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# Strong tightness as a condition of weak and almost sure convergence

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*Abstract.* A sequence of random elements $\{X_j, j \in J\}$ is called strongly tight if for an arbitrary $\varepsilon > 0$ there exists a compact set $K$ such that $P\left(\bigcap_{j\in J}[X_j \in K]\right) > 1-\varepsilon$. For the Polish space valued sequences of random elements we show that almost sure convergence of $\{X_n\}$ as well as weak convergence of randomly indexed sequence $\{X_\tau\}$ assure strong tightness of $\{X_n, n \in \mathbb{N}\}$. For $L^1$ bounded Banach space valued asymptotic martingales strong tightness also turns out to the sufficient condition of convergence. A sequence of r.e. $\{X_n, n \in \mathbb{N}\}$ is said to converge essentially with respect to law to r.e. $X$ if for all sets of continuity of measure $P \circ X^{-1}$, $P(\limsup_{n\to\infty}[X_n \in A]) = P(\liminf_{n\to\infty}[X_n \in A]) = P([x \in A])$. Conditions under which $\{X_n\}$ is essentially w.r.t. law convergent and relations to strong tightness are investigated.

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## 1. Notations and definitions

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $(S, \varrho)$ — a Polish space i.e. metric, complete and separable. A *random element* (r.e.) is any measurable mapping $X : \Omega \mapsto S$. For any sequence $\{X_n, n \in \mathbb{N}\}$ of random elements $\mathcal{F}_n$ will denote a smallest $\sigma$-algebra containing $X_1, \dots, X_n$. A mapping $\tau : \Omega \mapsto \mathbb{N}$ will be called a *stopping time* if $[\tau = n] \in \mathcal{F}_n$. Let $T$ be a collection of all bounded stopping times i.e. such stopping times that $P[\tau < M] = 1$. A *generalized sequence* $a_\tau$ is a mapping $f : T \mapsto S$ such that $f(\tau) = a_\tau$. A generalized sequence $a_\tau$ *converges* to $a$ if for any $\varepsilon > 0$ there exists $\nu \in T$ such that $\varrho(a_\tau, a) < \varepsilon$ for all $\tau \geq \nu$, a.s.

A sequence $\{X_n, n \geq 1\}$ of random elements is *randomly convergent in law* to a random element $X$ $\left(X_\tau \xrightarrow{D} X\right)$ if for any given $\varepsilon > 0$ there exists $\tau_0 \in T$ such that $L(X_\tau, X) < \varepsilon$ for every $\tau \in T, \tau \geq \tau_0$ a.s., where $L$ denotes the Lévy-Prokhorov metric.

**Definition 1.1.** A collection $\{P_j, j \in J\}$ of probability measures is tight if for any $\varepsilon > 0$ there exists a compact set $K \subset S$ such that for all $j \in J$

$$P_j(K) > 1 - \varepsilon.$$

**Definition 1.2.** A collection $\{X_j, j \in J\}$ of random elements is strongly tight if for any $\varepsilon > 0$ there exists a compact set $K \subset S$ such that

$$P\left(\bigcap_{j\in J}[X_j \in K]\right) > 1-\varepsilon.$$

Obviously if a collection $\{X_j, j \in J\}$ is strongly tight then the collection of probability measures $\{P \circ X_j^{-1}, j \in J\}$ is tight.

## 2. Essential with respect to law convergence of random elements

In this section we will consider random elements with values in a Polish space.

Let $\mathcal{C}_{P_X}$ denote a set of continuity of measure $P_X$, i.e.

$$\mathcal{C}_{P_X} = \{A \in \mathcal{B} : P[X \in \partial A] = 0\},$$

where $\partial A$ is a boundary of $A$.

**Definition 2.1.** A sequence of random elements $\{X_n, n \in \mathbb{N}\}$ is said to converge essentially w.r.t. law $\left(X_n \xrightarrow{ED} X\right)$ if for all $A \in \mathcal{C}_{P_X}$

$$P\left(\limsup_{n\to\infty}[X_n \in A]\right) = P\left(\liminf_{n\to\infty}[X_n \in A]\right) = P[X \in A].$$

This type of convergence was investigated in [10]. It seems to be worth mentioning that essential w.r.t. law convergence follows from a.s. convergence. On the other side if $X_n \xrightarrow{ED} X$ then there exists a r.e. $X'$ with the same distribution as $X$ such that $X_n \xrightarrow{a.s.} X'$.

The following theorem is analogous to Theorem 2.1 of [3].

