Homogeneous Einstein metrics on Stiefel manifolds

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Abstract. A Stiefel manifold $V_k\mathbf{R}^n$ is the set of orthonormal k-frames in $\mathbf{R}^n$, and it is diffeomorphic to the homogeneous space $SO(n)/SO(n-k)$. We study $SO(n)$-invariant Einstein metrics on this space. We determine when the standard metric on $SO(n)/SO(n-k)$ is Einstein, and we give an explicit solution to the Einstein equation for the space $V_2\mathbf{R}^n$.

Keywords: Riemannian geometry, homogeneous spaces, Einstein metrics, Stiefel manifolds

Classification: 53C20, 53C25, 53C30

1. Introduction

Homogeneous geometry studies various geometric quantities (geodesics, Einstein metrics, harmonic maps, Laplace transform, to name a few) on spaces of the form G/H, where G is a compact Lie group and H a closed subgroup of G. Well known classes of these spaces are the symmetric spaces and the generalized flag manifolds. Their geometry has been studied quite extensively in various aspects. However, little is known about more general homogeneous spaces such as Stiefel manifolds $V_k\mathbf{R}^n = SO(n)/SO(n-k)$ of orthonormal k-frames in $\mathbf{R}^n$, or Aloff-Wallach spaces $SU(3)/i_{k,l}(S^1)$ (where $(k,l) = 1$ and $i_{k,l}$ a certain embedding of the circle S^1 in $SU(3)$).

In the present paper we are interested in $SO(n)$-invariant Einstein metrics on Stiefel manifolds. An Einstein metric is a Riemannian metric g with the property $Ric(g) = cg$. Einstein metrics on Aloff-Wallach spaces have been studied by M. Wang [10] and O. Kowalski, Z. Vlášek [8]. The existence of $SO(n)$-invariant Einstein metrics on Stiefel manifolds is due to A. Sagle [9] and G. Jensen [5]. The first author reduced the problem to an algebraic system of equations, and the second used the method of Riemannian submersions. A special case is the unit tangent bundle $T_1S^n = V_2\mathbf{R}^n$ of S^n, which follows from [6] (see also [1, 9.77]). In our work we are interested in explicit solutions for the Einstein equation on the Stiefel manifolds $V_2\mathbf{R}^n$.

The paper is organized as follows: In Section 2 we provide some preliminary information about homogeneous spaces, and in Section 3 we examine when the standard metric on $V_k\mathbf{R}^n$ is Einstein. We give an elementary proof that this happens if and only if $V_k\mathbf{R}^n = S^n$ or $SO(n)$ (this result can also be deduced from [11]). In Section 4 we study a certain family of $SO(n)$-invariant metrics on $V_k\mathbf{R}^n$ and obtain a unique solution for the Einstein equation on $V_2\mathbf{R}^n$ for that particular family.

2. Preliminaries

Let $M = G/H$ be a compact, simply connected homogeneous space, where G is a semisimple, compact and connected Lie group, and H is a closed subgroup of G. Let $\mathbf{g}$ and $\mathbf{h}$ denote the Lie algebras of G and H respectively. We denote by $\mathrm{Ad}_G : G \to \mathrm{Aut}(\mathbf{g})$ (respectively $\mathrm{ad}_{\mathbf{g}} : \mathbf{g} \to \mathrm{End}(\mathbf{g})$) the adjoint representation of G (respectively $\mathbf{g}$).

