Comment.Math.Univ.Carolin. 37,2 (1996)207–216

Vector-valued sequence space $BMC(X)$ and its properties

Qing-Ying Bu

Abstract. In this paper, a vector topology is introduced in the vector-valued sequence space $BMC(X)$ and convergence of sequences and sequentially compact sets in $BMC(X)$ are characterized.

Keywords: vector-valued sequence space, topology, series, compact sets

Classification: 46A05, 40A05

1. Introduction

When A. Pietsch [4] gave characterizations for nuclearity of locally convex spaces in terms of vector-valued sequence spaces, he introduced a vector-valued sequence space $\ell_1(X)$ with values in a locally convex space X. And when Li Ronglu and Bu Qing-Ying [2] gave characterizations for a locally convex space which contains no copy of c_0, they introduced a vector-valued sequence space $BMC(X)$ with values in a locally convex space X. In fact, $\ell_1(X) = BMC(X)$, the space consisting of bounded multiplier convergent series in X. From [4], [2] it is obviously seen that the space $BMC(X)$ plays an important role in characterizing the structure of spaces in locally convex space theory. It will be seen in [1] that the space $BMC(X)$ also plays an important role in establishing Orlicz-Pettis type theorem for compact operators on locally convex spaces.

In Section 2 of this paper, we introduce a vector topology in the space $BMC(X)$ with values in a topological vector space X and characterize convergence of sequences in $BMC(X)$ and completeness of $BMC(X)$. In Section 3 we consider the space $BMC(X)$ with values in a locally convex space X and characterize sequentially compact subsets of $BMC(X)$ in different ways.

2. Convergence of sequences in $BMC(X)$

In this section, Let X be a separated topological vector space and U_X denote a local base of closed balanced neighbourhoods of 0 in X (see [5]). For a Banach space E, let $B(E)$ denote its closed unit ball. Let

$$BMC(X) = \Big\{\overline{x} = \{x_i\} \in X^{\mathbb{N}} : \text{series } \sum_i t_i x_i \text{ converges for each } \{t_i\} \in \ell_\infty\Big\}.$$

Then $BMC(X)$ is a sequence space with values in X. For a subset A of X, let

$$\widetilde{A} = \Big\{ \overline{x} \in BMC(X) : \sum_{i \geq 1} t_i x_i \in A \ \text{ for each } \ \{t_i\} \in B(\ell_\infty) \Big\}$$

and

$$\widetilde{U}_X = \Big\{ \widetilde{U} : U \in U_X \Big\}.$$

Proposition 2.1. *There is a unique vector topology for* $BMC(X)$ *for which* $\widetilde{U}_X$ *is a local base of neighbourhoods of* 0*. This vector topology will be denoted by* τ*.*

Proof: By Corollary 3 of [2], for each $\{x_i\} \in BMC(X)$ the set $\{\sum_{i \geq 1} t_i x_i : \{t_i\} \in B(\ell_\infty)\}$ is compact set in X and hence, is bounded. So it follows that $\widetilde{U}$ absorbs each $\overline{x}$ in $BMC(X)$ for each $U \in U_X$. In addition, it is easy to see that $\widetilde{U}$ is balanced for each $U \in U_X$. And for $U, V \in U_X$ such that $U + U \subset V$ it is easy to prove that $\widetilde{U} + \widetilde{U} \subset \widetilde{V}$. Thus we have proved $\widetilde{U}_X$ is an additive filterbase of balanced absorbing subsets of $BMC(X)$. Now, the proof follows from Theorem 5 of [5, p. 45]. □

For a net $\{\overline{x}^{\,\alpha}\}$ in $BMC(X)$, it is easy to see that

$$\tau - \lim_\alpha \overline{x}^{\,\alpha} = 0 \Longleftrightarrow \lim_\alpha \sum_{i \geq 1} t_i x_i^\alpha = 0 \tag{1}$$

uniformly for all $\{t_i\} \in B(\ell_\infty)$. Let

$$P_k : BMC(X) \longrightarrow X, \ P_k(\overline{x}) = x_k;$$

$$I_k : X \longrightarrow BMC(X), \ I_k(x) = (0, \ldots, 0, \overset{(k)}{x}, 0, 0, \ldots).$$

Then P_k and I_k are continuous linear maps, $k = 1, 2, \ldots$.

