A compact ccc non-separable space from a Hausdorff gap and Martin's Axiom

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Abstract. We answer a question of I. Juhasz by showing that MA $+\neg$ CH does not imply that every compact ccc space of countable π-character is separable. The space constructed has the additional property that it does not map continuously onto I^{ω_1}.

Keywords: ccc, non-separable, Hausdorff gap, π-character

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1. Introduction

I. Juhasz [Ju71] has proven that MA(ω_1) implies that every first countable compact ccc space is separable. This has been extended by Shapirovskii [Sh72] by replacing first countable with countable tightness. In Juhasz [Ju77], the question is raised whether tightness can be replaced by π-character, i.e., whether MA(ω_1) implies that every compact ccc space of countable π-character is separable. We will show not. We present our space as a space whose points are certain ideals because this is the way that we found it; although the inclined reader should easily be able to identify the base set as a rather simple subset of $2^\omega \times \kappa$ (where κ is a certain regular cardinal $> \omega_1$) using a Hausdorff gap as a parameter.

2. General theory of total ideal spaces

Let $P = \bigcup_{A\subset\omega} 2^A$ and put $p \preceq q$ if q extends p. Then $(P, \preceq)$ is a Dedekind complete partially ordered set. A subset F of P is **compatible** if $\bigcup F \in P$. We write $p \parallel q$ if $\{p, q\}$ is compatible and we write $p \perp q$ if $\{p, q\}$ is not compatible. A subset Q of P is **closed** in P if whenever F is a finite compatible subset of Q, then $\bigcup F \in Q$. For Q closed in P, a compatible and closed subset I of Q is called a **total ideal** of Q if

(a) $\bigcup I$ has domain all of ω and
(b) $p \in I$ and $q \in Q$ with $q \preceq p$ implies $q \in I$.

Let Fin $= \{p \in P : \text{dom}(p) \text{ is finite}\}$. The parameter for these ideal spaces will be a closed subset Q of P with Fin $\subset Q$. For such a Q, put $X(Q) = \{I \subset Q : I$ is a total ideal of $Q\}$. It is seen that $X(Q)$ is a closed subspace of 2^Q (where

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$2 = \{0, 1\}$ has the discrete topology, 2^Q has the product topology, and points of $X(Q)$ are identified with their characteristic functions). For $B \subset Q$, put $B^+ = \{I \in X(Q) : B \subset I\}$ and $B^- = \{I \in X(Q) : B \cap I = \varnothing\}$. If $B = \{q\}$, then we simply write q^+ and q^-. Then, since Q is closed, a base for $X(Q)$ consists of the clopen sets $q^+ \cap B^-$ where $q \in Q$ and B is a finite subset of $\{r \in Q : q \prec r\}$. We note the helpful facts that

(a) $q^+ \subset r^+$ iff $r \preceq q$,
(b) $q^+ \cap r^+ \neq \varnothing$ iff $q \parallel r$ iff $[q^{-1}(1) \cup r^{-1}(1)] \cap [q^{-1}(0) \cup r^{-1}(0)] = \varnothing$,
(c) $\lambda : X(Q) \longrightarrow 2^\omega$ given by $\lambda(I) = \bigcup I$ is a continuous surjection.

Put $\mathcal{Q} = \{q^+ : q \in Q\}$ and for each $f \in 2^\omega$, let M_f be the maximal total ideal $\{q \in Q : q \preceq f\}$.

Fact 2.1. *$\mathcal{Q}$ is a T_0-separating, binary π-base of $X(Q)$. Hence, $\pi w(X(Q)) = cf(Q, \preceq) = \min\{|D| : D \subset Q$ and $\forall\, q \in Q \colon \exists\, d \in D \colon (q \preceq d)\}$.*

This was proved in [Be88] and [Be89]. T_0-separating and binary are straightforward. The fact that $\mathcal{Q}$ is a π-base is crucial. The reader will see a proof of this in Lemma 3.2 where we must prove a little bit more in order to achieve countable π-character.

Now we show two facts which delineate the kinds of Souslinean examples that we can get from these ideal spaces.