Since G is semisimple and compact the Killing form $B(X,Y) = \mathrm{tr}\,\mathrm{ad}(X)\,\mathrm{ad}(Y)$ of $\mathbf{g}$ is nondegenerate and negative definite on $\mathbf{g}$, thus giving rise to an orthogonal decomposition of $\mathbf{g}$ as the direct sum $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$. This sum is $\mathrm{Ad}(H)$-invariant, i.e. $[\mathbf{m}, \mathbf{h}] \subset \mathbf{m}$. The tangent space T_oM of M at the identity coset $o = eH$ can be identified with $\mathbf{m}$ by the map

$$X \mapsto X^*(o) = \frac{\mathrm{d}}{\mathrm{d}\,t}(\exp tX \cdot o)\bigg|_{t=0}, \qquad X \in \mathbf{m}.$$

The restriction of $-B$ onto the complement $\mathbf{m}$ induces a G-invariant Riemannian metric g_B on M which is called the *standard metric* on M. Let $X, Y \in \mathbf{m}$, and $\{X_i\}$ be a B-orthonormal basis of $\mathbf{m}$. According to [3, Corollary 3.33] or [7, Theorem X.3.5(3)] the sectional curvature of (M, g_B) is given by

$$B(R(X,Y)X,Y) = B([X,Y]_{\mathbf{h}}, [X,Y]_{\mathbf{h}}) + \frac{1}{4}B([X,Y]_{\mathbf{m}}, [X,Y]_{\mathbf{m}})$$

and the Ricci curvature by

$$Ric(X,X) = -\frac{1}{4}B(X,X) + \frac{1}{2}\sum_i B([X,X_i]_{\mathbf{h}}, [X,X_i]_{\mathbf{h}}). \tag{1}$$

Finally, we recall the isotropy representation χ of H in T_oM. For any $\gamma \in G$, let L_γ denote the diffeomorphism of G/H given by left translation. The *isotropy representation* χ of H on T_oM is defined by $\chi(h) = (\mathrm{d}\,L_h)_0$. Using the identification of $\mathbf{m}$ with T_oM, this representation is identified with the adjoint representation of H on $\mathbf{m}$.

3. The standard metric on Stiefel manifolds

In this section we find the Stiefel manifolds $V_k\mathbf{R}^n = SO(n)/SO(n-k)$, $k \geq 1$ for which the standard homogeneous metric g_B is Einstein. For $k = 1$, $V_1\mathbf{R}^n = S^n$ which is isotropy irreducible (i.e. the isotropy representation is irreducible), thus g_B is the unique (up to homothety) $SO(n)$-invariant Riemannian metric which is Einstein [1, Theorem 7.44]. Hence assume that $k \geq 2$.

If $SO(n-k)$ is identified with the subgroup of $SO(n)$ consisting of matrices of the form

$$\begin{pmatrix} I_k & O \\ O & A \end{pmatrix}$$

with $A \in SO(n-k)$, then $\mathbf{m}$ may be identified with the subspace of $\mathbf{so}(n)$ consisting of matrices of the form

$$\begin{pmatrix} D_k & A \\ -A^t & O_{n-k} \end{pmatrix},$$

where A is a $k \times (n-1)$ real matrix and $D_k = \mathrm{diag}(0, \ldots, 0)$.

Let $Q(X,Y) = -\frac{B(X,Y)}{2(n-2)} = -\frac{\mathrm{tr}\, XY}{2}$ and let E_{ab} denote the $n \times n$ matrix with 1 at the (ab)-entry and 0 elsewhere. Then the set $\mathcal{B} = \{e_{ab} = E_{ab} - E_{ba} : 1 \leq a \leq k,\ 1 \leq a < b \leq n\}$ constitutes a Q-orthonormal basis of $\mathbf{m}$ (containing $\frac{k(2n-k-1)}{2} = \dim V_k\mathbf{R}^n$ elements). Equation (1) now becomes

$$Ric(X,X) = -\frac{1}{4}B(X,X) + \frac{1}{2}\sum_{e_{ab}\in\mathcal{B}} Q([X,e_{ab}]_{\mathbf{h}}, [X,e_{ab}]_{\mathbf{h}}). \tag{2}$$

To compute the sum above we need the following lemmas.