Lemma 2.2 ([3])**.** *Let* $x_{ij} \in X$ *for* $i, j \in \mathbb{N}$*. Suppose*

(I) $\lim_i x_{ij} = x_j$ *exists for each* $j \in \mathbb{N}$ *and*
(II) *for each increasing sequence* $\{m_j\}$ *of* $\mathbb{N}$ *there is a subsequence* $\{n_j\}$ *of* $\{m_j\}$ *such that* $\{\sum_{j \geq 1} x_{in_j}\}_{i=1}^\infty$ *is Cauchy.*

Then $\lim_i x_{ii} = 0$.

Theorem 2.3. *For* $\overline{x}^{\,(n)}, \overline{x} \in BMC(X)$*,* $n = 1, 2, \ldots$*, the following statements are equivalent:*

(i) $\tau\text{-}\lim_n \overline{x}^{\,(n)} = \overline{x}$.
(ii) $\lim_n \sum_{i \geq 1} t_i x_i^{(n)} = \sum_{i \geq 1} t_i x_i$ *for each* $\{t_i\} \in \ell_\infty$.
(iii) $\lim_n x_i^{(n)} = x_i$ *for* $i \in \mathbb{N}$*. And for each* $\{t_i\} \in \ell_\infty$*,* $\lim_k \sum_{i > k} t_i x_i^{(n)} = 0$ *uniformly for all* $n \in \mathbb{N}$.
(iv) $\lim_n x_i^{(n)} = x_i$ *for* $i \in \mathbb{N}$*. And* $\lim_k \sum_{i > k} t_i x_i^{(n)} = 0$ *uniformly for all* $n \in \mathbb{N}$ *and all* $\{t_i\} \in B(\ell_\infty)$.

Proof: (i) $\Rightarrow$ (iv). By (i), $\lim_n x_i^{(n)} = x_i$ obviously for $i \in \mathbb{N}$. Let $U, V \in U_X$ such that $V + V \subset U$. By (i), there is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\sum_{i \geq 1} t_i (x_i^{(n)} - x_i) \in V, \quad \{t_i\} \in B(\ell_\infty).$$

By Example 1 of [2], there is $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ and $n = 1, 2, \ldots, n_0$,

$$\sum_{i>k} t_i x_i^{(n)} \in U, \ \sum_{i>k} t_i x_i \in V, \ \{t_i\} \in B(\ell_\infty). \tag{2}$$

So for $k \geq k_0$ and $n > n_0$,

$$\sum_{i>k} t_i x_i^{(n)} = \sum_{i>k} t_i (x_i^{(n)} - x_i) + \sum_{i>k} t_i x_i \in V + V \subset U, \ \{t_i\} \in B(\ell_\infty). \tag{3}$$

Thus (iv) follows from (2) and (3).

(iv) $\Rightarrow$ (iii). Obviously.

(iii) $\Rightarrow$ (ii). Let $\{t_i\} \in \ell_\infty$, $U, V \in U_X$ such that $V + V + V \subset U$. By (iii), there is $k_0 \in \mathbb{N}$ such that

$$\sum_{i>k_0} t_i x_i^{(n)} \in V, \ \sum_{i>k_0} t_i x_i \in V, \ n = 1, 2, \ldots$$

and there is $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\sum_{i=1}^{k_0} t_i (x_i^{(n)} - x_i) \in V.$$

So for $n > n_0$,

$$\sum_{i \geq 1} t_i (x_i^{(n)} - x_i) = \sum_{i=1}^{k_0} t_i (x_i^{(n)} - x_i) + \sum_{i>k_0} t_i x_i^{(n)} - \sum_{i>k_0} t_i x_i \in V + V + V \subset U.$$

(ii) follows.

(ii) $\Rightarrow$ (i). By (ii), $\lim_n x_i^{(n)} = x_i$ obviously for $i \in \mathbb{N}$. If τ-$\lim_n \overline{x}^{(n)} \neq \overline{x}$, then there would exist $U \in U_X$, an increasing subsequence $\{n_k\}$ and $\{t_i^{(k)}\} \in B(\ell_\infty)$, $k = 1, 2, \ldots$ such that

$$\sum_{i \geq 1} t_i^{(k)} (x_i^{(n_k)} - x_i) \notin U, \quad k = 1, 2, \ldots .$$

