The sizes of relatively compact T_1-spaces

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Abstract. The relativization of Gryzlov's theorem about the size of compact T_1-spaces with countable pseudocharacter is false.

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Let X, Y be topological spaces such that $Y \subseteq X$. We say that Y is *relatively compact* (*relatively Lindelöf*) *in* X if for every open cover $\mathcal{U}$ of X there exists a finite (countable) subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}$ is a cover of Y. The notion of relative compactness was introduced (under a slightly different name) by D.V. Ranchin [R]. A.V Arhangel'skii showed that if Y is regular in X (i.e., for each closed subset B of X and each point $y \in X \backslash B$, there is an open neighborhood U of y in X such that $cl(U) \cap B = \emptyset$), X is first-countable at each point in Y, and Y is Lindelöf in X, then $|Y| \leq 2^{\aleph_0}$ (see Corollary 6 of [A] for a somewhat stronger result). Thus, Arhangel'skii's famous theorem on the size of first-countable regular Lindelöf spaces generalizes to the context of relative Lindelöfness.

In [G], A.A. Gryzlov proved that every compact T_1-space of countable pseudocharacter has cardinality at most $2^{\aleph_0}$. The question arises whether this theorem also generalizes to the context of relative compactness. Theorem 1 below shows that this is not the case.

1. Theorem. *Suppose λ is smaller than the first measurable cardinal. Then there exist first-countable T_1-spaces $Y \subset X$, such that $|Y| = \lambda$, $|X| = 2^\lambda$, and Y is compact in X.*

Proof: Let $\lambda \geq \aleph_0$ be as in the assumption. We let Y be λ itself with the discrete topology. Let Z be the set of all partitions $\bar{z} = (z_n)_{n\in\omega}$ of λ into countably many pairwise disjoint sets. The underlying set of the space X will be $Y \cup Z$. We define a topology on X as follows:

- the points in Y are isolated;
- the basic open neighborhoods of $\bar{z} \in Z$ are of the form $V_m^{\bar{z}} = \bigcup_{n\geq m} z_n \cup \{\bar{z}\}$.

Clearly, X and Y are first-countable T_1-spaces. It remains to show that Y is relatively compact in X. The latter is equivalent to the following:

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2. Claim. *For every function $f : Z \to \omega$ there exists a finite set $F \subset Z$ such that $|Y \setminus \bigcup_{\bar{z} \in F} \bigcup_{n \geq f(\bar{z})} z_n| < \aleph_0$.*

PROOF: Suppose not and let f be a counterexample. For each $F \in [Z]^{<\aleph_0}$ define: $Y_F = Y \setminus \bigcup_{\bar{z} \in F} \bigcup_{n \geq f(\bar{z})} z_n$. Then the family $\{Y_F : F \in [Z]^{<\aleph_0}\}$ generates a filter $\mathcal{F}$ of subsets of λ with the following properties:

(1) $\mathcal{F}$ contains no finite subset of λ;
(2) for every partition $\{z_n : n \in \omega\}$ of λ into countable many pairwise disjoint subsets there exists an $m \in \omega$ such that $\bigcup_{n \leq m} z_n \in \mathcal{F}$.

The existence of such an $\mathcal{F}$ leads to a contradiction, since it implies that λ is at least as big as the first measurable cardinal. To see the latter, consider the characteristic function of $\mathcal{F}$, i.e., the function $\chi_{\mathcal{F}} : \mathcal{P}(\lambda) \to \{0,1\}$ that takes the value $\chi_{\mathcal{F}}(a) = 1$ if and only if $a \in \mathcal{F}$. If we identify $\mathcal{P}(\lambda)$ and ${}^{\lambda}\{0,1\}$ with the product topology, then (1) implies that $\chi_{\mathcal{F}}$ is not continuous. On the other hand, if $\lim_{n\to\infty} a_n = a$ and $z_n = \{\xi \in \lambda : n = \min\{k \in \omega : \forall m \geq k\, (\xi \in a_m \leftrightarrow \xi \in a)\}\}$, then $\{z_n : n \in \omega\}$ is a partition of λ. Applying (2) to this partition, we can see that $\chi_{\mathcal{F}}$ is sequentially continuous. Now a result of Antonovskii and Chudnovskii [AC, Theorem 1.3] implies that λ is as least as big as the first measurable cardinal. □

In Theorem 1, the assumption that λ is smaller than the first measurable cardinal cannot be dropped, since the following relativized version of a well-known classical result holds:

3. Proposition. *Suppose X is a T_1-space and $\psi(x, X) < \kappa$ for every $x \in X$. If Y is relatively Lindelöf in X, then $|Y| < \kappa$.*

PROOF: Suppose X, Y form a counterexample. Let $\mu : \mathcal{P}(\kappa) \to \{0,1\}$ be a κ-additive measure that vanishes on singletons, and assume without loss of generality that $\kappa \subseteq Y$. For each $x \in X$, choose $\{U^x_\xi : \xi < \lambda_x < \kappa\}$ such that U^x_ξ is an open neighborhood of x and $\bigcap_{\xi < \lambda_x} U^x_\xi = \{x\}$. By κ-additivity of μ, we can pick for every $x \in X$ a ξ_x such that $\mu(\kappa \cap U^x_{\xi_x}) = 0$. Now let $\mathcal{U} = \{U^x_{\xi_x} : x \in X\}$. Then $\mathcal{U}$ is an open cover of X, but if $\mathcal{V} = [\mathcal{U}]^{<\kappa}$, then $\mu(\bigcup \mathcal{V} \cap \kappa) = 0$, hence $\mathcal{V}$ does not cover Y. □

References

[A] Arhangel'skii A.V., *A generic theorem in the theory of cardinal invariants of topological spaces*, Comment. Math. Univ. Carolinae **36.2** (1995), 303–325.

[AC] Antonovskii M.Y., Chudnovskii D.V., *Some questions of general topology and Tikhonov semifields II*, Russian Math. Surveys **31.3** (1976), 69–128.

[G] Gryzlov A.A., *Two theorems on the cardinality of topological spaces*, Soviet Math. Dokl. **21.2** (1980), 506–509.

[R] Ranchin D.V., *On compactness modulo an ideal*, Soviet Math. Dokl. **13.1** (1972), 193–197.

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