Lemma 1. *The multiplication table of the elements in $\mathcal{B}$ is given as follows:*

$$[e_{ab}, e_{cd}] = \begin{cases} 0 & a \neq c,d \quad b \neq c,d \\ -e_{ac} & a \neq c,d \quad b = d \\ e_{ad} & a \neq c,d \quad b = c \\ -e_{bd} & a = c, \quad\ \ b \neq c,d \\ e_{bc} & a = d, \quad\ \ b \neq c,d. \end{cases}$$

Proof: The computation is straightforward. □

Let $\mathcal{B}_1 = \{e_{ab} : 1 \leq a < b \leq k\}$, $\mathcal{B}_2 = \{e_{ab} : 1 \leq a \leq k,\ k+1 \leq b \leq n\}$, $\mathcal{B}_3 = \{e_{ab} : k+1 \leq a < b \leq n\}$. Then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{B}_1$, $\mathcal{B}_3$ are Q-orthonormal bases for $\mathbf{o}(k)$ and $\mathbf{o}(n-k)$ respectively. Furthermore, the set $\mathcal{B}_2$ constitutes a Q-orthonormal basis for $T_o(Gr_k\mathbf{R}^n)$ the tangent space to the Grassmanian $Gr_k\mathbf{R}^n = SO(n)/S(O(k) \times O(n-k))$ at the identity coset.

Lemma 2. *The sets $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ defined above satisfy the following relations:*

(a) $[\mathcal{B}_1, \mathcal{B}_1] \subset \mathcal{B}_1$,
(b) $[\mathcal{B}_1, \mathcal{B}_2] \subset \mathcal{B}_2$,
(c) $[\mathcal{B}_2, \mathcal{B}_2] \subset \mathcal{B}_1 \oplus \mathcal{B}_3$.

Proof: Relation (a) follows from Lemma 1. For (b), the sets $\mathcal{B}_1 \oplus \mathcal{B}_3$ and $\mathcal{B}_2$ constitute Q-orthonormal bases for $\mathbf{so}(k) \oplus \mathbf{so}(n-k)$ and $T_o(Gr_k\mathbf{R}^n)$ respectively. Hence $[\mathcal{B}_1, \mathcal{B}_2] \subset [\mathcal{B}_1 \oplus \mathcal{B}_3, \mathcal{B}_2] \subset \mathcal{B}_2$, since $Gr_k\mathbf{R}^n$ is a symmetric space. For the same reason we obtain (c). □

Let $I = \{(i,j) : 1 \leq i < j \leq k\}$, $II = \{(i,j) : 1 \leq i \leq k,\ k+1 \leq j \leq n\}$, and $X = \sum_{e_{ij}\in\mathcal{B}} X_{ij}e_{ij} \in \mathbf{m}$.

Proposition 1. *The Ricci curvature for g_Q on $V_k\mathbf{R}^n$ is given by:*

$$Ric(X,X)=\frac{1}{2}(n-2)\sum_{(i,j)\in I}X_{ij}^2+\frac{1}{2}(2n-k-3)\sum_{(i,j)\in II}X_{ij}^2.$$

Hence the standard metric on $V_k\mathbf{R}^n$ $(k\geq 2)$ is Einstein if and only if $(n-2)=(2n-k-3)$ or $n-k=1$.

Combining Proposition 1 with the remarks at the beginning of this section we obtain

Theorem 1. *The standard metric on $V_k\mathbf{R}^n$ $(k\geq 1)$ is Einstein if and only if $V_k\mathbf{R}^n=S^n$ or $SO(n)$.*