Fact 2.2. *If $X(Q)$ is σ-linked, then $X(Q)$ is separable.*

If $X(Q)$ is σ-linked, then $\mathcal{Q} = \bigcup_{n<\omega} \mathcal{Q}_n$ where for each $n < \omega$, $\mathcal{Q}_n$ is linked. Since $\mathcal{Q}$ is binary, by choosing $I_n \in \bigcap \mathcal{Q}_n$ for each $n < \omega$, we get that $\{I_n : n < \omega\}$ is dense in $X(Q)$.

We refer the reader to Todorcevic [To89] for the definition of the Open Colouring Axiom OCA.

Fact 2.3 (OCA). *If $X(Q)$ is ccc, then $X(Q)$ is separable.*

For each $q \in Q$ put $A_q = q^{-1}(1)$ and $B_q = q^{-1}(0)$. Then A_q and B_q are disjoint subsets of ω. For each $q \in Q$ let a_q, b_q be the characteristic functions of A_q, B_q respectively. Let $S = \{(a_q, b_q) : q \in Q\}$ have the subspace topology from $2^\omega \times 2^\omega$. Define a partition of $[S]^2$ by $\{(a_q, b_q), (a_r, b_r)\} \in K_0$ iff $q^+ \cap r^+ = \varnothing$ iff $(A_q \cup A_r) \cap (B_q \cup B_r) \neq \varnothing$. K_0 is open in $[S]^2$. Since $X(Q)$ is ccc, there does not exist a K_0-homogeneous subset of S which has cardinality ω_1. Hence, by OCA, $Q = \bigcup_{n<\omega} Q_n$ where for every n and for every q,r in Q_n, $q^+ \cap r^+ \neq \varnothing$, i.e., $\{q^+ : q \in Q_n\}$ is linked. We get that $\mathcal{Q}$ is σ-linked, hence Fact 2.2 implies that $X(Q)$ is separable.

Remarks: We have learned that Fact 2.3 follows from a more general result Theorem 10.3* in Todorcevic and Farah [TF95]. We see from Fact 2.3, that if we want a ccc but not separable space $X(Q)$, then we must be in a model of set theory contradicting OCA. We did this in [Be89] under CH producing a first countable Corson compact space which was ccc but did not have Property K. We also point out that interesting separable spaces $X(Q)$ of uncountable π-weight can be achieved in every model of set theory. In [Be88], for each regular cardinal κ for which there exists a κ-chain of clopen sets in $\beta\omega \setminus \omega$, we produced a separable space $X(Q)$ of π-weight κ that did not continuously map onto I^{ω_1}. So Problem 2 in Shapirovskii [Sh93] has a negative answer. Referring to the last comment in this paper, it seems that the "last word" in a large part of the theory of compact spaces has not yet been spoken.

3. The Hausdorff gap space

Our example will use a (κ, κ) Hausdorff gap where $\omega_1 < \kappa = cf(\kappa) \leq \mathfrak{c}$. Let $(A_\alpha, B_\alpha)_{\alpha<\kappa}$ be such that

Q1: $A_0 = \varnothing = B_0$ and $A_\alpha \cup B_\alpha \subset \omega$

Q2: $\alpha < \beta \Rightarrow (A_\alpha \subset^* A_\beta$ and $B_\alpha \subset^* B_\beta)$ (strict almost inclusion)

Q3: $A_\alpha \cap B_\alpha = \varnothing$

Q4: $\nexists\, A \subset \omega$ such that $\forall\, \alpha < \kappa(A_\alpha \subset^* A$ and $B_\alpha \subset^* \omega \setminus A)$.

Put $Q = \{p \in P : \exists\, \alpha < \kappa$ with $\mathrm{dom}(p) =^* A_\alpha \cup B_\alpha$ and $p^{-1}(1) =^* A_\alpha\}$ and let $X = X(Q)$. For each $q \in Q$ define $\delta(q) =$ the unique $\alpha < \kappa$ with $\mathrm{dom}(q) =^* A_\alpha \cup B_\alpha$ and extend δ so that $\delta : X \longrightarrow \kappa$ by $\delta(I) = \sup\{\delta(q) : q \in I\}$. This definition of δ is well-defined because if $I \in X$, then by Q4, $\exists\, \alpha < \kappa$ such that either $A_\alpha \not\subset^* \lambda(I)^{-1}(1)$ or $B_\alpha \not\subset^* \lambda(I)^{-1}(0)$, hence if $\delta(q) > \alpha$, then $q \notin I$.