For convenience, we can suppose that

$$\sum_{i\geq 1} t_i^{(n)}(x_i^{(n)} - x_i) \notin U, \quad n = 1, 2, \dots .$$

Let $V, W \in \mathcal{U}_X$ such that $V + V \subset W$ and $W + W \subset U$. Pick $m_1 \in \mathbb{N}$ such that $\sum_{i>m_1} t_i^{(1)}(x_i^{(1)} - x_i) \in V$. Then

$$\sum_{i=1}^{m_1} t_i^{(1)}(x_i^{(1)} - x_i) \notin V.$$

Set $n_1 = 1$. Since $\lim_n x_i^{(n)} = x_i$ for $i \in \mathbb{N}$, there is $n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that $\sum_{i=1}^{m_1} s_i(x_i^{(n_2)} - x_i) \in W$ for all $\{s_i\} \in B(\ell_\infty)$. It follows that $\sum_{i=1}^{m_1} t_i^{(n_2)}(x_i^{(n_2)} - x_i) \in W$. So $\sum_{i>m_1} t_i^{(n_2)}(x_i^{(n_2)} - x_i) \notin W$. Pick $m_2 \in \mathbb{N}$ with $m_2 > m_1$ such that $\sum_{i>m_2} t_i^{(n_2)}(x_i^{(n_2)} - x_i) \in V$. Then

$$\sum_{i=m_1+1}^{m_2} t_i^{(n_2)}(x_i^{(n_2)} - x_i) \notin V.$$

Proceeding in this manner we produce increasing sequences $\{n_k\}$ and $\{m_k\}$ such that

$$\sum_{i=m_k+1}^{m_{k+1}} t_i^{(n_{k+1})}(x_i^{(n_{k+1})} - x_i) \notin V, \quad k = 0, 1, 2, \dots, \tag{4}$$

here set $m_0 = 0$. Let

$$y_{kj} = \sum_{i=m_j+1}^{m_{j+1}} t_i^{(n_{j+1})}(x_i^{(n_{k+1})} - x_i).$$

Then $\lim_k y_{kj} = 0$ for $j \in \mathbb{N}$. Set $t_i = t_i^{(n_{j+1})}$ for $m_j < i \leq m_{j+1}$, $j = 0, 1, 2, \dots$, and $t_i = 0$ elsewhere. Then $\{t_i\} \in \ell_\infty$ and $\sum_{j\geq 0} y_{kj} = \sum_{i\geq 1} t_i(x_i^{(n_{k+1})} - x_i)$. By (ii), $\lim_k \sum_{j\geq 0} y_{kj} = 0$. So it follows from Lemma 2.2 that $\lim_k y_{kk} = 0$. This contradicts (4) and (i) follows.

The proof of Theorem 2.3 is complete. □

Proposition 2.4. *$BMC(X)$ is complete (or sequentially complete) space if and only if X is complete (or sequentially complete) space.*

Proof: If $BMC(X)$ is complete space, then it is easy to prove that X is complete. Conversely, if X is complete space, we will prove that $BMC(X)$ is complete space.

Let $\{\overline{x}^{\alpha}\}$ be Cauchy net in $BMC(X)$ and $U,V \in U_X$ such that $V+V+V \subset U$. Then for $\widetilde{V} \in \widetilde{U}_X$ there is α_0 such that for $\alpha,\beta \geq \alpha_0$, $\overline{x}^{\alpha} - \overline{x}^{\beta} \in \widetilde{V}$, i.e. for $\alpha,\beta \geq \alpha_0$,

$$\sum_{i\geq 1} t_i(x_i^{\alpha} - x_i^{\beta}) \in V, \quad \{t_i\} \in B(\ell_\infty). \tag{5}$$

By the continuity of P_i, $\{x_i^{\alpha}\}$ is Cauchy net in X and hence, there is $x_i \in X$ such that

$$\lim_{\alpha} x_i^{\alpha} = x_i, \quad i = 1,2,\dots\ . \tag{6}$$

From (5) it follows that for $\alpha,\beta \geq \alpha_0$ and each $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} t_i(x_i^{\alpha} - x_i^{\beta}) \in V, \quad \{t_i\} \in B(\ell_\infty).$$

So by (6) for $\alpha \geq \alpha_0$ and $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} t_i(x_i^{\alpha} - x_i) \in V, \quad \{t_i\} \in B(\ell_\infty).$$

Because of Example 1 of [2], there is $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\sum_{i>n} t_i x_i^{\alpha_0} \in V, \quad \{t_i\} \in B(\ell_\infty).$$