Proof of Proposition 1: Let $X=\sum_{e_{ij}\in\mathcal{B}=\mathcal{B}_1\cup\mathcal{B}_2}X_{ij}e_{ij}$. Then equation (2) implies that

$$\begin{aligned}
Ric(X,X)&=-\frac{1}{4}(n-2)\operatorname{tr}X^2+\frac{1}{2}\sum_{e_{ab}\in\mathcal{B}}Q([\sum_{e_{ij}\in\mathcal{B}}X_{ij}e_{ij},e_{ab}]_{\mathbf{h}},[\sum_{e_{ij}\in\mathcal{B}}X_{ij}e_{ij},e_{ab}]_{\mathbf{h}})\\
&=\frac{1}{2}(n-2)\sum_{(i,j)\in I\cup II}X_{ij}^2\\
&\quad+\frac{1}{2}\sum_{e_{ab}\in\mathcal{B}_1}Q(\sum_{e_{ij}\in\mathcal{B}_1}X_{ij}[e_{ij},e_{ab}]_{\mathbf{h}},\sum_{e_{ij}\in\mathcal{B}_1}X_{ij}[e_{ij},e_{ab}]_{\mathbf{h}})\\
&\quad+\frac{1}{2}\sum_{e_{ab}\in\mathcal{B}_1}Q(\sum_{e_{ij}\in\mathcal{B}_2}X_{ij}[e_{ij},e_{ab}]_{\mathbf{h}},\sum_{e_{ij}\in\mathcal{B}_2}X_{ij}[e_{ij},e_{ab}]_{\mathbf{h}})\\
&\quad+\frac{1}{2}\sum_{e_{ab}\in\mathcal{B}_2}Q(\sum_{e_{ij}\in\mathcal{B}_1}X_{ij}[e_{ij},e_{ab}]_{\mathbf{h}},\sum_{e_{ij}\in\mathcal{B}_1}X_{ij}[e_{ij},e_{ab}]_{\mathbf{h}})\\
&\quad+\frac{1}{2}\sum_{e_{ab}\in\mathcal{B}_2}Q(\sum_{e_{ij}\in\mathcal{B}_2}X_{ij}[e_{ij},e_{ab}]_{\mathbf{h}},\sum_{e_{ij}\in\mathcal{B}_2}X_{ij}[e_{ij},e_{ab}]_{\mathbf{h}}).
\end{aligned}$$

By Lemma 2 (a), (b) the second, the third and the fourth sum are zero. A calculation that uses Lemmas 1 and 2 (c) gives that the fifth sum is equal to $(n-k-1)\sum_{(i,j)\in II}X_{ij}^2$. Thus the Ricci curvature is given by

$$\begin{aligned}
Ric(X,X)&=\frac{1}{2}(n-2)\sum_{(i,j)\in I\cup II}X_{ij}^2+\frac{1}{2}(n-k-1)\sum_{(i,j)\in II}X_{ij}^2\\
&=\frac{1}{2}(n-2)\sum_{(i,j)\in I}X_{ij}^2+\frac{1}{2}(2n-k-3)\sum_{(i,j)\in II}X_{ij}^2.
\end{aligned}$$

□

Alternatively, one may use representation theory to obtain another proof of Theorem 1 as follows. For a homogeneous space G/H let $\chi = \chi_o \oplus \chi_1 \oplus \cdots \oplus \chi_s$ be the decomposition of the isotropy representation of H in $\mathbf{m}$ as a direct sum of a trivial representation χ_o and $\mathrm{Ad}(H)$-irreducible representations χ_i $(i = 1, \ldots, s)$. Let $\mathbf{m} = \mathbf{m}_o \oplus \mathbf{m}_1 \oplus \cdots \oplus \mathbf{m}_s$ be the corresponding decomposition of $\mathbf{m}$. According to [11, Corollary 1.3] if the metric g_B is Einstein, then either H is trivial or $\mathbf{m}_o = 0$.

