OCA and towers in $\mathcal{P}(\mathbb{N})/fin$

Ilijas Farah*

Abstract. We shall show that Open Coloring Axiom has different influence on the algebra $\mathcal{P}(\mathbb{N})/fin$ than on $\mathbb{N}^{\mathbb{N}}/fin$. The tool used to accomplish this is forcing with a Suslin tree.

Keywords: Open Coloring Axiom, dense sets of reals, towers, forcing, Suslin trees

Classification: 03E05, 03E35, 04A20, 03E50, 06A05

The Open Coloring Axiom is a natural Ramseyan statement about sets of reals which can also be considered as a natural two-dimensional version of the classical Perfect Set Property of sets of reals. It states that if X is a set of reals and $[X]^2 = K_0 \cup K_1$ is a partition such that K_0 is open, then one of the following applies:

(1) there is an uncountable K_0-homogeneous subset of X, or
(2) X is the union of countably many K_1-homogeneous sets (i.e. it is σ-K_1-*homogeneous*).

It is a quite useful principle with strong influence on structures in close relationship with the set of reals, such as $\langle \mathbb{N}^{\mathbb{N}}, \leq^* \rangle$ or the algebra $\mathcal{P}(\mathbb{N})/fin$ (see [7], [9]). Deeper applications of OCA frequently need some additional help of Martin's Axiom which is its natural context because both of the axioms can be considered as parts of the stronger Proper Forcing Axiom. The purpose of this note is to explain why some parts of MA are needed at least in some of the well-known applications. We shall choose one of the most famous consistency results about the real numbers: the Baumgartner's statement BA which says that all $\aleph_1$-dense sets of reals are isomorphic (a set of reals X is $\aleph_1$-*dense* iff for all $x < y$ in X the interval $(x, y) \cap X$ has size $\aleph_1$). This is a natural generalization of Cantor's theorem that all $\aleph_0$-dense sets of reals are isomorphic. Clearly, BA is at least formally stronger than the statement that every set of reals of size $\aleph_1$ is isomorphically embeddable into any other uncountable set of reals. We can weaken this still further and say (A is σ-*embeddable* into B if there are countably many partial increasing maps from A into B such that their domains cover A):

(*) If A and B are sets of reals of size $\aleph_1$ then A is σ-embeddable into B.

*Partially supported by Ontario Graduate Scholarship.

In [7, §8] it is proved that (*) plus a little bit of MA implies BA. The form of MA that is needed can be stated as saying that $\mathfrak{t} > \omega_1$, namely that every tower of height ω_1 in $\langle [\mathbb{N}]^\omega, \subseteq^* \rangle$ can be extended (or equivalently, that $\mathrm{MA}_{\aleph_1}$ for σ-centered posets is true — see [5]). This leads to following natural questions some of which are answered here:

(a) What if we use OCA instead of (*)? Or, does OCA imply (*)?
(b) What about removing $\mathfrak{t} > \omega_1$, even if (*) is supported by OCA?
(c) Does BA or OCA imply $\mathfrak{t} > \omega_1$?

[Note that both OCA and BA have a strong influence on $\mathfrak{b}$, the corresponding cardinal of the structure $\langle \mathbb{N}^{\mathbb{N}}, \leq^* \rangle$ (see [7, §§0,1,7,8]). For another explanation of difference between $\mathfrak{t}$ and $\mathfrak{b}$, see [8, Example 1].] We give a negative answer to **(b)** and to OCA-part of **(c)** in Theorem 1 below. The question whether OCA implies BA was independently from us asked by M. Weese [10]. It should be remarked that MA alone does not imply BA (see [1]).

Theorem 1. *There is a forcing extension of L in which OCA and* (*) *are true,* $\mathfrak{t} = \omega_1$ *and BA fails.*

Consider the following strengthening of (*):

(**) For every uncountable family $\mathcal{F}$ of (nonempty) countable pairwise disjoint sets of reals and for every set of reals A of size $\aleph_1$ there are $h: A \to \mathcal{F}$ and partial increasing maps $f_n: A \to \bigcup \mathcal{F}$ such that $\{f_n(a) \mid n \in \mathbb{N}, a \in \operatorname{dom} f_n\} = h(a)$.

Proof of Theorem 1: It will obviously suffice to prove the following lemmas. $\square$

Lemma 1. *Forcing with a complete ccc Boolean algebra which does not add reals* (*i.e. a Suslin algebra*) *preserves OCA and* (**).

Lemma 2. *Forcing with a Suslin tree makes* $\mathfrak{t} = \omega_1$.