Thus for $n > n_0$ and $\alpha \geq \alpha_0$,

$$\begin{aligned}\sum_{i=1}^{n} t_i x_i - \sum_{i\geq 1} t_i x_i^{\alpha} = \sum_{i=1}^{n} t_i(x_i - x_i^{\alpha}) - \sum_{i>n} t_i(x_i^{\alpha} - x_i^{\alpha_0}) -& \\ - \sum_{i>n} t_i x_i^{\alpha_0} \in V+V+V \subset U, \quad \{t_i\} \in B(\ell_\infty).&\end{aligned}$$

It follows that the series $\sum_i t_i x_i$ converges for each $\{t_i\} \in \ell_\infty$, i.e. $\overline{x} = \{x_i\} \in BMC(X)$ and for $\alpha > \alpha_0$,

$$\sum_{i\geq 1} t_i x_i - \sum_{i\geq 1} t_i x_i^{\alpha} \in U, \quad \{t_i\} \in B(\ell_\infty).$$

So $\tau\text{-}\lim_{\alpha} \overline{x}^{\alpha} = \overline{x}$ and we have proved that $BMC(X)$ is complete. The proof is complete. $\square$

3. Compact sets in $BMC(X)$

In this section, let X be a locally convex space and X' its dual space. Then (X, X') forms a dual pair. Let U_X denote a local base of barrelled neighbourhoods of 0 in X. The gauge of $U \in U_X$ will be denoted by p_U and the polar of U will be denoted by U^0 (see [5]). It is easy to see that

$$p_U(x) = \sup\{|f(x)| : f \in U^0\}, \quad x \in X. \tag{7}$$

For each $U \in U_X$ and each $\overline{x} = \{x_i\} \in BMC(X)$, let

$$\varepsilon_U(\overline{x}) = \sup\Big\{p_U\Big(\sum_{i\geq 1} t_i x_i\Big) : \{t_i\} \in B(\ell_\infty)\Big\}. \tag{8}$$

Then $\varepsilon_U(\cdot)$ is a seminorm on $BMC(X)$ and the topology generated by the family of seminorms $\{\varepsilon_U(\cdot) : U \in U_X\}$ on $BMC(X)$ is just the original topology τ.

Proposition 3.1. *For each $U \in U_X$ and each $\overline{x} \in BMC(X)$,*

$$\varepsilon_U(\overline{x}) = \sup\Big\{\sum_{i\geq 1} |f(x_i)| : f \in U^0\Big\}. \tag{9}$$

The proof follows from (7) and (8).

For $t = \{t_i\} \in \ell_\infty$, let

$$\varphi_t : BMC(X) \longrightarrow X, \quad \varphi_t(\overline{x}) = \sum_{i\geq 1} t_i x_i.$$

Then for each $U \in U_X$, $p_U(\varphi_t(\overline{x})) \leq \varepsilon_U(\overline{x})$. So φ_t is a continuous linear map.

By Example 1 of [2], it is known that each $\{x_i\} \in BMC(X)$ has the following property:

$$\tau\text{-}\lim_n \sum_{i>n} I_i(x_i) = 0. \tag{$*$}$$

In order to consider a subset of $BMC(X)$, we give

Definition 3.2. A subset A of $BMC(X)$ is called uniformly convergent if τ-$\lim_n \sum_{i>n} I_i(x_i) = 0$ uniformly for all $\{x_i\} \in A$; A is called $\sigma(X, X')$-uniformly convergent if for each $f \in X'$, $\lim_n \sum_{i>n} |f(x_i)| = 0$ uniformly for all $\{x_i\} \in A$.

Theorem 3.3. *Let X be a sequentially complete space and A a subset of $BMC(X)$. Then A is relatively sequentially compact if and only if*

(a) *A is uniformly convergent and*
(b) *for each $i \in \mathbb{N}$, $P_i(A)$ is relatively sequentially compact subset of X.*

PROOF: If A is relatively sequentially compact, then (b) holds obviously. Next we will prove that (a) holds.

Suppose that (a) does not hold. Then there is $U \in U_X$ such that

$$\lim_n \sup \Big\{ \varepsilon_U \Big(\sum_{i>n} I_i(x_i) \Big) : \overline{x} = \{x_i\} \in A \Big\} \neq 0,$$

i.e.

$$\lim_n \sup \Big\{ p_U \Big(\sum_{i>n} t_i x_i \Big) : \{t_i\} \in B(\ell_\infty),\ \overline{x} = \{x_i\} \in A \Big\} \neq 0.$$