**Theorem 2.1.** *Let $\{X_n\}$ be a sequence of r.e., and $X$ – a r.e. Then the following conditions are equivalent:*

1. $X_n \xrightarrow{ED} X$, *as* $n \to \infty$,
2. *for all* $A \in \mathcal{C}_{P_X}$ $P\left(\limsup_{n\to\infty}[X_n \in A]\right) = \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty}[X_k \in A]\right) = P[X \in A]$,
3. *for any closed set* $F$ $\lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty}[X_k \in F]\right) \le P[X \in F]$,
4. *for any open set* $G$ $\lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty}[X_k \in F]\right) \ge P[X \in G]$.

Proof: Implication $((1) \Rightarrow (2))$ is obvious.

$((2) \Rightarrow (1))$. Consider condition (2) for a complement $A^c$ of the set $A \in \mathcal{C}_{P_X}$

$$\lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty}[X_k \in A^c]\right) = P[X \in A^c].$$

Then, obviously

$$\lim_{n\to\infty} P\left(\left(\bigcup_{k=n}^{\infty}[X_k \in A^c]\right)^c\right) = P[X \in A]$$

and finally

$$\lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty}[X_k \in A]\right) = P[X \in A].$$

((2) $\Rightarrow$ (3)). Let $F^\delta = \{x : \varrho(x, F) \le \delta\}$. Then $\partial F^\delta \subset \{x : \varrho(x, F) = \delta\}$. For any closed set $F$ there exists a sequence $\delta^k \downarrow 0$ such that the sets $F^{\delta_k} \in \mathcal{C}_{P_X}$ and $\bigcap_{k=n}^{\infty} F^{\delta_k} = F$. Take a closed set $F$. Moreover, there exists $F^\delta \in \mathcal{C}_{P_X}$ such that $P_X(F^\delta \setminus F) < \varepsilon$. Then

$$\begin{aligned}\lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty}[X_k \in F]\right) &\le \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty}[X_k \in F^\delta]\right)\\ &= P[X \in F^\delta] \le P[X \in F] + \varepsilon.\end{aligned}$$

Since $\varepsilon$ is an arbitrary positive number

$$\lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty}[X_k \in F]\right) \le P[X \in F].$$

((3) $\Rightarrow$ (4)). For an open set $G$ we have

$$\begin{aligned}\lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty}[X_k \in G]\right) &= \lim_{n\to\infty}\left(1 - P\left(\left(\bigcap_{k=n}^{\infty}[X_k \in G]\right)^c\right)\right)\\ &= 1 - \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty}[X_k \in G^c]\right) \ge 1 - P[X \in G^c]\\ &= P[X \in G].\end{aligned}$$

The case ((4) $\Rightarrow$ (3)) can be proved in the similar way.

Now we need only (((3) and (4)) $\Rightarrow$ (2)). Let $A \in \mathcal{C}_{P_X}$ and let $\operatorname{Int} A$ denote interior of $A$. Then

$$\begin{aligned}P[X \in \operatorname{Int} A] &\le \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty}[X_k \in \operatorname{Int} A]\right)\\ &\le \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty}[X_k \in \operatorname{Int} A]\right) \le \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty}[X_k \in A]\right)\\ &\le \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty}[X_k \in \bar{A}]\right) \le P[X \in \bar{A}].\end{aligned}$$

Since $A \in \mathcal{C}_{P_X}$, (2) holds. □

There is a connection between essential w.r.t. law convergence and strong tightness.

**Theorem 2.2.** *If a sequence of random elements $\{X_n, n \in \mathbb{N}\}$ converges essentially w.r.t. law to a random element $X$, then it is strongly tight.*

Proof: Since $S$ is separable there exists a countable dense set $\{x_i, i \in \mathbb{N}\}$. Let $K(x_i, \delta) = \{x : \varrho(x, x_i) < \delta\}$. Define

$$B_m(\delta) = \bigcup_{i=1}^{m} K(x_i, \delta).$$

For any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$P[X \in B_m(\delta)] > 1 - \frac{\varepsilon}{2}.$$

By (4) of Theorem 2.1

$$\lim_{n \to \infty} P\left(\bigcap_{k=n}^{\infty} [X_k \in B_m(\delta)]\right) \geq P[X \in B_m(\delta)] > 1 - \frac{\varepsilon}{2}$$

and, by the definition of the limit, there exists an $n_0 \in \mathbb{N}$ such that

$$P\left(\bigcap_{k=n_0}^{\infty} [X_k \in B_m(\delta)]\right) > 1 - \frac{3\varepsilon}{4}.$$

