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A sufficient condition of full normality

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Abstract. We present a direct constructive proof of full normality for a class of spaces (locales) that includes, among others, all metrizable ones.

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Paracompactness of a topological space is equivalent to a range of other properties. Among them is full normality, introduced by Tukey in [11]. The equivalence was proved by A.H. Stone in [9]. In this paper, we investigate full normality in pointfree setting.

All metrizable locales are known to be fully normal. Our investigation was motivated originally by the aim to show as directly as possible how the existence of a countable admissible system of covers (the metrizability) provides an arbitrary cover with a star-refinement (the full normality). It turned out that the point is in the existence of a (downwards) well-ordered admissible system of covers, countability being a special case. If a well-ordered admissible system exists then we can present an explicit formula for a star-refinement of a given cover. The proof is constructive in the sense that it uses neither the axiom of choice nor the law of excluded middle.

1. Preliminaries

A *locale* is a complete lattice satisfying the distributivity law

$$\bigvee_{i\in J}(x_i \wedge y) = (\bigvee_{i\in J} x_i) \wedge y.$$

An example is the lattice ΩX of all open sets of a topological space X. Another example is a complete Boolean algebra. For more detail on locales, the reader can consult [2].

Let L be a locale. A *cover* of L is a subset C of L such that $\bigvee C$ equals the unit 1. In the rest of this section, C, D will denote covers of L. We say that C *refines* D, and write $C \leq D$, if for each $c \in C$ there exists $d \in D$ such that $c \leq d$. For arbitrary covers C, D we put $C \wedge D = \{c \wedge d \mid c \in C, d \in D\}$. This is a cover by virtue of the distributivity law; it refines both C and D.

Let $x \in L$. Put
$$C \cdot x = \bigvee \{c \in C \mid c \wedge x \neq 0\}.$$
(Cf. [5].) This element is called the *star* of x under C. One usually writes Cx instead of $C \cdot x$.

Further, define
$$x/C = \bigvee \{y \in C \mid Cy \leq x\}.$$
Sometimes we will omit parentheses in expressions like
$$(\dots(x/C_1)/C_2\dots)/C_n.$$
For instance, $x/C/D = (x/C)/D$.

We list some properties of these operations. They follow immediately from the definitions.

Fact 1.1. (1) *$x \leq Cx$ and $x/C \leq x$,*
(2) *if $x \leq y$ then $Cx \leq Cy$ and $x/C \leq y/C$,*
(3) *$Cx \leq y$ iff $x \leq y/C$,*
(4) *$x \geq C(x/C)$ and $x \leq (Cx)/C$,*
(5) *$C(\bigvee_{i\in J} x_i) = \bigvee_{i\in J}(Cx_i)$,*
(6) *if $C \leq D$ then $Cx \leq Dx$ and $x/C \geq x/D$.*

Define $C \cdot D = \{Cd \mid d \in D\}$ (cf. [4]). By (1) of Fact 1.1, we obtain a cover as $\bigvee_{d\in D} Cd \geq \bigvee_{d\in D} d = 1$.

Observation 1.2. *If $C \leq C'$ and $D \leq D'$ then $C \cdot D \leq C' \cdot D'$.*

As an operation on covers, $\cdot$ is neither associative nor commutative. We have the following fact, though.

Proposition 1.3. *$C(Dx) \leq (CD)x = C(D(Cx))$.*

Proof: Since $x \leq Cx$, it suffices to use (2) of Fact 1.1 twice to get $C(Dx) \leq C(D(Cx))$. We now wish to prove $(CD)x = C(D(Cx))$.
$$\begin{aligned}
(CD)x &= \bigvee_{d\in D} \{Cd \mid Cd \wedge x \neq 0\}\\
&= \bigvee_{d\in D} \{Cd \mid \exists\, c \in C \text{ such that } c \wedge x \neq 0 \text{ and } c \wedge d \neq 0\}\\
&= \bigvee_{d\in D} \{Cd \mid Cx \wedge d \neq 0\}\\
&= C(\bigvee_{d\in D} \{d \mid Cx \wedge d \neq 0\}) \quad \text{(by (5) of Fact 1.1)}\\
&= C(D(Cx)).
\end{aligned}$$

□

We say that C is a *star-refinement* of D (written $C \leq^* D$) if $C \cdot C \leq D$. A locale whose every cover has a star-refinement is called *fully normal*.

