Characterizing algebras of C^∞-functions on manifolds

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Abstract. Among all C^∞-algebras we characterize those which are algebras of C^∞-functions on second countable Hausdorff C^∞-manifolds.

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0. Introduction

It is well known (see e.g. [2]) that a completely regular topological space is determined by the algebra of all continuous functions on it. More precisely, let X_1, X_2 be completely regular topological spaces, and let $C(X_1)$, $C(X_2)$ be the corresponding algebras of continuous functions on them. If there is an isomorphism of algebras (in the purely algebraic sense) $\varphi : C(X_2) \to C(X_1)$, then there exists a unique homeomorphism $h : X_1 \to X_2$ such that $\varphi = h^*$, where h^* denotes the homomorphism induced by h. Similarly, a second countable Hausdorff C^∞-manifold is completely determined by the algebra of C^∞-functions on it. Namely, a theorem of the exactly same type as above holds, where, of course, the spaces are substituted by second countable Hausdorff C^∞-manifolds, algebras are substituted by algebras of all C^∞-functions, and there is diffeomorphism instead of homeomorphism. (See [6], [7], and [8].) In the framework of topology we can also find interesting results of another type, namely results describing which algebras can be realized as algebras of continuous functions on completely regular topological spaces (see [1]). The goal of this paper is to present a theorem showing which algebra can be realized as the algebra of C^∞-functions on a second countable Hausdorff C^∞-manifold. We characterize these algebras only among C^∞-algebras. But it is possible to recognize C^∞-algebras among locally-m-convex algebras (see [3, 6.6]).

1. C^∞-algebras

An $\mathbb{R}$-algebra is a commutative ring A with unit together with a ring homomorphism $\mathbb{R} \to A$. Then every map $p : \mathbb{R}^n \to \mathbb{R}^m$ which is given by an m-tuple of real polynomials $(p_1, \dots, p_m)$ in n variables can be interpreted as a mapping $A(p) : A^n \to A^m$ in such a way that projections, composition, and identity are preserved, by just evaluating each polynomial p_i on an n-tuple $(a_1, \dots, a_n) \in A^n$.

A C^∞-algebra A is a real algebra in which we can moreover interpret all smooth mappings $f:\mathbb{R}^n\to\mathbb{R}^m$. There is a corresponding map $A(f):A^n\to A^m$, and again projections, composition, and the identity mapping are preserved.

More precisely, a C^∞-algebra A is a product preserving functor from the category C^∞ to the category of sets, where C^∞ has as objects all spaces $\mathbb{R}^n$, $n\ge 0$, and all smooth mappings between them as arrows. Morphisms between C^∞-algebras are then natural transformations; they correspond to those algebra homomorphisms which preserve the interpretation of smooth mappings.

This definition of C^∞-algebras is due to Lawvere [4], for a thorough account see Moerdijk-Reyes [5], for a discussion from the point of view of functional analysis see [3].

2. Theorem. *Let A be a C^∞-algebra. Then A is the algebra of smooth functions on some finite dimensional second countable Hausdorff manifold M if and only if the following conditions are satisfied:*

(1) *A is* point determined ([5, 4.1])*, so A can be embedded as algebra into a power $\prod_{x\in X}\mathbb{R}$ of copies of $\mathbb{R}$. Equivalently, the intersection of all ideals of codimension 1 in A is 0.*
(2) *A is* finitely generated*, so $A=C^\infty(\mathbb{R}^n)/I$ for some ideal $I\subset C^\infty(\mathbb{R}^n)$.*
(3) *For each ideal $\mathfrak{m}_x$ of codimension 1 in A the localization $A_{\mathfrak{m}_x}$ is isomorphic to the C^∞-algebra $C^\infty_0(\mathbb{R}^m)$ consisting of all germs at 0 of smooth functions on $\mathbb{R}^m$, for some m.*

Proof: The conditions are obviously necessary. Let us prove they are also sufficient.