We have discovered Lemma 2 while reading Dordal's paper [4] which uses similar arguments. A. Dow has pointed out that Lemma 2 was known to Booth.

Lemma 3. *Forcing with a Suslin tree adds to the universe an* $\aleph_1$*-dense set of reals B such that*
(a) *no uncountable ground-model set of reals can be mapped to B by a continuous* 1–1 *map,*
(b) *there is no continuous* 1–1 *map from B into itself other than the identity.*
So in particular BA fails in a forcing extension by a Suslin tree.

Lemma 4. *There is a forcing extension of L in which OCA and* (**) *are true and Suslin Hypothesis fails.*

Proof of Lemma 1: Suppose that OCA is true, and that $\mathcal{S}$ is a Suslin algebra. Let $\dot{X}$ be an $\mathcal{S}$-name for a set of reals, and let $[\dot{X}]^2 = K_0 \cup K_1$ be an open partition. [Since $\mathcal{S}$ is not adding reals, assuming that K_0 is in the ground model is not a loss of generality. Notice also that $\dot{X}$ is a set of ground-model reals.] All we have to prove is that if $\dot{X}$ is forced to be non-σ-K_1-homogeneous, then the set of all $p \in \mathcal{S}$ which force that there is an uncountable 0-homogeneous subset of $\dot{X}$ is dense in $\mathcal{S}$.

Work in the ground model: fix $q \in \mathcal{S}$, let X^q be the set of all possibilities below q for $\dot{X}$, i.e. the set $\{x \in \mathbb{R} \mid p \Vdash \check{x} \in \dot{X}$, for some $p \leq q\}$. Since q forces that X^q includes $\dot{X}$ it is also non-σ-1-homogeneous, and therefore (by OCA) it has an uncountable 0-homogeneous subset $Y = \{y_\xi \mid \xi < \omega_1\}$. Let $p_\xi \leq q$ be a condition which forces that y_ξ is in $\dot{X}$, for $\xi < \omega_1$. By ccc-ness of $\mathcal{S}$ we can pick $p \leq q$ which forces that there are uncountably many p_ξ's in a generic filter $\dot{G}$, so p forces that $Y \cap \dot{X}$ is an uncountable 0-homogeneous subset of $\dot{X}$.

To see that (**) is preserved, first observe that it is enough to check the case when A is a ground-model set of reals and $\dot{\mathcal{F}} = \{\dot{F}_\alpha \mid \alpha < \omega_1\}$ is a $\mathcal{S}$-name for a family as in (**). Let $\hat{F}_\alpha$ be the set of all possibilities for $\dot{F}_\alpha$, set $\{x \in \mathbb{R} \mid$ some $p \in \mathcal{S}$ forces that $x \in \dot{F}_\alpha\}$, and let $\hat{\mathcal{F}} = \{\hat{F}_\alpha \mid \alpha < \omega_1\}$. By going to a subset we can assume that $\hat{\mathcal{F}}_\alpha$ is a family of pairwise disjoint sets, so fix h and $\{f_n\}$ as in (**). Go to a forcing extension by $\mathcal{S}$, and for $a \in A$ define

$$h'(a) = \dot{F}_\alpha, \qquad \text{if } h(a) = \hat{F}_\alpha,$$

$$f'_n(a) = \begin{cases} f_n(a), & \text{if } f_n(a) \in h'(a), \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Then h and $\{f'_n\}$ witness (**). □

Remark. Similarly, forcing with $\mathcal{S}$ preserves $\mathrm{OCA}^{[\mathrm{ARS}]}$ and SOCA (i.e. versions of Open Coloring Axiom defined in [1]): In the case of $\mathrm{OCA}^{[\mathrm{ARS}]}$, just notice that the set of possibilities for $\dot{X}$ is σ-homogeneous, and therefore X is such as well. For SOCA, the proof is analogous to the case when X^q is not σ-1-homogeneous in the proof of OCA. Note that in these proofs the use of ccc-ness of $\mathcal{S}$ was necessary, because a natural poset for collapsing $\mathfrak{c}$ to $\aleph_1$ does not add reals, while all of the above axioms contradict CH.