So there are $\varepsilon_0 > 0$, increasing subsequence $\{n_k\}$ of $\mathbb{N}$, $\{t_i^{(k)}\} \in B(\ell_\infty)$ and $\overline{x}^{(k)} \in A$ such that

$$p_U \Big(\sum_{i>n_k} t_i^{(k)} x_i^{(k)} \Big) \geq \varepsilon_0, \quad k = 1, 2, \dots \,. \tag{10}$$

Since A is relatively sequentially compact, there are a subsequence $\{\overline{x}^{(k_j)}\}_1^\infty$ of $\{\overline{x}^{(k)}\}_1^\infty$ and $\overline{x} \in BMC(X)$ such that $\tau\text{-}\lim_j \overline{x}^{(k_j)} = \overline{x}$. By Theorem 2.3,

$$\lim_m \sup \Big\{ p_U \Big(\sum_{i>m} t_i x_i^{(k_j)} \Big) : \{t_i\} \in B(\ell_\infty),\ j \in \mathbb{N} \Big\} = 0.$$

So there is n_{k_j} such that

$$p_U \Big(\sum_{i>n_{k_j}} t_i^{(k_j)} x_i^{(k_j)} \Big) < \varepsilon_0.$$

This contradicts (10). Thus we have proved that (a) holds.

Conversely, if the conditions (a) and (b) hold, we will prove that A is relatively sequentially compact. Let $\{\overline{x}^{(n)}\}_1^\infty \subset A$. By (b), using the diagonal method we can find a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_k x_i^{(n_k)}$ exists for $i \in \mathbb{N}$. For convenience, we can suppose that $n_k = k$, i.e.

$$\lim_n x_i^{(n)} \ \text{ exists}, \quad i = 1, 2, \dots \,. \tag{11}$$

By (a) for each $U \in U_X$ and $\varepsilon > 0$, there is $k_0 \in \mathbb{N}$ such that

$$\varepsilon_U \Big(\sum_{i>k_0} I_i(x_i) \Big) < \varepsilon/4 \ \text{ for } \ \{x_i\} \in A.$$

And furthermore, by (11) there is $n_0 \in \mathbb{N}$ such that for $n, m > n_0$,

$$p_U\big(x_i^{(n)} - x_i^{(m)}\big) < \varepsilon/2k_0, \quad i = 1, 2, \dots, k_0.$$

Thus for $n, m > n_0$,

$$\begin{aligned}\varepsilon_U\big(\overline{x}^{(n)} - \overline{x}^{(m)}\big) \le \sum_{i=1}^{k_0} p_U\big(x_i^{(n)} - x_i^{(m)}\big) + \varepsilon_U\Big(\sum_{i>k_0} I_i(x_i^{(n)})\Big) \\ + \varepsilon_U\Big(\sum_{i>k_0} I_i(x_i^{(m)})\Big) < \varepsilon.\end{aligned}$$

So $\{\overline{x}^{(n)}\}_1^\infty$ is a Cauchy sequence of $BMC(X)$ and hence, τ- $\lim_n \overline{x}^{(n)}$ exists in $BMC(X)$ by Proposition 2.4. Thus we have proved that A is relatively sequentially compact. The proof is complete.

□

Lemma 3.4. *For each $t = \{t_i\} \in \ell_\infty$, φ_t is c.c.t. – $\sigma(X, X')$ continuous on each $\sigma(X, X')$-uniformly convergent subset of $BMC(X)$, where c.c.t. denotes the coordinatewise convergence topology on $BMC(X)$.*

Proof: Let A be an $\sigma(X, X')$-uniformly convergent subset of $BMC(X)$ and $\{\overline{x}^\alpha\}$ be a net of A such that $\lim_\alpha x_i^\alpha = 0$ for $i \in \mathbb{N}$. Thus for $\varepsilon > 0$ and $f \in X'$, there is $n_0 \in \mathbb{N}$ such that

$$\sum_{i>n_0} |f(x_i)| < \varepsilon/2, \quad \text{for} \quad \overline{x} = \{x_i\} \in A.$$

And hence, there is α_0 such that for $\alpha > \alpha_0$,

$$|f(x_i^\alpha)| < \varepsilon/2n_0, \quad i = 1, 2, \dots, n_0.$$

So for $\alpha > \alpha_0$,

$$|f(\varphi_t(\overline{x}^\alpha))| \le \sum_{i=1}^{n_0} |f(x_i^\alpha)| + \sum_{i>n_0} |f(x_i^\alpha)| < \varepsilon.$$

Thus we have proved that $\sigma(X, X') - \lim_\alpha \varphi_t(\overline{x}^\alpha) = 0$. The proof is complete.

