the inequality $|X| \leq 2^{\psi(X)}$ is true also for compact Hausdorff space, the question arises whether this formula is true for H-closed spaces. A well known example (see 4.8 in [PW]) is the Katětov H-closed extension $\kappa\omega$ of the discrete set of nonnegative integers ω. $\kappa\omega$ has the underlying set of $\beta\omega$ with the finer topology generated by $\tau(\beta\omega) \cup \{\omega \cup \{p\} : p \in \beta\omega\}$. $\kappa\omega$ has pseudocharacter $\aleph_0$, but $|\kappa\omega| = 2^{\mathfrak{c}}$. It seems reasonable to conjecture that $|X| \leq 2^{2^{\psi(X)}}$ should hold for any H-closed space X. Of course, this would then follow if we could establish that $\overline{\psi}(X) \leq 2^{\psi(X)}$ for an H-closed space X. It turns out that this is not the case and in the present paper we produce an H-closed space X such that $|X| > 2^{2^{\psi(X)}}$ and $\overline{\psi}(X) > 2^{\psi(X)}$. The H-closed space $\kappa\omega$ has the property $\psi(\kappa\omega) < \overline{\psi}(\kappa\omega) = \chi(\kappa\omega)$. If Y is the underlying set of unit interval I with the finer topology generated by $\tau(I) \cup \{U \backslash C : U \in \tau(I), C \in [I]^{\leq\omega}\}$ then the space Y is H-closed and satisfies $\psi(Y) = \overline{\psi}(Y) < \chi(Y)$. A natural question is whether there is an H-closed space Z satisfying (*) $\psi(Z) < \overline{\psi}(Z) < \chi(Z)$. If Z is a semiregular, H-closed space, i.e. Z is minimal Hausdorff, then $\overline{\psi}(Z) = \chi(Z)$. So obtaining a H-closed space Z satisfying (*) will be delicate, as the semiregularization Z_s of Z does not have this property. In this paper we develop a H-closed space Z satisfying (*). Another very important cardinal relation for compact Hausdorff spaces is the equality $nw(X) = w(X)$, where nw and w are the cardinal functions netweight and weight respectively. Examining the proof of this equality (see e.g. p. 170 of [E]) it is not difficult to realize that it actually holds for any minimal Hausdorff space X. Again the question arises whether such a relation can be true also for every H-closed space. The last example presented here, namely a countable H-closed space with uncountable character, will provide a negative answer to this question.

Henceforth all the spaces under consideration are assumed to be Hausdorff. For a space X and a point $p \in X$, let $\mathcal{U}_p$ denote the set $\{U \in \tau(X) : p \in U\}$. Recall that $\psi(p, X) = \min\{|\mathcal{U}| : \mathcal{U} \subseteq \mathcal{U}_p, \cap\mathcal{U} = \{p\}\}$, $\overline{\psi}(p, X) = \min\{|\mathcal{U}| : \mathcal{U} \subseteq \mathcal{U}_p, \bigcap_{U\in\mathcal{U}} cl_X U = \{p\}\}$ and $\chi(p, X) = \min\{|\mathcal{U}| : \mathcal{U} \subseteq \mathcal{U}_p, \mathcal{U}$ is a local base at $p\}$. Moreover $\psi(X) = \sup\{\psi(p, X) : p \in X\}$, $\overline{\psi}(X) = \sup\{\overline{\psi}(p, X) : p \in X\}$ and $\chi(X) = \sup\{\chi(p, X) : p \in X\}$. $w(X)$ is the smallest cardinality of a base of X and $nw(X)$ is the smallest cardinality of a network of X. A network of the space X is a family $\mathcal{S}$ of subsets such that every open set of X is an union of members of $\mathcal{S}$. Let X_s be the underlying set of X with the topology generated by the regular open sets of X. A subset A of X is regular open if $\mathrm{int}_X cl_X A = A$.

Example 1. *A large H-closed space of small pseudocharacter.*

Let Y be a discrete space such that $|Y|$ is not an Ulam measurable cardinal, e.g. Y is the set of all real numbers with the discrete topology. The space κY is H-closed and $(\kappa Y)_s = \beta Y$ (see [PW]). The points of $\kappa Y\backslash Y$ are the free ultrafilters on Y and $|\kappa Y\backslash Y| = |\beta Y\backslash Y| = 2^{2^{|Y|}}$. Therefore $|\kappa Y| = 2^{2^{|Y|}}$. Since $|Y|$ is not Ulam measurable, every ultrafilter p on Y does not have the countable intersection property; that is, there exists a countable family $\{P_n : n \in \omega\} \subseteq p$ such that $\bigcap_{n\in\omega} P_n = \varnothing$.

