A short note on separable frames

Themba Dube

Abstract. Following the introduction of separability in frames ([2]) we investigate further properties of this notion and establish some consequences of the Urysohn metrization theorem for frames that are frame counterparts of corresponding results in spaces. In particular we also show that regular subframes of compact metrizable frames are metrizable.

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The notion of separability for frames was introduced in Dube [2] where, among other results, the Urysohn metrization theorem for frames was proved. In this paper we investigate further results among which is the frame analogue of the spatial result that a metrizable space is (uniformly) separable (i.e. the uniform covers have a basis consisting of countable covers) if and only if it is separable as a topological space. We also add to Sun's ([5]) list of conditions equivalent to having a countable base for metrizable frames.

We refer to Johnstone [3] for general background on frames. Recall that a frame L is called *separable* if there is a countable set $S \subseteq L - \{1\}$, called a *separator*, such that each nonzero element of L joins some member of S at the top. We take this opportunity to rectify a slight slip in [2] where it is mentioned that any countable chain is an example of a frame which has a countable base but is not separable. This of course is not entirely true since $\{0\}$ is a separator for the two-element chain. So one should exclude this case.

We showed in [2] that a paracompact separable frame is Lindelöf whence one deduces that a separable metrizable frame is Lindelöf. In fact for metrizable frames (as in spaces) separability and Lindelöfness are equivalent as we show below. First some terminology: Call a uniform frame *uniformly separable* in case the uniformity has a basis consisting of countable covers, and say it is of *countable type* (Banaschewski and Pultr [1]) if the uniformity has a countable basis. One notes the subtle difference between a metrizable frame and a uniform frame of countable type.

Proposition 1. *A uniform frame of countable type is uniformly separable iff its underlying frame is separable.*

Proof: (Sufficiency): Let (L, μ) be a uniformly separable frame of countable type and let ν be a countable basis for μ. For each $A \in \nu$ find a countable uniform cover B_A that refines A. Then $\bigcup\{B_A \mid A \in \nu\}$ is a countable base for L by Lemma 4.1 in Pultr [4]. Since L is regular the Urysohn metrization theorem yields the result.

(Necessity): Let L be the underlying frame of a uniform frame (uniformity μ) of countable type and assume L is separable. Note that L is metrizable and therefore paracompact (Sun [5]); so, by Proposition 3.6 in [2], L is Lindelöf. Now let ν be as above. Given $A \in \nu$ let B be a uniform star-refinement of A and let C be a countable subcover of B. The countable cover BC is refined by B since C is a cover; so it is uniform. Since $C \subseteq B$ and B star-refines A it follows that BC refines A. Consequently (L, μ) is uniformly separable. □

Corollary. *A Lindelöf metrizable frame is separable.*

Proof: Let L be a Lindelöf metrizable frame. So in Pultr's [4] notation, $L = L_\nu$ for some countable uniformity basis ν which generates some uniformity, μ, say. Now arguing as in the necessity part of Proposition 1 we see that (L, μ) is of countable type. Consequently L is separable since it is the underlying frame of a uniformly separable uniform frame of countable type. □

Adding the results above into Sun's [5] list, we have:

Proposition 2. *For a metrizable frame L, the following conditions are equivalent.*

(a) *L has a countable base.*
(b) *Each pairwise disjoint family in L is countable.*
(c) *L is separable.*
(d) *L is Lindelöf.*

Proof: The equivalence of (a) and (b) is Sun's [5] result. Since the implication (c) $\Rightarrow$ (a) appears in the Urysohn metrization theorem we are left with showing (a) $\Rightarrow$ (c). For this let A be a countable base for L and assume, without loss of generality, that $0 \notin A$. By regularity (from metrizability) choose, for each $a \in A$, an element $b_a \in L - \{1\}$ such that $a \vee b_a = 1$ and put $S = \{b_a \mid a \in A\}$. Now if $0 \neq x \in L$, then $x = \bigvee D$ for some nonempty $D \subseteq A$. Thus $x \vee s = 1$ for some $s \in S$, and so S is a separator. □

Remark 1. It was shown in [2] that, generally, quotients of separable frames are not separable. If however a frame is metrizable then, in view of the fact that quotients of metrizable frames are metrizable (Pultr [4]) and that a countable base for a frame determines a countable base for a quotient, we have that:

Quotients of separable metrizable frames are separable.

Remark 2. In view of Proposition 2 we immediately have that, as in spaces, compact metrizable frames are separable. Thus a frame with a metrizable compactification is separable.

Subframes of metrizable (or even compact metrizable) frames are generally not metrizable. Take for instance the three-element chain as a subframe of the complete Boolean algebra of four elements. If however one restricts to regular subframes of compact metrizable frames then the result holds. We note that this is, in a way, a frame analogue of the topological theorem that says the continuous image of a compact metric space in a Hausdorff space is metrizable. In order to show this we first note that if M is a subframe of L and r is the right adjoint of the inclusion map $M \to L$, then for any $a \in M$, $r(a) = a$.

Proposition 3. *A regular subframe of a compact metrizable frame is metrizable.*

Proof: Let M be a regular subframe of a compact metrizable frame L. Since a countable base is σ-discrete it suffices, by Theorem 4.3 in Pultr [4], to produce a countable base for M. Since L is compact metrizable it is separable and hence has a countable base, B, say. Let $r : L \to M$ be the right adjoint of the inclusion $M \to L$. We will show that the countable set $C = \{r(\bigvee F) \,|\, F \subseteq B \text{ is finite}\}$ is a base for M. Let $u \in M$ and take t in M which is rather below u in M. Then $t \wedge s = 0$ and $s \vee u = 1$ for some $s \in M$. Since B is a base for L, $u = \bigvee G$ for some $G \subseteq B$. So by compactness there exists a finite $F \subseteq G$ such that $s \vee \bigvee F = 1$. Thus $t \leq \bigvee F$, and therefore $t = r(t) \leq r(\bigvee F) \leq r(u) = u$. The regularity of M now shows that C is a base for M. □

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Department of Mathematics, University of Zululand, Private Bag X1001, KwaDlangezwa, 3886, South Africa

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