Let $\mathcal{T}$ be a system of covers. We write $x \lhd^{\mathcal{T}} y$ if there exists $S \in \mathcal{T}$ such that $Sx \leq y$. $\mathcal{T}$ is said to be *admissible* on a locale L (cf. [5]) if for all $y \in L$, $y = \bigvee \{x \in L \mid x \lhd^{\mathcal{T}} y\}$. We say that $\mathcal{T}$ is a *uniformity basis* if for each $S \in \mathcal{T}$ there exists $T \in \mathcal{T}$ such that $T \leq^* S$. A *uniform locale* is a couple $(L, \mathcal{T})$, where L is a locale and $\mathcal{T}$ an admissible uniformity basis.

One easily checks the following fact.

Fact 1.4. *$\mathcal{T}$ is admissible iff for all $y \in L$, $y = \bigvee_{S \in \mathcal{T}} y/S$.*

We recall several definitions of basic properties a locale L may have. For $x, y \in L$ we write $x \lhd y$ if there is $z \in L$ such that $z \wedge x = 0$ and $z \vee y = 1$. L is *regular* if for all $y \in L$, $y = \bigvee \{x \in L \mid x \lhd y\}$.

L is *compact* if for each cover $C \subseteq L$ there is a finite cover $D \subseteq C$.

L is *metrizable* if it has a countable admissible uniformity basis. (This is essentially the original extension of the notion of metrizability due to Isbell [1]. Later it has been shown ([6],[7]) that it is indeed equivalent to the existence of a metric structure on L. It is also equivalent to just possessing a countable admissible system of covers.)

A subset T of L is *locally finite* if there is a cover W with the property that the set $\{t \in T \mid w \wedge t \neq 0\}$ is finite for each $w \in W$. L is *paracompact* if every cover of L has a locally finite refinement. On paracompactness, consult [3] for the point-set view and, e.g., [8] or [10] for the pointfree one.

2. Powers of admissible systems

Let C, D denote covers. Define a cover C^n for $n \geq 1$ inductively by putting

$$C^1 = C \quad \text{and} \quad C^{n+1} = C \cdot C^n.$$

Lemma 2. *For any $n \geq 1$, $C \leq D$ implies $C^n \leq D^n$.*

PROOF: By induction, using 1.2. □

For $x \in L$ and $n \geq 1$, define $C^{(n)}(x)$ as follows:

$$C^{(1)}(x) = Cx; \quad C^{(n+1)}(x) = C \cdot (C^{(n)}(x)).$$

Similarly, define $x/^{(n)}C$ as

$$x/^{(1)}C = x/C; \quad x/^{(n+1)}C = (x/^{(n)}C)/C.$$

Observation 2.2. (1) $C^{(n+1)}(x) = C^{(n)}(Cx)$,
(2) $x/^{(n+1)}C = (x/C)/^{(n)}C$.

Fact 2.3. $C^{(n)}(x) \leq y$ *iff* $x \leq y/^{(n)}C$.

Proof: For $n = 1$, this is (3) of Fact 1.1. Assume validity for a given n. Then

$$\begin{aligned} &C^{(n+1)}(x) = C(C^{(n)}(x)) \le y \\ &\iff C^{(n)}(x) \le y/C \\ &\iff x \le (y/C)/^{(n)}C \\ &\iff x \le y/^{(n+1)}C \quad \text{by the preceding Observation.} \end{aligned}$$

□

The operations $C^n \cdot x$ and $C^{(n)}(x)$, as well as their adjoints, are closely related.