Lemma 3.1. *X can be partitioned into $\mathfrak{c}$ many closed G_δ subspaces each of which is homeomorphic to an ordinal space $[0, \alpha]$ where $|\alpha| < \kappa$. Thus, X is G_δ-scattered (i.e., scattered in the G_δ topology) and so cannot map continuously onto I^{ω_1}.*

PROOF: Q1–Q4 allows us to easily identify, for each $f \in 2^\omega$, the closed G_δ subspace $\lambda^{-1}(f)$. If $\delta(M_f)$ is an isolated ordinal or if $\delta(M_f)$ is a limit ordinal which is not attained (i.e., $\delta(M_f) \notin M_f$), then $\lambda^{-1}(f) \approx$ the ordinal space $[0, \delta(M_f)]$. If $\delta(M_f)$ is a limit ordinal which is attained, then $\lambda^{-1}(f) \approx [0, \delta(M_f)+1]$. Thus, X is partitioned into $\mathfrak{c}$ many closed G_δ ordinal subspaces and so every non-empty subspace of X contains a relative G_δ-point. By a result of Shapirovskii [Sh80], X cannot map continuously onto I^{ω_1}. □

We now partition Q into horizontal sections. For each $\alpha < \kappa$ put $Q^\alpha = \{q \in Q : \delta(q) = \alpha\}$ and put $\mathcal{Q}^\alpha = \{q^+ : q \in Q^\alpha\}$.

Lemma 3.2. *For each $\alpha < \kappa$, $\mathcal{Q}^\alpha$ is a π-base for $\{q^+ \cap B^- : \delta(q) \leq \alpha\}$, i.e., for every q and finite B with $q^+ \cap B^- \neq \varnothing$ and $\delta(q) \leq \alpha$ there exists $r \in Q^\alpha$ with $r^+ \subset q^+ \cap B^-$.*

PROOF: Assume $q^+ \cap B^- \neq \varnothing$ and let $\delta(q) \leq \alpha$. Choose $I \in q^+ \cap B^-$ and put $f = \lambda(I)$. Put $A = \{p \in B : p \not\preceq f\}$ and $C = \{p \in B : p \preceq f\}$. For each $p \in A$ choose $n_p \in \mathrm{dom}(p)$ with $p(n_p) \neq f(n_p)$. Put $R = \mathrm{dom}(q) \cup \{n_p : p \in A\}$

and put $r = f \restriction R$. Since $B \cap I = \varnothing$, for each $p \in C$ and for each finite $H \subset \omega$ we have that $\mathrm{dom}(p) \not\subset R \cup H$. This implies that we can choose, for each $p \in C$, an $m_p \in \mathrm{dom}(p) \setminus R$ such that distinct p's yield distinct m_p's. Let s have domain $\{m_p : p \in C\}$ and satisfy that for each $p \in C$, $s(m_p) \neq p(m_p)$. Then, $\delta(r \cup s) = \delta(r) = \delta(q) \leq \alpha$ and $\varnothing \neq (r \cup s)^+ \subset q^+ \cap B^-$. □

