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A note on finite sets of terms closed under subterms and unification

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Abstract. The paper contains two remarks on finite sets of groupoid terms closed under subterms and the application of unifying pairs.

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By a term we shall mean a groupoid term. Let us write $u+v \sim t$ if $t = f(u) = g(v)$ for a unifying pair f, g of the terms u and v, i.e., if t is a substitution instance of both u and v and any term that is a substitution instance of both u and v, is a substitution instance of t. (A survey of unification theory is contained, for example, in Dershowitz and Jouannaud [1].)

Let us call a set S of terms SU-closed if it is closed with respect to subterms and whenever $u+v \sim t$ for two terms $u, v \in S$, then $t \sim t' \in S$ for some t'.

Theorem 1. *There is no finite, SU-closed set of terms containing the following three terms:*

$$(xy \cdot z)x, \qquad x(yz \cdot u), \qquad x \cdot yx.$$

Proof: Let us define an infinite sequence $a_0, a_1, \dots$ of terms as follows: a_0 is a variable; $a_{i+1} = a_i x$ for a variable x not occurring in a_i. So, $a_i = (((x_0 x_1)x_2)\dots)x_i$, where $x_0, \dots, x_i$ is a sequence of pairwise distinct variables. Also, put $b_i = y a_i$, where y is a variable not occurring in a_i. Hence $b_2 \sim x(yz \cdot u)$. It is easy to see that

$$(xy \cdot z)x + b_i \sim ((a_i x)y)a_i \supseteq a_{i+2}$$

for $i \geq 2$ (where x, y are two distinct variables not occurring in a_i, and

$$x \cdot yx + a_{i+1} \sim a_i \cdot x a_i \supseteq b_i$$

for $i \geq 3$. □

The depth of a term is defined inductively as follows: the depth of a variable is 0; the depth of $t_1 t_2$ is $1 + \max(d_1, d_2)$, where d_1 is the depth of t_1 and d_2 is the depth of t_2. So, $xy \cdot zu$ is of depth 2.

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Theorem 2. *There exists a finite, SU-closed set of terms containing* (*up to similarity*) *all terms of depth at most* 2.

Proof: The set consists of the terms of depth 2, plus the twelve terms

$$\begin{array}{lcl}
xx\cdot(xx\cdot xx) & \sim & x\cdot xx \;+\; xx\cdot y \\
xy\cdot(xy\cdot xy) & \sim & x\cdot xx \;+\; xy\cdot z \\
xx\cdot(xx\cdot y) & \sim & x\cdot xy \;+\; xx\cdot y \\
xy\cdot(xy\cdot z) & \sim & x\cdot xy \;+\; xy\cdot z \\
xx\cdot(xx\cdot x) & \sim & x\cdot xy \;+\; xx\cdot yx \\
xy\cdot(xy\cdot x) & \sim & x\cdot xy \;+\; xy\cdot zx \\
xy\cdot(xy\cdot y) & \sim & x\cdot xy \;+\; xy\cdot zy \\
xx\cdot(y\cdot xx) & \sim & x\cdot yx \;+\; xx\cdot y \\
xy\cdot(z\cdot xy) & \sim & x\cdot yx \;+\; xy\cdot z \\
xx\cdot(x\cdot xx) & \sim & x\cdot yx \;+\; xx\cdot xy \\
xy\cdot(x\cdot xy) & \sim & x\cdot yx \;+\; xy\cdot xz \\
xy\cdot(y\cdot xy) & \sim & x\cdot yx \;+\; xy\cdot yz
\end{array}$$

and their duals. □

References

[1] Dershowitz N., Jouannaud J.-P., *Rewrite systems*, Chapter 6, 243–320 in J. van Leeuwen, ed., Handbook of Theoretical Computer Science, B: Formal Methods and Semantics, North Holland, Amsterdam, 1990.

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