Fact 2.4. (1) $C^n \cdot x = C^{(2n-1)}(x)$,
(2) $x/C^n = x/^{(2n-1)}C$.

Proof: (1) For $n = 1$ the proposition holds. Let it hold for a given n. Then using 1.3,

$$\begin{aligned} C^{n+1}x &= (C \cdot C^n)x = C(C^n(Cx)) \\ &= C(C^{(2n-1)}(Cx)) = C^{(2n)}(Cx) \\ &= C^{(2(n+1)-1)}(x). \end{aligned}$$

(2) By 2.3, for any y,

$$\begin{aligned} y \le x/C^n &\iff C^n y \le x \\ &\iff C^{(2n-1)}(y) \le x \\ &\iff y \le x/^{(2n-1)}C. \end{aligned}$$

Putting first $y = x/C^n$ and then $y = x/^{(2n-1)}C$, we obtain the desired equality. □

Let $\mathcal{T}$ be a system of covers. It is said to be *down-directed* if it contains, with any $S, T \in \mathcal{T}$, a common refinement; that is, a cover R such that $R \le T$ and $R \le S$.

Lemma 2.5. *If $\mathcal{T}$ is down-directed then*

$$\bigvee_{S\in\mathcal{T}} x/S/S/S = \bigvee_{V\in\mathcal{T}} \bigvee_{U\in\mathcal{T}} \bigvee_{T\in\mathcal{T}} x/T/U/V.$$

Proof: For "$\le$", take $S = T = U = V$. On the other hand, for given T, U, V, consider a common refinement S. Then by (6) of Fact 1.1, $a/S/S/S \ge a/T/U/V$. □

Let $\mathcal{T}$ be a system of covers and $k \ge 1$. Put $\mathcal{T}^k = \{S^k \mid S \in \mathcal{T}\}$.

Proposition 2.6. *If $\mathcal{T}$ is a down-directed admissible system then $\mathcal{T}^k$ is admissible and down-directed as well.*

Proof: *Directedness.* Let $S^k, T^k \in \mathcal{T}^k$ be given. Let $R \in \mathcal{T}$ be a common refinement of S and T. By Lemma 2.1, R^k refines both S^k and T^k.

Admissibility. By induction. Basis ($k = 2$): Using (2) of Fact 2.4,

$$x \geq \bigvee_{S\in\mathcal{T}} x/S^2 = \bigvee_{S\in\mathcal{T}} x/S/S/S = \bigvee_{V\in\mathcal{T}} \bigvee_{U\in\mathcal{T}} \bigvee_{T\in\mathcal{T}} x/T/U/V = x$$

by admissibility of $\mathcal{T}$ used three times. Consequently, $\mathcal{T}$ is admissible.

Step: Since $S^{k+1} = S \cdot S^k \leq (S^k)^2$, we get

$$\bigvee_{S\in\mathcal{T}} x/S^{k+1} = \bigvee_{S\in\mathcal{T}} x/(S^k)^2 = \bigvee_{T\in\mathcal{T}^k} x/T^2$$

and since we assume $\mathcal{T}^k$ to be admissible (and know that it is down-directed), we may use the Basis to show that this expression equals x. Hence $\mathcal{T}^{k+1}$ is admissible. □

3. Construction of a star-refinement

Let $\mathcal{T}$ be a fixed system of covers and A a cover. We will now construct a subset $D_{\mathcal{T},A}$ of L, which will be, under certain assumptions on $\mathcal{T}$, a cover that star-refines A. Put

$$D_{\mathcal{T},A} = \{x \mid \exists\ S \in \mathcal{T},\ \exists\ a \in A \quad \text{s.t. } x \in S \text{ and } a/S^3 \wedge x \neq 0\}.$$

For $S \in \mathcal{T}$ and $a \in A$, $d_S(a)$ denotes the set $\{x \in S \mid a/S^3 \wedge x \neq 0\}$. Thus,

$$D_{\mathcal{T},A} = \bigcup \{d_S(a) \mid S \in \mathcal{T}, a \in A\}.$$

Observation 3.1. $\bigvee d_S(a) = S(a/S^3)$.