We compute the isotropy representation for the Stiefel manifolds $V_k\mathbf{R}^n = SO(n)/SO(n-k)$, $k \geq 1$. Let λ_n denote the standard representation of $SO(n)$ (given by the natural action of $SO(n)$ on $\mathbf{R}^n$). If $\wedge^2\lambda_n$ denote the second exterior power of λ_n, then $\mathrm{Ad}_{SO(n)} = \wedge^2\lambda_n$. The isotropy representation χ of $SO(n-k)$ is characterized by the property $\mathrm{Ad}_{SO(n)}\Big|_{SO(n-k)} = \mathrm{Ad}_{SO(n-k)} \oplus \chi$. We compute

$$\mathrm{Ad}_{SO(n)}\Big|_{SO(n-k)} = \wedge^2\lambda_n\Big|_{SO(n-k)} = \wedge^2(\lambda_{n-k} \oplus k) = \wedge^2\lambda_{n-k} \oplus \wedge^2 k \oplus (\lambda_{n-k} \otimes k),$$

where k denotes the trivial k-dimensional representation. The subrepresentation $\wedge^2\lambda_{n-k}$ is $\mathrm{Ad}_{SO(n-k)}$, thus $\chi = \wedge^2 k \oplus \lambda_{n-k} \oplus \cdots \oplus \lambda_{n-k}$ (k summands). If we further decompose $\wedge^2 k$ into a sum of $\binom{k}{2}$ 1-dimensional subrepresentations, we obtain

$$\chi = 1 \oplus \cdots \oplus 1 \oplus \lambda_{n-k} \oplus \cdots \oplus \lambda_{n-k}. \tag{3}$$

Hence, in our case, if g_B is Einstein, then either H is trivial (i.e. $V_k\mathbf{R}^n = SO(n)$) or $\mathbf{m}_o = 0$ in which case $k = 1$ hence $\chi = \lambda_{n-1}$ (i.e. $V_k\mathbf{R}^n = SO(n)/SO(n-1) = S^n$).

4. Homogeneous Einstein metrics on $V_2\mathbf{R}^n$

In this section we are interested in $SO(n)$-invariant Einstein metrics on the Stiefel manifolds $V_2\mathbf{R}^n = SO(n)/SO(n-2)$.

Let $M = G/H$ be a homogeneous space and

$$\mathbf{m} = \mathbf{m}_1 \oplus \cdots \oplus \mathbf{m}_s \tag{4}$$

the decomposition of $\mathbf{m}$ into $\mathrm{Ad}(H)$-invariant subspaces. We consider the family of $\mathrm{Ad}(H)$-invariant *diagonal* metrics

$$\langle\ ,\ \rangle = y_1\, Q|_{\mathbf{m}_1} \oplus \cdots \oplus y_s\, Q|_{\mathbf{m}_s}, \qquad y_i > 0 \tag{5}$$

where Q is the multiple of the Killing form used in Section 3. If the $\mathbf{m}_i$'s are pairwise inequivalent representations, then any G-invariant metric on M is determined by $\langle\ ,\ \rangle$. Otherwise, each decomposition (4) of $\mathbf{m}$ will yield a family of G-invariant metrics on M. We will see later on that we face this problem with the Stiefel manifolds.

Let $\{X_i\}$ be a $\langle\ ,\ \rangle$-orthonormal basis of $\mathbf{m}$. According to [1, Corollary 7.38] the Ricci curvature of M is given by

$$\begin{aligned} Ric(X,X) &= -\frac{1}{2}\sum_i |[X,X_i]_{\mathbf{m}}|^2 - \frac{1}{2}B(X,X) \\ &+ \frac{1}{4}\sum_{i,j}\langle [X_i,X_j]_{\mathbf{m}},X\rangle^2 - \langle [Z,X]_{\mathbf{m}},X\rangle \end{aligned} \tag{6}$$

where $Z=\sum_{i,j}\langle X_i,[X_j,X_i]_{\mathbf{m}}\rangle X_j$.

We now come to the Stiefel manifolds. As shown in Section 2, the isotropy representation of $V_k\mathbf{R}^n$ is decomposed as $\chi = 1\oplus\cdots\oplus 1\oplus\lambda_{n-k}\oplus\cdots\oplus\lambda_{n-k}$. We observe that the first $\frac{k(k-1)}{2}$ summands are equivalent to each other and the same is true for the remaining k summands.