Since $\{p\} \cup P_n$ is an open neighbourhood of p in κY, it follows that $\psi(p, \kappa Y) = \aleph_0$. Thus $|\kappa Y| > 2^{2^{\psi(\kappa Y)}}$ and consequently, by the inequality in [DP], also $\overline{\psi}(\kappa Y) > 2^{\psi(\kappa Y)}$.

Comment: We do not know any example of a minimal Hausdorff space M such that $|M| > 2^{2^{\psi(M)}}$ or $\overline{\psi}(M) > 2^{\psi(M)}$.

Example 2. *A H-closed space X such that $\psi(X) < \overline{\psi}(X) < \chi(X)$.*

We start by modifying the topology of $\kappa\omega$. Let $\mathcal{F}$ be a uniform ultrafilter on the set $\kappa\omega\backslash\omega$ such that $\mathcal{F}$ has no base of size smaller than $2^{2^{\mathfrak{c}}}$. Clearly $\cap\mathcal{F} = \varnothing$. Since $\kappa\omega$ and $\beta\omega$ have the same underlying set, we can consider $\mathcal{F}$ as a filter on $\beta\omega$. The compactness of $\beta\omega$ implies that $\bigcap_{F\in\mathcal{F}} cl_{\beta\omega}(F) \neq \varnothing$ and the maximality of $\mathcal{F}$ guarantees that the previous intersection consists of a singleton. Let us denote by u the unique cluster point of $\mathcal{F}$ in $\beta\omega$. As $\beta\omega$ is compact, we see that $\mathcal{F}$ actually converges to u. Let $\kappa_{\mathcal{F}}\omega$ have the same underlying set of $\kappa\omega$ with the topology defined by declaring $U \subseteq \kappa_{\mathcal{F}}\omega$ be open if $p \in U\backslash(\omega \cup \{u\})$ implies there is some $A \in p$ such that $A \subseteq U$ and $u \in U$ implies there is some $F \in \mathcal{F}$ and $A \in u$ such that $A \cup F \subseteq U$. We have that $\tau(\beta\omega) \subset \tau(\kappa_{\mathcal{F}}\omega) \subset \tau(\kappa\omega)$. Now, since $\tau(\beta\omega) \subset \tau(\kappa_{\mathcal{F}}\omega)$ we have that $\kappa_{\mathcal{F}}\omega$ is Hausdorff and since $\tau(\kappa_{\mathcal{F}}\omega) \subset \tau(\kappa\omega)$ we have that $\kappa_{\mathcal{F}}\omega$ is H-closed. The topology of $\kappa_{\mathcal{F}}\omega$ differs from the topology of $\kappa\omega$ only at the point u and hence $\psi(p, \kappa_{\mathcal{F}}\omega) \leq \aleph_0$ for any $p \in \kappa_{\mathcal{F}}\omega\backslash\{u\}$. The argument in Example 1 yields $\psi(u, \kappa_{\mathcal{F}}\omega) = \aleph_0$ and therefore $\psi(\kappa_{\mathcal{F}}\omega) = \aleph_0$. Since $(\kappa_{\mathcal{F}}\omega)_s = \beta\omega$, the argument in Example 1 also shows that $\overline{\psi}(\kappa_{\mathcal{F}}\omega) = \overline{\psi}(\beta\omega) = \mathfrak{c}$. Let $\mathcal{B} \subset \mathcal{U}_u$ be a local base for u. The trace of $\mathcal{B}$ on the set $\kappa\omega\backslash\omega$ is a base for the filter $\mathcal{F}$ and therefore, size not smaller than $2^{2^{\mathfrak{c}}}$, we see that the character of u in $\kappa_{\mathcal{F}}\omega$ must be equal to $2^{|\kappa\omega|}$. Since $|\kappa\omega| = 2^{\mathfrak{c}}$, we have shown that $\chi(\kappa_{\mathcal{F}}\omega) = 2^{2^{\overline{\psi}(\kappa_{\mathcal{F}}\omega)}}$.

While the gap between pseudocharacter and closed pseudocharacter in an H-closed space can be arbitrarily large (Example 1), there is a certain link between closed pseudocharacter and character. In fact for any space X we have $\chi(X) \leq 2^{|X|}$ and consequently, by the inequality $|X| \leq 2^{\overline{\psi}(X)}$ mentioned at the beginning, we see that $\chi(X) \leq 2^{2^{\overline{\psi}(X)}}$ must hold for any H-closed space X. Notice that Example 2 also shows that the previous inequality is the best possible.