Fact 3.2. $a/S^3 \leq \bigvee d_S(a) \leq a/S^2$.

Proof: The first inequality follows from (1) of Fact 1.1. As for the second one, by (1) of Fact 2.4,

$$\begin{aligned} S(a/S^3) &= S(a/^{(5)}S) = S((a/^{(4)}S)/S) \\ &\leq a/^{(4)}S \quad \text{(by (4) of Fact 1.1)} \\ &\leq a/^{(3)}S = a/S^2. \end{aligned}$$

□

Remark. The lower bound we have just obtained will ensure that the elements of $D_{\mathcal{T},A}$ do constitute a cover while by virtue of the upper bound they are still small enough to star-refine A.

Lemma 3.3. *If $\mathcal{T}$ is down-directed and admissible then $D_{\mathcal{T},A}$ is a cover.*

Proof:

$$\begin{aligned}\bigvee D_{\mathcal{T},A} &= \bigvee_{a\in A}\bigvee_{S\in\mathcal{T}}\bigvee d_S(a) \geq \bigvee_{a\in A}\bigvee_{S\in\mathcal{T}} a/S^3 \\ &= \bigvee_{a\in A} a\end{aligned}$$

by Proposition 2.6. □

A system $\mathcal{T}$ of covers is *well-decreasing* if every subset of $\mathcal{T}$ has a maximum element with respect to the refinement relation. We will now state the main theorem of this article.

Theorem 3.4. *Let $\mathcal{T}$ be a well-decreasing admissible system of covers and A a cover. Then $D_{\mathcal{T},A}$ is a cover star-refining A. In particular, any locale that has such a system is fully normal.*

Proof: Since any well-decreasing system is down-directed, $D_{\mathcal{T},A}$ is a cover by the preceding lemma. We wish to prove that $D_{\mathcal{T},A} \leq^* A$.

Let $x \in D_{\mathcal{T},A}$; we investigate $D_{\mathcal{T},A} \cdot x$. Certainly $x \in d_S(a)$ for some $S \in \mathcal{T}$, $a \in A$. Put

$$\mathcal{M} = \{U \in \mathcal{T} \mid \exists\, c \in A, \exists\, y \in d_U(c) \text{ s.t. } y \wedge x \neq 0\}.$$

First of all, $x \in d_S(a)$ and $x \neq 0$ imply (1) $S \in \mathcal{M}$. In particular, $\mathcal{M}$ is not void. Let M be the maximum element of $\mathcal{M}$ with respect to $\leq$. Denoting the c and y one has for M as c_0 and y_0, we get $y_0 \wedge x \neq 0$ and $y_0 \in d_M(c_0)$. Since by (1), $S \leq M$, there exists $x' \in M$ such that $x \leq x'$ and $y_0 \wedge x' \neq 0$. Hence, (2) $x' \leq My_0$.

Consider $u \in D_{\mathcal{T},A}$ such that $u \wedge x \neq 0$. Necessarily $u \in d_U(b)$ for some $b \in A$ and $U \in \mathcal{T}$. Since $u \wedge x \neq 0$, $U \in \mathcal{M}$ and $U \leq M$. This implies that there is also $u' \in M$ such that $u' \geq u$. Clearly $u' \wedge x' \neq 0$, which, combined with (2), yields $u' \leq M(My_0)$. Now

$$\begin{aligned} u \leq u' &\leq M(My_0) \leq M^2 y_0 \\ &\leq M^2\left(\bigvee d_M(c_0)\right) \leq M^2(c_0/M^2) \quad \text{(by 3.2)} \\ &\leq c_0. \end{aligned}$$

Thus, for any $u \in D_{\mathcal{T},A}$, $u \wedge x \neq 0$ implies $u \leq c_0$, i.e. $D_{\mathcal{T},A} \cdot x \leq c_0$.