On the other side, for each random element $X_i$ $(i = 1, \ldots, n_0 - 1)$ there exists $m_i$ such that

$$P[X_i \in B_{m_i}(\delta)] > 1 - \frac{\varepsilon}{4 \cdot 2^i}.$$

Put $m(\varepsilon, \delta) = \max\{m, m_1, m_2, \ldots, m_{n_0 - 1}\}$. Then

$$P\left(\bigcap_{i=1}^{\infty} [X_i \in B_{m(\varepsilon,\delta)}(\delta)]\right) > 1 - \varepsilon.$$

Define a set

$$K = \overline{\bigcap_{i=1}^{\infty} B_{m(\frac{\varepsilon}{2^k}, \frac{1}{k})}(\frac{1}{k})},$$

which is compact (it is closed and contains a finite $\varepsilon$-net). Moreover,

$$P\left(\bigcap_{i=1}^{\infty} [X_i \in K]\right) > 1 - \varepsilon. \tag{1}$$

Indeed,

$$\begin{aligned}
P\left(\bigcap_{i=1}^{\infty}[X_i \in K]\right) &= 1 - P\left(\bigcup_{i=1}^{\infty}[X_i \notin K]\right)\\
&= 1 - P\left(\bigcup_{i=1}^{\infty}\left[X_i \in \bigcap_{k=1}^{\infty} B_{m(\frac{\varepsilon}{2^k},\frac{1}{k})}(\frac{1}{k})\right]\right)\\
&= 1 - P\left(\bigcup_{i=1}^{\infty}\bigcup_{k=1}^{\infty}\left[X_i \notin B_{m(\frac{\varepsilon}{2^k},\frac{1}{k})}(\frac{1}{k})\right]\right)\\
&\geq 1 - \sum_{k=1}^{\infty} P\left(\bigcup_{i=1}^{\infty}\left[X_i \notin B_{m(\frac{\varepsilon}{2^k},\frac{1}{k})}(\frac{1}{k})\right]\right)\\
&= 1 - \sum_{k=1}^{\infty}\left(1 - P\left(\bigcap_{i=1}^{\infty}\left[X_i \in B_{m(\frac{\varepsilon}{2^k},\frac{1}{k})}(\frac{1}{k})\right]\right)\right)\\
&\geq 1 - \sum_{k=1}^{\infty}\frac{\varepsilon}{2^k} = 1 - \varepsilon.
\end{aligned}$$

Condition (1) assures strict tightness of the sequence $\{X_i\}$. □

Essential w.r.t. law convergence of random elements sequence $\{X_n\}$ is equivalent to the weak convergence of $\{X_\tau\}$ for all $\tau \to \infty$ $(\tau \in T)$. It is easy to see that the following theorem holds.

**Theorem 2.3.** *Suppose that for all $\tau \to \infty$ $(\tau \in T)$ $X_\tau \xrightarrow{D} X$, then a collection of probability measures $P_{X_\tau} = PX_\tau^{-1}$ is tight.*

By the Prokhorov theorem ([3]) if a sequence $\{X_n, n \geq 1\}$ of random elements converges in law to a random element $X$, then the sequence of their distributions is tight, i.e. for any $\varepsilon > 0$ there exists a compact $K_\varepsilon$ such that

$$P[X_n \in K_\varepsilon] > 1 - \varepsilon.$$

By the Theorem 2.3 we have

**Corollary 2.1.** *If for any $\tau \to \infty$, $(\tau \in T)$ $X_\tau \xrightarrow{D} X$, then the sequence $\{X_n, n \geq 1\}$ is strongly tight.*

## 3. Strong tightness in Polish spaces

**Theorem 3.1.** *Let $(S, \varrho)$ be a Polish space and let $\{X_n, n \geq 1\}$ be a sequence of $S$-valued random elements. If $X_n \xrightarrow{a.s.} X$ as $n \to \infty$, for some r.e. $X$, then the sequence $\{X_n\}$ is strongly tight.*

Proof: By the Theorem 2 in [5], $X_\tau \xrightarrow{D} X$ for any $\tau \in T$, such that $\tau \to \infty$. This combined with Corollary 2.1 completes the proof. □

Some properties of the metric space $(S, \varrho)$ carry over to the space of random elements $E_S$ with the Lévy-Prokhorov metric $L$ or with the Ky-Fan metric

$$K(X,Y) = \inf\{\varepsilon : P[\varrho(X,Y) > \varepsilon] < \varepsilon\}.$$

Examples of those properties are separability and completeness (see [3]). Unfortunately, compactness of the space $S$ does not assure compactness of the $(E_S, K)$.