Proof of Lemma 2: Let $T \subseteq \mathcal{P}(\mathbb{N})/fin$ be such that $\langle T, \supseteq^* \rangle$ is a Suslin tree with the property that incomparable nodes are almost disjoint. Work in a forcing extension by T: let B be a cofinal branch of T. Then it is a decreasing ω_1-tower in $[\mathbb{N}]^\omega$; we claim that no infinite $a \subseteq \omega$ extends it. Suppose that it is not true and fix such a. Then a is in the ground model, and the ground-model set $\{s \in T \mid s \supseteq^* a\}$ is, by (2), equal to B — a contradiction. □

Proof of Lemma 3: Assume that T is a set of reals, and that $<_T$ is a Suslin tree ordering on it. Since for every set of reals $\{x_\xi \mid \xi < \omega_1\}$ in its bijective enumeration there is an α such that $\{x_\xi \mid \alpha \leq \xi < \omega_1\}$ is $\aleph_1$-dense, we can assume that T forces that the generic branch $\dot{B}$ is an $\aleph_1$-dense set of reals. We shall prove

that $\dot{B}$ has the required properties. Notice that if $Y \subseteq \dot{B}$ is uncountable, then $\dot{B}$ can be defined as the downward closure of Y in T, so in particular Y is not in the ground model.
(a) Suppose that $\dot{Z}$ is a T-name for a ground-model set of reals and that $\dot{f}$ is a T-name for a continuous real function mapping $\dot{Z}$ into B with an uncountable range. Work in V^T: we can extend f to a Borel function $\bar{f}$ whose domain includes $\dot{Z}$. Since T is not adding reals, $\bar{f}$ is coded by a ground-model real. But then $\bar{f}''Z$ is also in the ground model, and so is $\dot{B}$ — a contradiction.
(b) Work in the extension: Suppose that $f: \dot{B} \to \dot{B}$ is a continuous 1–1 map, different from the identity. Again $f = \bar{f} \restriction \dot{B}$ for some ground-model function $\bar{f}$. Since $\dot{B}$ is $\aleph_1$-dense, f moves uncountably many points. We claim that $t <_T f(t)$ for uncountably many $t \in \dot{B}$: let $\dot{B} = \{t_\xi \mid \xi < \omega_1\}$ be an increasing enumeration of $\dot{B}$, and let $\hat{f}: \omega_1 \to \omega_1$ be the function defined by

$$\hat{f}(\xi) = \eta \qquad \text{iff} \qquad f(t_\xi) = t_\eta.$$

Then if α is an ordinal such that $\hat{f}''\alpha \subseteq \alpha$, $\hat{f}^{-1}\alpha \subseteq \alpha$ and $\beta > \alpha$ is the least such that $\hat{f}(\beta) \neq \beta$, we indeed have $t_\beta <_T f(t_\beta)$, so there are uncountably many such β. Let s be a T-condition which decides this fact and $\bar{f}$. Then the set B^s of possibilities for $\dot{B}$ below s (i.e. the cone $\{t \in T \mid s <_T t\}$) is included in the domain of $\bar{f}$. Notice that $\bar{f}(t)$ and t are $\leq_T$-compatible for all t in B^s. By our choice of S, the set

$$\{t \in T[s] \mid t <_T \bar{f}(t)\}$$

is dense in T below s, so the set

$$\{r \in T \mid \text{ for some } t <_T r \text{ nodes } \bar{f}(t) \text{ is incomparable with } r\}$$

is dense as well. But $\dot{B}$ avoids this set — otherwise $f''\dot{B}$ would not be a subset of $\dot{B}$. This contradicts to the assumed genericity of $\dot{B}$. $\square$

Proof of Lemma 4: By $[X]^{<\omega}_{K_0}$ we denote a poset of all finite subsets of X which are 0-homogeneous. Assume that $[Y]^2 = K_0 \cup K_1$ is an open partition, fix a bijective enumeration $Y = \{y_\xi \mid \xi < \omega_1\}$, let $Y_{(\cdot,\alpha)} = \{y_\xi \mid \xi < \alpha\}$ and $Y_{[\alpha,\cdot)} = \{y_\xi \mid \alpha \leq \xi\}$, and consider the following property (for the definition of the oscillation mapping ω^*_{f,K_0} see [7, p. 39]) of the set Y:

(†) Y is not σ-K_1-homogeneous and for every countable $D \subseteq Y$ and $f: D^n \to Y$ for $n \in \mathbb{N}$ there are at most countably many $\alpha < \omega_1$ such that $\omega^*_{f,K_0}(\vec{x}) \cap Y_{[\alpha,\cdot)}$ is nonempty and σ-K_1-homogeneous for some $\vec{x} \in Y^n_{(\cdot,\alpha)}$.