We have shown that for each $x \in D_{\mathcal{T},A}$ there exists $c_0 \in A$ with the property that $D_{\mathcal{T},A} \cdot x \leq c_0$, that is, $D_{\mathcal{T},A} \leq^* A$. This completes the proof. □

4. Remarks and examples

Two well-known facts are corollaries to the preceding theorem.

Remark 4.1. Let a locale L have a countable admissible system $\mathcal{T} = \{S_i\}_{i<\omega}$. Then the system $\widehat{\mathcal{T}} = \{S_0 \wedge S_1 \wedge \cdots \wedge S_k \mid k < \omega\}$ is a well-decreasing admissible system. Theorem 3.4 can be applied, yielding the fact that L is fully normal. Consequently, all metrizable locales are fully normal.

Remark 4.2. By metrization theorems (see [7]), existence of a countable admissible system of covers is equivalent to existence of a countable admissible uniformity basis. This fact can now be seen more directly.

Remark 4.3. *Constructivity.* The only point in the argument which depended on the axiom of choice was the proof of Lemma 2.5. There we considered a common refinement as a function of a triple T, U, V, but we did not specify the values. However, we use the lemma only for systems that are well-decreasing (and hence linear orders), in which case we can take for the common refinement the finest of the three covers. With this modification, the argument requires neither the axiom of choice nor the law of excluded middle.

Example 4.4. *A non-metrizable locale that has a well-decreasing admissible system.* In this and the following example, for an ordinal number α, the symbol $[\alpha]$ denotes the set $\{\beta \mid \beta < \alpha\}$. (We use a different name for the same thing to emphasize we have in mind a *set* of ordinals.) The successor of α is denoted by $\alpha + 1$. The symbol ω_1 stands for the first uncountable ordinal, ω for the first infinite one. Axiom of choice is assumed from now on.

Let $J = \{\{\alpha\} \mid \alpha < \omega_1\}$ and $\uparrow\!\beta = \{\alpha \mid \beta \leq \alpha \leq \omega_1\}$ for $\beta \leq \omega_1$.

Let X be the space $[\omega_1 + 1]$ with the topology generated by $J \cup \{\uparrow\!\beta \mid \beta < \omega_1\}$. Then ΩX has a well-decreasing admissible system but it is not metrizable. To see the former, put $S_\beta = J \cup \uparrow\!\beta$ and observe that $\{S_\beta\}_{\beta<\omega_1}$ meets the condition.

Suppose that ΩX is metrizable. Let $\mathcal{T}$ be the countable admissible system. Define

$$g : \mathcal{T} \to [\omega_1 + 1] : S \mapsto \min\{\alpha \mid \exists\, u \in S \text{ s.t. } \alpha \in u \text{ and } \omega_1 \in u\}.$$

Lemma 4.5. $\sup_{S\in\mathcal{T}} g(S) = \omega_1$.

Proof: Let $\beta \geq g(S)$ for each $S \in \mathcal{T}$ and suppose $\beta < \omega_1$. Then for any S and u such that $\omega_1 \in u$ we have $Su \subseteq \uparrow\!(\beta+1)$. Therefore $\omega_1 \notin \bigcup\{u \mid u \lhd^{\mathcal{T}} \uparrow\!(\beta + 1)\}$. But, as $\omega_1 \in \uparrow\!(\beta + 1)$, this contradicts the admissibility of $\mathcal{T}$. □

Since $g(S) < \omega_1$ for all $S \in \mathcal{T}$, and since $\mathcal{T}$ is countable, we have shown that ω_1 has to be a supremum of countably many countable ordinals, which is impossible. Hence ΩX is not metrizable.