**Example 3.1.** Let $\xi$ be a random variable uniformly distributed on $[0,1]$. Let $0, \delta_1\delta_2\delta_3\dots$ be an infinite dyadic representation of $\xi$, i.e. $\xi = \frac{\delta_1}{2} + \frac{\delta_2}{2^2} + \frac{\delta_3}{2^3} + \dots$. For any integer number $n$

$$[\delta_n = 0] = \bigcup_{i=1}^{2^{n-1}} \left[\frac{2(i-1)}{2^n} \le \xi < \frac{2i-1}{2^n}\right],$$

$$[\delta_n = 1] = \bigcup_{i=1}^{2^{n-1}} \left[\frac{2i-1}{2^n} \le \xi < \frac{2i}{2^n}\right].$$

Obviously,

$$P[\delta_n = 0] = \sum_{i=1}^{2^{n-1}} P\left[\frac{2(i-1)}{2^n} \le \xi < \frac{2i-1}{2^n}\right] = \sum_{i=1}^{2^{n-1}} \frac{1}{2^n} = \frac{1}{2}.$$

Analogously, $P[\delta_n = 1] = \frac{1}{2}$. Random variable $\delta_n$ are also independent. Indeed, take any finite sequence $\{i_1, i_2, \dots, i_n\} \subset \mathbb{N}$. Let $m = i_n$ and $\eta^{(m)} = \frac{\delta_{i_1}}{2^{i_1}} + \frac{\delta_{i_2}}{2^{i_2}} + \dots + \frac{\delta_{i_n}}{2^{i_n}}$ be an $m$-digital dyadic number. (This does not affect the above assumption of $\xi$ having infinite representations.) Let $\{\varepsilon_i\}$ be a 0-1 sequence.

$$\begin{aligned}
&P\left([\delta_{i_1} = \varepsilon_1] \cap [\delta_{i_2} = \varepsilon_2] \cap \dots \cap [\delta_{i_n} = \varepsilon_n]\right) \\
&= P\left[\eta^{(m)} = \frac{\delta_{i_1}}{2^{i_1}} + \frac{\delta_{i_2}}{2^{i_2}} + \dots + \frac{\delta_{i_n}}{2^{i_n}}\right] = \frac{1}{2^m} \\
&= P[\delta_{i_1} = \varepsilon_1] \cdot P[\delta_{i_2} = \varepsilon_2] \cdot \ldots \cdot P[\delta_{i_n} = \varepsilon_n].
\end{aligned}$$

Consider now the matrix

$$\begin{matrix} \delta_1 & \delta_3 & \delta_6 & \dots \\ \delta_2 & \delta_5 & \dots & \dots \\ \delta_4 & \dots & \dots & \dots \end{matrix}$$

and random dyadic numbers

$$\xi_1 = 0, \delta_1\delta_3\delta_6 \dots = \frac{\delta_1}{2} + \frac{\delta_3}{2^2} + \frac{\delta_6}{2^3} + \dots$$

$$\xi_2 = 0, \delta_2\delta_5\delta_9 \dots = \frac{\delta_2}{2} + \frac{\delta_5}{2^2} + \frac{\delta_9}{2^3} + \dots$$

$$\xi_3 = 0, \delta_4\delta_8 \dots = \frac{\delta_4}{2} + \frac{\delta_8}{2^2} + \dots$$

$\xi_i$ are independent for $\delta_i$ are. Now we will prove that $\xi_i$ are uniformly distributed on $[0,1]$. Indeed, for any $n$

$$\xi_i^{(n)} = \sum_{k=1}^{n} \frac{\delta_k}{2^k}$$

may take values from the set $\{0, \frac{1}{2^n}, \frac{2}{2^n}, \dots, \frac{2^n-1}{2^n}\}$ with probabilities $\frac{1}{2^n}$. As $n \to \infty$, $\xi_i^{(n)} \to \xi_i$ and the distribution of $\xi_i^{(n)}$ converges to the uniform distribution.