We restrict ourselves to $V_2\mathbf{R}^n$ and we look for $SO(n)$-invariant Einstein metrics of the form (5). The isotropy representation is decomposed as $\chi = 1\oplus\lambda_{n-2}\oplus\lambda_{n-2}$, hence these metrics depend on three parameters $y_1,\ y_2,\ y_3,\ y_i>0$. Let $\{e_{ij}=E_{ij}-E_{ji} : 1\le i\le 2,\ 1\le i<j\le n\}$ be the Q-orthonormal basis described in Section 2 adapted to the decomposition $\mathbf{m}=\mathbf{m}_1\oplus\mathbf{m}_2\oplus\mathbf{m}_3$, so that $\mathbf{m}_1=\operatorname{span}\{e_{12}\}$, $\mathbf{m}_2=\operatorname{span}\{e_{1j} : 3\le j\le n\}$, $\mathbf{m}_3=\operatorname{span}\{e_{2j} : 3\le j\le n\}$. Then the set $\{X_{ij}=e_{ij}/\sqrt{y_k} : e_{ij}\in\mathbf{m}_k\ (k=1,2,3)\}$ constitutes a $\langle\ ,\ \rangle$-orthonormal basis of $\mathbf{m}$.

Lemma 3. *The elements X_{ij} satisfy the following relations:*

$$\begin{aligned} [X_{12},X_{1j}]_{\mathbf{m}} &= -\sqrt{\frac{y_3}{y_1y_2}}X_{2j}\quad (j=3,\dots,n) \\ [X_{12},X_{2j}]_{\mathbf{m}} &= \sqrt{\frac{y_2}{y_1y_3}}X_{1j}\quad (j=3,\dots,n) \\ [X_{1j},X_{2j}]_{\mathbf{m}} &= -\sqrt{\frac{y_1}{y_2y_3}}X_{12}\quad (j=3,\dots,n) \end{aligned} \tag{7}$$

Proof: It is straightforward from Lemma 1. □

Using the above lemma a calculation shows that the vector Z in (6) is zero, thus the Ricci curvature of $V_2\mathbf{R}^n$ is given by

$$\begin{aligned} Ric(X,X) = &-\frac{1}{2}\sum_{\substack{i=1,2\\ 3\le j\le n}} |[X,X_{ij}]_{\mathbf{m}}|^2 - \frac{1}{2}(n-2)\operatorname{tr}X^2 \\ &+ \frac{1}{2}[\sum_{j=3}^n\langle [X_{12},X_{1j}]_{\mathbf{m}},X\rangle^2 + \sum_{j=3}^n\langle [X_{12},X_{2j}]_{\mathbf{m}},X\rangle^2 \\ &+ \sum_{j=3}^n\langle [X_{1j},X_{2j}]_{\mathbf{m}},X\rangle^2] \end{aligned}$$

or equivalently

$$
\begin{aligned}
Ric(X,X) = &-\frac{1}{2}[|[X,X_{12}]_{\mathbf{m}}|^2 + \sum_{j=3}^{n}|[X,X_{1j}]_{\mathbf{m}}|^2 \\
&+\sum_{j=3}^{n}|[X,X_{2j}]_{\mathbf{m}}|^2] - \frac{1}{2}(n-2)\operatorname{tr}X^2 \\
&+\frac{1}{2}[\frac{y_3}{y_1y_2}\sum_{j=3}^{n}\langle X_{2j},X\rangle^2 + \frac{y_2}{y_1y_3}\sum_{j=3}^{n}\langle X_{1j},X\rangle^2 \\
&+(n-2)\frac{y_1}{y_2y_3}\langle X_{12},X\rangle^2].
\end{aligned} \tag{8}
$$