Example 4.6. *A compact regular locale that has no well-decreasing admissible system.* Let Y be $[\omega_1+1]$ with the usual interval topology. ΩY is compact regular but we will show that does not have a well-decreasing admissible system.

Assume it does and call the system $\mathcal{T}$. For a limit ordinal $\kappa \le \omega_1$, put

$$g_\kappa : \mathcal{T} \to [\kappa] :$$
$$S \mapsto \min\{\alpha \in [\omega_1+1] \mid \exists\, u \in S \text{ s.t. } \alpha \in u \text{ and } \kappa \in u\} \quad \text{and}$$
$$f_\kappa : [\kappa] \to \mathcal{T} :$$
$$\alpha \mapsto \max\{S \in \mathcal{T} \mid g_\kappa(S) \ge \alpha\}.$$

Verification. We have to make sure the definitions are correct, that is, to check the domain and range of (a) g_κ and (b) f_κ.

(a) g_κ is certainly defined on the whole of $\mathcal{T}$. By the definition, $g_\kappa(S) \le \kappa$, but since κ is not the infimum of any open set (being a limit ordinal), the values are really in $[\kappa]$.

(b) First, observe that $\sup_{S\in\mathcal{T}} g_\kappa(S) = \kappa$. This can be proved in the same fashion as the lemma in the preceding example. Therefore, no $\alpha < \kappa$ is an upper bound for $\{g_\kappa(S)\}_{S\in\mathcal{T}}$. From this it follows that f_κ is well defined on the whole of $[\kappa]$. Its values clearly lie in $\mathcal{T}$.

Lemma 4.7. *Consider f_κ and g_κ as mappings between the posets $([\kappa], \le)$ and $(\mathcal{T}, \le)$.*

(1) *f_κ is non-decreasing and g_κ is non-increasing,*
(2) *$\alpha \le g_\kappa(S)$ iff $S \le f_\kappa(\alpha)$,*
(3) *$g_\kappa f_\kappa(\alpha) \ge \alpha$ and*
(4) *$g_\kappa(S) > g_\kappa(T)$ implies $S < T$.*

Proof: (1) and (2) follow straight from the definition. So does (3) from (2). It remains to prove (4). Let $S \not< T$. Then since $\mathcal{T}$ is a linear order, $S \ge T$. By (1), $g_\kappa(S) \le g_\kappa(T)$, which means $g_\kappa(S) \not> g_\kappa(T)$ and we are finished. □

Proposition 4.8. $\sup_{n<\omega} g_{\omega_1} f_\omega(n) = \omega_1$.

Proof: Let $\beta \ge g_{\omega_1} f_\omega(n)$ for all $n < \omega$. Suppose $\beta < \omega_1$. Fix $n < \omega$. Then by (3), $g_{\omega_1} f_{\omega_1}(\beta+1) \ge \beta+1 > \beta \ge g_{\omega_1} f_\omega(n)$. According to (4), $f_{\omega_1}(\beta+1) < f_\omega(n)$. But by (1), $g_\omega f_{\omega_1}(\beta+1) > g_\omega f_\omega(n) \ge n$, where the last inequality follows from (3).

We conclude that for all $n < \omega$, $g_\omega f_{\omega_1}(\beta+1) > n$. On the other hand, for any S we have $g_\omega(S) \in [\omega]$, so that $g_\omega f_{\omega_1}(\beta+1) < \omega$. This is a contradiction. □

Let us return to the example. Since $g_{\omega_1}(S) < \omega_1$ for each S, we have shown that ω_1 has to be the supremum of a countable set of countable ordinals. This cannot happen and so ΩY has no well-decreasing admissible system.

Note. A similar argument can be employed to prove the following stronger statement: If ΩZ has a well-decreasing admissible system and if there is a point in Z with an infinite countable basis of neighbourhoods, then all $x \in Z$ have countable bases of neighbourhoods.

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