Let $\{\xi_n, n \geq 1\}$ be a sequence of i.i.d. random variables uniformly distributed on $[0,1]$ defined above. By the Borel-Cantelli Lemma a sequence of i.i.d. r.v. converges in law (and, equivalently, in the Ky-Fan metric) to a degenerated r.v. Indeed, let

$$F_n(x) = \begin{cases} 0, & \text{for } x \leq 0, \\ x, & \text{for } x \leq 1, \\ 1, & \text{for } x > 1, \end{cases}$$

be the distribution function of $\xi_n$. Let $A_n = [\xi_n < x]$. Then

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} P\xi_n^{-1}((-\infty, x)) = \sum_{n=1}^{\infty} F_n(x) = \begin{cases} 0, & \text{for } x \leq 0, \\ \infty, & \text{for } x > 0. \end{cases}$$

For $x \leq 0$, obviously $F_n(x) \to 0$. If $x > 0$, then, since $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \xi_k^{-1}((-\infty, x))$ is a decreasing sequence,

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \xi_k^{-1}((-\infty, x))\right) = \lim_{n\to\infty} P\xi_n^{-1}((-\infty, x)) = \lim_{n\to\infty} F_n(x) = 1$$

which equals 1, by the Borel-Cantelli Lemma.

## 4. Convergence in Banach spaces

Let $\mathcal{E}$ denote a Banach space with the norm $\|\cdot\|$ and let $\mathcal{E}^*$ be its dual with the norm $\|\cdot\|_*$.

We have the following result similar to the one obtained in [1].

**Lemma 4.1.** *Let $\mathcal{E}$ be a separable Banach space. Suppose $Y$ is an integrable cluster point of the sequence $\{X_n, n \geq 1\} \subset \mathcal{E}$. Then there exists an increasing sequence of stopping times $\{\tau_n, n \in \mathbb{B}\} \subset T$, such that*

$$X_{\tau_n} \to Y \quad \text{a.s.}$$

*as $n \to \infty$.*

Proof: We have to show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $m \geq 1$ we can choose $\tau_k \geq m$ so that

$$P[\varrho(X_{\tau_k}, Y) > \delta] \leq \varepsilon. \tag{4}$$

For $N \geq m$ define a random element

$$Z = E(Y|\mathcal{F}_N)$$

measurable with respect to $\mathcal{F}_N$. Then $P[\varrho(Y,Z) < \frac{\delta}{2}] > 1 - \frac{\varepsilon}{2}$, (see Proposition V-2-6 in [6]), and for all $N \geq 1, 2, \ldots$ there exists $n > N$ such that $\varrho(X_n, Y) < \frac{\delta}{2}$. Moreover $\varrho(X_n, Z) \leq \varrho(X_n, Y) + \varrho(Y, Z)$, therefore

$$[\varrho(Y,Z) < \frac{\delta}{2}] \subset [\varrho(X_n, Z) < \frac{\delta}{2},\ n \geq N].$$

Thus there exists $N_0 > N$ such that

$$P[\varrho(X_n, Z) < \frac{\delta}{2} \quad \text{for some} \quad N \leq n \leq N_0] > 1 - \frac{\varepsilon}{2}\,.$$

Define the set $\Phi_n = [\varrho(X_n, Z) < \frac{\delta}{2}]$ and a stopping time

$$\tau_{k+1}(\omega) = \begin{cases} m & k = 0, \\ \inf\{n > \tau_k(\omega) : \omega \in \Phi_n & \text{for some} \quad N \leq n \leq N_0\} \\ N_0 & \omega \notin \Phi_n. \end{cases}$$

Now $P[\varrho(X_{\tau_k}, Z) < \frac{\delta}{2}] \geq 1 - \frac{\varepsilon}{2}$ and

$$P[\varrho(X_{\tau_k}, Z) < \delta] \geq 1 - \varepsilon.$$

$\square$

Uniform boundness of $E\|X_n\|$ is one of the conditions that assure almost sure convergence of real-valued amarts. However this condition is not sufficient in Banach spaces. It turns out that strong tightness is necessary and sufficient condition of almost sure convergence of the $L^1$ bounded Banach space valued amarts.

Let us outline the proofs of these facts.

**Lemma 4.2.** *Let $\mathcal{E}$ be a Banach space and let $K$ be a compact subset of $\mathcal{E}$. There exists a countable sequence $\{x_k^*\} \subset \mathcal{E}^*$ such that for an arbitrary sequence $\{x_n\} \subset K$, $x_n \to x$ (in the norm) for some $x$ if and only if for all $k$, $x_k^*(x_n)$ converges* ([6]).

*Remark 4.1.*. In general, even the convergence of $\{x^*(x_n),\ n \in \mathbb{N}\}$ for all $x^* \in \mathcal{E}^*$ does not imply even weak convergence of $\{x_n,\ n \in \mathbb{N}\}$. Consider the following sequence $x_n = (\underbrace{1, 1, \ldots, 1}_{n}, 0, \ldots)$ in the space $c_0$ of all real-valued sequences converging to zero.