Proposition 2. *The Ricci curvature for the metric (5) on $V_2\mathbf{R}^n$ is given by*

$$
\begin{aligned}
&Ric(X_{12},X_{12}) = \frac{n-2}{2}[\frac{y_1}{y_2y_3} - \frac{y_2}{y_1y_3} - \frac{y_3}{y_1y_2} + \frac{2}{y_1}] \\
&Ric(X_{1j},X_{1j}) = -\frac{1}{2}[\frac{y_1}{y_2y_3} - \frac{y_2}{y_1y_3} + \frac{y_3}{y_1y_2} - \frac{2(n-2)}{y_2}] \\
&Ric(X_{2j},X_{2j}) = -\frac{1}{2}[\frac{y_1}{y_2y_3} + \frac{y_2}{y_1y_3} - \frac{y_3}{y_1y_2} - \frac{2(n-2)}{y_3}] \\
&Ric(X_{12},X_{1j}) = Ric(X_{12},X_{2j}) = Ric(X_{1j},X_{2k}) = 0, \qquad (j,k=3,\ldots,n)
\end{aligned}
$$

Proof: The first three equalities are obtained from equation (8) and relations (7) by a straightforward calculation. For the remaining equalities we use polarization and the first three equalities. □

Setting $Ric(X_{12},X_{12}) = Ric(X_{1j},X_{1j}) = Ric(X_{2j},X_{2j})$ we obtain the following

Proposition 3. *The Einstein equation for $V_2\mathbf{R}^n$ reduces to the following system of algebraic equations:*

$$
\begin{aligned}
(n-1)y_1^2 - (n-1)y_2^2 - (n-3)y_3^2 &= 2(n-2)y_3(y_1-y_2) \\
y_2^2 - y_3^2 &= (n-2)y_1(y_2-y_3)
\end{aligned}
$$

The system of equations in Proposition 3 has only one solution, namely $y_2 = y_3 = \frac{n-1}{2(n-2)}y_1$. Indeed, if $y_2 \neq y_3$, the second equation gives (by normalizing $y_1 = 1$) $y_2 = (n-2) - y_3$, and by substituting to the first equation we obtain the equation $4(n-2)y_3^2 - 4(n-2)^2y_3 + (n-1)^2(n-3) = 0$. The discriminant of this quadratic is equal to $-16(n-2)(n^2-5n+5)$, which is negative for $n \geq 4$. Hence there is no solution in this case.

Furthermore, one can check that as long as the Q-orthonormal basis $\mathcal{B}$ of $\mathbf{m}$ described in Section 2 is adapted to the decomposition $\mathbf{m} = \mathbf{m}_1 \oplus \mathbf{m}_2 \oplus \mathbf{m}_3$

so that $\mathbf{m}_1 = \mathrm{span}\{e_{12}\}$, $\mathbf{m}_2 = \mathrm{span}\{e_{ij} : 1 \leq i, j \leq n,\ j \neq 2\}$, and $\mathbf{m}_3 = \mathrm{span}\{e_{kl} : e_{kl} \neq e_{ij}\}$, then the Einstein equation for $V_2\mathbf{R}^n$ reduces to the same algebraic system given in Proposition 3. (Simply use the expression (6) for the Ricci tensor and Lemma 3, where the numbers under the square roots are modified appropriately). Thus we have obtained

Theorem 2. *The Stiefel manifolds $V_2\mathbf{R}^n = SO(n)/SO(n-2)$ admit (up to scale) at least one $SO(n)$-invariant Einstein metric, explicitly given as $y_2 = y_3 = \frac{n-1}{2(n-2)} y_1$. If the Q-orthonormal basis $\mathcal{B}$ is adapted to the decomposition* (4) *the way described above, then this metric is unique.*

As mentioned earlier there may be additional $SO(n)$-invariant Einstein metrics on $V_2\mathbf{R}^n$ for a different adaption of the basis $\{e_{ij}\}$ to the decomposition (4) (in fact there are $(2n-3)(n-2)(2n-5)$ such adaptions). Furthermore, the metric we obtained is not necessarily the same as the one predicted by Sagle [9].

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