Lemma 3.3. *X is ccc.*

PROOF: Assume not and choose an uncountable (meaning of cardinality ω_1 throughout this proof) $R \subset Q$ such that $r \neq s$ in $R \Rightarrow r \perp s$. Since $\delta : Q \longrightarrow \kappa$ is $\leq \omega$–to–1, choose an uncountable $R' \subset R$ such that $\delta \restriction R'$ is 1–1. Since there exist only countably many finite collections of finite subsets of ω, choose an uncountable $R'' \subset R'$ and finite $F, A, G, B, H, C \subset \omega$ such that $p \in R''$ and $\delta(p) = \alpha \Rightarrow \mathrm{dom}(p) = (A_\alpha \setminus F) \cup (B_\alpha \setminus G) \cup H$ and $p^{-1}(1) = (A_\alpha \setminus A) \cup B \cup C$ where F and A are disjoint subsets of A_α, G and B are disjoint subsets of B_α, and H and C are disjoint subsets of $\omega \setminus (A_\alpha \cup B_\alpha)$. Let $E = \{\delta(p) : p \in R''\}$. Since R'' consists of pairwise incompatible elements, we see that for $\alpha \neq \beta$ in E, $(A_\alpha \cup A_\beta) \cap (B_\alpha \cup B_\beta) \neq \varnothing$. Since $\mathrm{cf}(\kappa) > \omega_1$, choose $\gamma < \kappa$ such that $\gamma > \sup(E)$. Choose $n < \omega$ and an uncountable $K \subset E$ such that for each $\alpha \in K$, $A_\alpha \setminus n \subset A_\gamma$ and $B_\alpha \setminus n \subset B_\gamma$. Since $A_\gamma \cap B_\gamma = \varnothing$, for every $\alpha \neq \beta$ in K, $(A_\alpha \cup A_\beta) \cap (B_\alpha \cup B_\beta) \cap n \neq \varnothing$. So we have a finite partition $[K]^2 = \bigcup_{i \in n} \{\{\alpha, \beta\} : i \in (A_\alpha \cup A_\beta) \cap (B_\alpha \cup B_\beta)\}$. By Ramsey's Theorem, get $j < n$ and $\alpha < \beta < \eta$ in K such that $\{\alpha, \beta, \eta\}$ is homogeneous for j. This contradicts that $A_\iota \cap B_\iota = \varnothing$ for $\iota = \alpha, \beta, \eta$. Lemma 3.3 is proved.

□

Lemma 3.4. *$\mathcal{Q}$ is a point–$< \kappa$ collection, i.e., if $I \in X$, then $\{q^+ : I \in q^+\}$ has cardinality $< \kappa$. Consequently, X does not have Property K_κ ($\mathcal{Q}$ is a collection of κ many clopen sets which does not have a linked subcollection of cardinality κ).*

PROOF: If $I \in q^+$ for κ many q's, then $\lambda(I)^{-1}(1)$ would fill our Hausdorff gap $(A_\alpha, B_\alpha)_{\alpha<\kappa}$ contradicting Q4. □

The above lemma tells us that X is not separable and also that X is not the support of a measure algebra (as these all have Property K_κ for every regular κ).

Lemma 3.5. *X has countable π-character.*

PROOF: Let $I \in X$ and put $\alpha = \delta(I)$. Lemma 3.2 implies that for every neighbourhood $q^+ \cap B^-$ of I, there exists $r^+ \in \mathcal{Q}^\alpha$ such that $r^+ \subset q^+ \cap B^-$. Since $|\mathcal{Q}^\alpha| = \omega$, we are done. □

So, we have shown

Theorem 3.6. *If there exists a (κ, κ) Hausdorff gap where $\kappa = \mathrm{cf}(\kappa) > \omega_1$, then there exists a compact, ccc, non-separable space X which has countable π-character, character $= \sup\{\lambda : \lambda < \kappa\}$, and which does not continuously map onto I^{ω_1}.*

Corollary. MA(ω_1) *does not imply any of the following:*

(a) *Every compact ccc space of countable π-character is separable.*
(b) *Every compact ccc space of tightness (or even character) $\leq \omega_1$ is separable.*
(c) *Every compact ccc non-separable space continuously maps onto I^{ω_1}.*

Proof: We can apply the theorem because Kunen (cf. Baumgartner [Ba84]) has proved that Martin's Axiom is consistent with $\mathfrak{c} = \omega_2$ + there exists a (ω_2, ω_2) Hausdorff gap. □

The above (a) answers the question of Juhasz [Ju77] (this question was also repeated on page 209 in Fremlin [Fr84]). The above (b) is a different kind of example showing that the theorem of Shapirovskii [Sh72]:
MA$(\omega_1) \Rightarrow$ Every compact ccc space of tightness $< \omega_1$ is separable
cannot be improved in the tightness direction. It is quite different from the first published example (Bell [Be80]); that one was covered by Cantor cubes of uncountable weight. The above (c) is of interest because of the following: Let **A** represent the axiom of Todorcevic "Every compact ccc non-separable space maps onto I^{ω_1}". One use of axiom **A** is that it resolves several problems in the literature. S. Todorcevic has shown that **A** $\Rightarrow$ MA(ω_1). What we have shown is that MA$(\omega_1) \not\Rightarrow$ **A**.

In conclusion, we mention that the question of whether every model of set theory contains an example of a compact ccc non-separable space with countable π-character remains open.

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