**Lemma 4.3.** *Suppose $\{X_n,\ n \geq 1\}$ is strongly tight sequence of random elements. Then there exists a countable subset $\{x_k^*\} \subset \mathcal{E}^*$ such that $X_n \xrightarrow{a.s.} X$ if and only if for any $k \in \mathbb{N}$ the sequence $\{x_k^*(X_n),\ n \in \mathbb{N}\}$ converges for $n \to \infty$.*

Proof: If $X_n \xrightarrow{a.s.} X$ then for any $x^* \in \mathcal{E}^*$ $x^*(X_n) \xrightarrow{a.s.} x^*(X)$.

Consider now *sufficiency.* Take any $p \in \mathbb{N}$, then there exists a compact set $K_{\frac{1}{p}}$ such that

$$P\left(\bigcap_{n=1}^{\infty} [X_n \in K_{\frac{1}{p}}]\right) > 1 - \frac{1}{p}.$$

By Lemma 4.2 for any $\{x_n,\ n \in \mathbb{N}\} \subset K_{\frac{1}{p}}$, $x_n$ converges to some $x$ if and only if there exists a countable set $\{x_l^{*(p)}\} \subset \mathcal{E}^*$ such that $x_l^{*(p)}(x_n)$ converges. Let

$$\{x_k^*\} = \{x_l^{*(p)},\ p, l \in \mathbb{N}\}.$$

Suppose that for all $k \in \mathbb{N}$ the sequence $\{x_k^*(X_n)\}$ converges a.s. for $n \to \infty$. Let $\Omega_0$ be a set where $\{x_k^*(X_n(\omega)),\ n \in \mathbb{N}\}$ converges for any $k$. Define

$$\Omega_p = \bigcap_{n=1}^{\infty} [X_n \in K_{\frac{1}{p}}] \cap \Omega_0$$

and $\Omega' = \bigcup_{p=1}^{\infty} \Omega_p$. Obviously, $P(\Omega_p) > 1 - \frac{1}{p}$ and $P(\Omega') = 1$. Take $\omega \in \Omega'$, then $\omega \in \Omega_p$ for some $p$. The sequence $x_l^{*(p)}(X_n(\omega))$ converges for all $l$. The limit is measurable. Thus, by Lemma 4.3, $X_n(\omega)$ converges, therefore $X_n$ converges a.s. □

### 4.1 Almost sure convergence of asymptotic martingales

**Definition 4.1** ([5])**.** A sequence $\{(X_n, \mathcal{A}_n);\ n \geq 1\}$ of Pettis integrable r.v.s. is called an asymptotic martingale (amart) iff $X_n$ is $\mathcal{A}_n$-measurable for every $n \in \mathbb{N}$ and if for every $\varepsilon > 0$ there exists $\tau_0 \in T$ such that for every $\tau, \nu \in T$ $\tau, \nu \geq \tau_0$ we have

$$\|EX_\tau - EX_\nu\| < \varepsilon.$$

**Theorem 4.1.** *Let $\{(X_n, \mathcal{A}_n)\}$ be an $L^1$-bounded asymptotic martingale. The necessary and sufficient condition for a.s. convergence of $X_n$ to an integrable random element $X$ is strong tightness of the sequence $\{X_n\}$.*

Proof: *Necessity* of the above condition follows from the Theorem 3.1. For *sufficiency*, assume that $\{X_n\}$ is strictly tight. For any $x^* \in \mathcal{E}^*$ the sequence $x^*(X_n)$ is an $L^1$-bounded real-valued asymptotic martingale. Indeed $\sup_n E|x^*(X_n)| \leq \sup_n \|x^*\|_* \cdot E\|X_n\| < \infty$ and $|Ex^*(X_\tau) - Ex^*(X_\sigma)| = |(EX_\tau) - x^*(EX_\sigma)| \leq \|x^*\|_* \|EX_\tau - EX_\sigma\|$. Since $\{x^*(X_n)\}$ is an $L^1$-bounded asymptotic martingale it converges a.s. ([1]) and, by Lemma 4.3 $X_n$ converges a.s. The limit $X$ of $\{X_n\}$ is integrable. Indeed, by Fatou lemma

$$\int X dP = \int \lim_{n\to\infty} X_n = \lim_{n\to\infty} \int X_n dP < \infty.$$

$\square$

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