□

Theorem 3.5. *Let X be a sequentially complete space which contains no copy of c_0. Then a subset A of $BMC(X)$ is relatively sequentially compact if and only if*

(c) *A is $\sigma(X, X')$-uniformly convergent and*
(d) *for each $t \in \ell_\infty$, $\varphi_t(A)$ is relatively sequentially compact subset of X.*

PROOF: If A is relatively sequentially compact, then by the continuity of φ_t and Theorem 3.3, the conditions (c) and (d) hold.

Conversely, if the conditions (c) and (d) hold, we will prove that A is relatively sequentially compact. Let $\{\overline{x}^{(n)}\}_1^\infty \subset A$. By use of the proof of Theorem 3.3, we can suppose that

$$\lim_n x_i^{(n)} = x_i^{(0)} \in X, \quad i = 1, 2, \dots . \tag{12}$$

Next we will prove that $\overline{x}^{(0)} = \{x_i^{(0)}\} \in BMC(X)$.

For $f \in X'$, by (c) there is $k_0 \in \mathbb{N}$ such that $\sum_{i>k_0} |f(x_i)| \leq 1$ for each $\overline{x} \in A$. Since (d) implies (b), $\bigcup_{i=1}^{k_0} P_i(A)$ is a relatively sequentially compact subset of X and hence bounded. So there is a constant $c > 0$ such that

$$|f(P_i(\overline{x}))| = |f(x_i)| \leq c, \ \overline{x} = \{x_i\} \in A, \ i = 1, 2, \dots, k_0.$$

Thus

$$\sum_{i \geq 1} |f(x_i)| \leq k_0 c + 1, \quad \overline{x} = \{x_i\} \in A.$$

Now for a fixed $m \in \mathbb{N}$, by (12) there is an $n_0 \in \mathbb{N}$ such that

$$|f(x_i^{(n_0)} - x_i^{(0)})| < 1/m, \quad i = 1, 2, \dots, m.$$

So

$$\sum_{i=1}^m |f(x_i^{(0)})| \leq \sum_{i=1}^m |f(x_i^{(n_0)} - x_i^{(0)})| + \sum_{i=1}^m |f(x_i^{(n_0)})| \leq k_0 c + 2.$$

Since $m \in \mathbb{N}$ is arbitrary, we have $\sum_{i \geq 1} |f(x_i^{(0)})| \leq k_0 c + 2 < \infty$. Therefore, the series $\sum_i x_i^{(0)}$ is a weakly unconditionally Cauchy series in X. It follows from Theorem 4 of [2] that the series $\sum_i x_i^{(0)}$ is unconditionally convergent and hence bounded multiplier convergent. Thus we have proved that $\overline{x}^{(0)} \in BMC(X)$.

Now let $D = A \cup \{\overline{x}^{(0)}\}$. For each $t = \{t_i\} \in \ell_\infty$, since Lemma 3.4 implies that φ_t is c.c.t.$-\sigma(X, X')$ continuous on D, by (12) we have $\sigma(X, X')-\lim_n \varphi_t(\overline{x}^{(n)}) = \varphi_t(\overline{x}^{(0)})$. By use of the condition (d), we have $\lim_n \varphi_t(\overline{x}^{(n)}) = \varphi_t(\overline{x}^{(0)})$, i.e. $\lim_n \sum_{i \geq 1} t_i x_i^{(n)} = \sum_{i \geq 1} t_i x_i^{(0)}$. It follows from Theorem 2.3 that $\tau\text{-}\lim \overline{x}^{(n)} = \overline{x}^{(0)}$. So we have proved that A is relatively sequentially compact. The proof is complete. □

Remark 3.6. Condition (d) in Theorem 3.5 cannot be replaced by condition (b) in Theorem 3.3. For example, let $X = \ell_p$ $(1 < p < \infty)$, $e_i = (0, \dots, 0, \overset{(i)}{1}, 0, 0, \dots)$ and $A = \{(0, \dots, 0, e_n, 0, 0, \dots)\}_1^\infty$. Then $A \subset BMC(X)$ and it is easy to prove that A satisfies the conditions (b) and (c) but does not satisfy (a) and (d), and so is not relatively sequentially compact.

Acknowledgement. The author thanks his supervisors Prof. Wu Congxin and Prof. Li Ronglu for good guidance and help.

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Department of Mathematics, Harbin Institute of Technology, Harbin 150006, China

(*Received* October 19, 1994)