A well known property of a compact space X says that if $\chi(p, X) \geq \kappa$ for all $p \in X$ then $|X| \geq 2^\kappa$ (see the Cech-Pospisil's Theorem in [H]). It was shown in [DP] that an H-closed space can fail to have the previous property. The example developed in [DP] has the nice feature to be first countable, but its existence is only consistent. Here we present an easy example in ZFC when $\kappa = \mathfrak{c}$.

Example 3. *An H-closed space X satisfying $\chi(p, X) = \mathfrak{c}$ for all $p \in X$ and $|X| = \mathfrak{c}$.*

Let I be the unit interval with the usual topology. Enlarge the topology of I by declaring that all subsets of I having cardinality less than $\mathfrak{c}$ are closed and let X be the space so obtained. We have $X_s = I$ and so X is H-closed. Fix a point $p \in X$ and suppose there exists a local base $\mathcal{B}$ for p in X such that $|\mathcal{B}| < \mathfrak{c}$. Picking a point in each member of $\mathcal{B}$ other than p and letting C be the set so obtained, we see that no element of $\mathcal{B}$ can be contained in $(X \setminus C) \cup \{p\}$. Since the latter set is a neighbourhood of p in X, it follows that $\chi(p, X) \geq \mathfrak{c}$.

Example 4. *A countable H-closed space of uncountable character.*

Let $Y = \{(0,0)\} \cup \{(\frac{1}{n}, 0) : n \in N\} \cup \{(\frac{1}{n}, \frac{1}{m}) : n \leq m, n \in N, m \in N\}$. Y with the topology inherited from the Euclidean topology of the plane is compact. For any $A \subset N$ denote by $\hat{A}$ the set $\{(\frac{1}{n}, 0) : n \in A\}$. Fix a free ultrafilter $\mathcal{F}$ on N and for any $F \in \mathcal{F}$ put $F* = \hat{F} \cup (Y \setminus \hat{N})$. Every $F*$ is dense in Y. Now let X be the space obtained by enlarging the topology of Y in such a way that every set of the form $F*$ is open. Since the family $\{F* : F \in \mathcal{F}\}$ is a filter of dense subsets of Y, it follows that the semiregularization of X is just Y. This implies that X is H-closed. To finish we have to check that X is not first countable. This will be achieved by showing that the character of the point $(0,0)$ is not countable. Assume the contrary and let $\mathcal{G}$ be a countable fundamental system of neighbourhoods of $(0,0)$. For every $G \in \mathcal{G}$ let $H_G = \{n : (\frac{1}{n}, 0) \in G\}$. Since every $F*$ is a neighbourhood of $(0,0)$, it follows that for every $F \in \mathcal{F}$ there exists some $G \in \mathcal{G}$ such that $G \subset F*$ and consequently $H_G \subset F$. The next step is to verify that every H_G is actually a member of $\mathcal{F}$. Fix an element $G \in \mathcal{G}$. By the definition of the topology on X, there exist some neighbourhood U of $(0,0)$

in Y and some $F \in \mathcal{F}$ such that $U \cap F* \subset G$. Taking into account what the definition of the topology on Y, we see that there exists some $n^* \in N$ such that $\{(\frac{1}{n}, 0) : n > n^*\} \subset U$ and consequently $\{n : n > n^*\} \cap F \subset H_G$. As $\mathcal{F}$ is an ultrafilter, the set $\{n : n > n^*\} \cap F$ must belong to $\mathcal{F}$ and so $H_G \in \mathcal{F}$. In conclusion we have shown that the set $\{H_G : G \in \mathcal{G}\}$ is a base for $\mathcal{F}$. This is a contradiction, as it is well known that no free ultrafilter on N has a countable base. The proof is then complete. $\square$

It is not possible to have a countable H-closed space in which every point has uncountable character since every countable H-closed space has a dense set of isolated points (see [PW]).

Recalling that for any space X we always have $nw(X) \leq |X|$ and $\chi(X) \leq w(X)$, it is obvious that if X is the space in the above example then $nw(X) < w(X)$.

References

[DP] Dow A., Porter J., *Cardinalities of H-closed spaces*, Topology Proceedings **7** (1982), 27–50.

[E] Engelking R., *General Topology*, Warsaw, 1977.

[H] Hodel R.G., *Cardinal function I*, in Handbook of Set-theoretical Topology, K. Kunen and J. Vaughan ed., North Holland-Amsterdam, 1984.

[PW] Porter J.R., Woods R.G., *Extensions and Absolutes of Hausdorff Spaces*, Springer-Verlag, Berlin, 1988.

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