By condition (2) A is finitely generated, $A=C^\infty(\mathbb{R}^n)/I$; so by [5, 4.2] the C^∞-algebra A is point determined (1) if and only if the ideal I has the following property:

$$\text{(4)}\qquad \text{For } f\in C^\infty(\mathbb{R}^n),\quad f|Z(I)=0 \ \text{ implies }\ f\in I,$$

where $Z(I)=\bigcap\{f^{-1}(0):f\in I\}\subset\mathbb{R}^n$. Let us denote by $\{\mathfrak{m}_x:x\in M\}$ the set of all ideals $\mathfrak{m}_x$ of codimension 1 in A. Then $A/\mathfrak{m}_x\cong\mathbb{R}$ and we write $a(x)$ for the projection of $a\in A$ in $A/\mathfrak{m}_x$. In particular we identify the elements of A with functions on M. Let $c_1,\dots,c_n\in A$ be a set of generators. Then we may view $c=(c_1,\dots,c_n):M\to\mathbb{R}^n$ as a mapping such that the pullback $c^*(f)=f\circ c=A(f)(c)$ is the quotient mapping $C^\infty(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n)/I=A$. By condition (1) $c:M\to\mathbb{R}^n$ is injective, and the image $c(M)$ equals $Z(I)=\bigcap\{f^{-1}(0):f\in I\}$, by (4). In particular, $c(M)$ is closed. The initial topology on M with respect to all functions in A coincides with the subspace topology induced via the embedding $c:M\to\mathbb{R}^n$, so this topology is metrizable and locally compact.

Let us fix a 'point' $x\in M$. The codimension 1 ideal $\mathfrak{m}_x$ is a prime ideal, so the subset $A-\mathfrak{m}_x\subset A$ is closed under multiplication and without divisors of 0, thus the localization $A_{\mathfrak{m}_x}$ may be viewed as the set of fractions $\frac{a}{b}$ with $a\in A$, $b\in A-\mathfrak{m}_x$; it is a local algebra with maximal ideal $\tilde{\mathfrak{m}}_x=\{\frac{a}{b}:a\in\mathfrak{m}_x,b\in$

$A - \mathfrak{m}_x\}$. Note that $\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 \cong T_0^*\mathbb{R}^m = \mathbb{R}^m$ by condition (3). Now choose $a_1, \dots, a_m \in \mathfrak{m}_x$ such that $\frac{a_1}{1}, \dots, \frac{a_m}{1} \in A_{\mathfrak{m}_x}$ form a basis of $\tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 = \mathbb{R}^m$, and choose $g_1, \dots, g_m \in C^\infty(\mathbb{R}^n)$ with $c^*(g_i) = a_i$. Then $g_i(c(x)) = 0$, so g_i is in the codimension 1 ideal $\mathfrak{m}_{c(x)} = \{f \in C^\infty(\mathbb{R}^n) : f(c(x)) = 0\}$. Since $c^* : C^\infty(\mathbb{R}^n) \to A$ induces in turn homomorphisms

$$C^\infty_{c(x)}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)_{\mathfrak{m}_{c(x)}} \to A_{\mathfrak{m}_x}$$
$$\mathbb{R}^n = T^*_{c(x)}\mathbb{R}^n = \tilde{\mathfrak{m}}_{c(x)}/\tilde{\mathfrak{m}}^2_{c(x)} \to \tilde{\mathfrak{m}}_x/\tilde{\mathfrak{m}}_x^2 = \mathbb{R}^m$$

and since $\mathfrak{m}_{c(x)} \cong \mathfrak{m}_x \oplus I$ as vector spaces, we may find functions $g_{m+1}, \dots, g_n \in I$ such that the quotients $\frac{g_1}{1}, \dots, \frac{g_n}{1} \in C^\infty_{c(x)}(\mathbb{R}^n)$ map to a basis of $\tilde{\mathfrak{m}}_{c(x)}/\tilde{\mathfrak{m}}^2_{c(x)} = T^*_{c(x)}\mathbb{R}^n$. By the implicit function theorem on $\mathbb{R}^n$ the functions $g_{m+1}, \dots, g_n$ are near $c(x)$ an equation of maximal rank for $c(M) = Z(I)$, and the functions $g_1, \dots, g_m$ restrict to smooth coordinates near $c(x)$ on the closed submanifold $c(M) = Z(I)$ of $\mathbb{R}^n$, and the number m turns out to be a locally constant function on M. Also the functions $a_1, \dots, a_m$ restrict to smooth coordinates near x of M. $\square$

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