In [7, Theorem 4.4] the following was proved:

Proposition 1 (CH). **(a)** *If $[X]^2 = K_0 \cup K_1$ is an open partition and X is not σ-K_1-homogeneous then there is an uncountable $Y \subseteq X$ with the property* (†).
(b) *If Y has the property* (†)*, then the poset $[Y]^{<\omega}_{K_0}$ is ccc.* $\square$

For $A, \mathcal{F}$ as in (**) and $h: A \to \bigcup \mathcal{F}$ let $\mathcal{P}_h$ be the poset of all finite strictly increasing functions p from A into $\bigcup \mathcal{F}$ such that $p(a) \in h(a)$ for all $a \in \mathrm{dom}(p)$. Then the poset $\mathcal{P}_h^{<\omega}$ of all finite powers of $\mathcal{P}_h$ generically adds sequence $\{f_n\}$ as in (**), so it remains only to assure that this poset is ccc. Fix a 1–1 enumeration $A = \{a_\alpha \mid \alpha < \omega_1\}$ and consider the following property of the function h:

(††) For every G_δ subset G of $\mathbb{R}^n$ (for some $n \in \mathbb{N}$) and a continuous $g: G \to \mathbb{R}$ there is a countable ordinal α such that $g'' A^n_{[\alpha,\cdot)}$ is disjoint from $A_{[\alpha,\cdot)}$.

Proposition 2 (CH). **(a)** *For all $A, \mathcal{F}$ as in* (**) *there is a mapping $h: A \to \mathcal{F}$ such that the poset $\mathcal{P}_h$ has the property* (††).
(b) *If h has the property* (††)*, then the poset $\mathcal{P}_h^{<\omega}$ is ccc.*

PROOF: This proof is essentially the same as [7, Theorem 4.2], so we will just outline the construction and refer the reader there if he or she wants more details. Let g_ξ ($\xi < \omega_1$) be the enumeration of all continuous functions as in (††). We inductively define $h(a_\alpha)$ for all α. At the αth step of our construction consider a countable set

$$\{g_\xi(p\hat{\ }x_\alpha) \mid \xi < \alpha \text{ and } p \text{ is a finite sequence in } \{a_\eta, h(a_\eta) \mid \eta < \alpha\}\}$$

and pick $h(a_\alpha) \in \mathcal{F}$ to be disjoint from it. Then h is as required. $\square$

We always assume that poset $[X]^{<\omega}_{K_0}$ has the property (†) and that poset $\mathcal{P}_h$ has the property (††). In the universe which satisfies combinatorial principles CH and $\Diamond_{\omega_2}$ one can define a forcing notion $\mathcal{P}$ such that OCA and (**) are true in the extension by $\mathcal{P}$ (this construction which uses $\Diamond_{\omega_2}$ as a refection device is described in [6]). Poset $\mathcal{P}$ is a finite support iteration of posets $[X]^{<\omega}_{K_0}$ and $\mathcal{P}_h^{<\omega}$.

Claim. **(a)** *Forcing with the poset $[X]^{<\omega}_{K_0}$ does not destroy Suslin trees.*
(b) *Forcing with the poset $\mathcal{P}_h^{<\omega}$ does not destroy Suslin trees.*
(c) *Forcing with a finite support iteration $\mathcal{P}$ of posets $[X]^{<\omega}_{K_0}$ and $\mathcal{P}_h^{<\omega}$ does not destroy Suslin trees.*

PROOF: **(a)** Let T be a Suslin tree. Since "$[X]^{<\omega}_{K_0}$ destroys T" means "$[X]^{<\omega}_{K_0}$ forces that $\langle T, <_T\rangle$ is not ccc", it is enough to prove that forcing with T cannot destroy ccc-ness of $[X]^{<\omega}_{K_0}$. But all objects mentioned in (†) are coded by reals (note that every K_1-homogeneous set is included in a closed K_1-homogeneous set) and a fixed enumeration of X, so forcing with T does not change the situation, and therefore the proof that $[X]^{<\omega}_{K_0}$ is ccc is the same in the extension by T as it is in the ground model.

(b) Similarly, statement (††) mentions only reals and a fixed enumeration of A.

(c) This statement is equivalent to "Forcing with a Suslin tree T does not destroy ccc-ness of $\mathcal{P}$", and it follows from **(a)**, **(b)**, and the fact that a finite support iteration of ccc posets is ccc, applied in the extension by T. $\square$

So if we perform a natural forcing of OCA and (**), then there is a Suslin tree in an extension.

Remark. It should be noted that we can get a model as in Theorem 1 in which there are $2^{\aleph_1} = \aleph_2$ many pairwise nonisomorphic $\aleph_1$-dense sets of reals. To get this, fix a Suslin algebra $\mathcal{S}$ of size $\aleph_2$ in the ground model. (By a result of Jensen, the existence of such object follows from combinatorial principles $\diamondsuit_{\omega_1}$ and $\square_{\omega_1}$ which are true in L.) The properties of $\mathcal{S}$ that we need are (see [3]):

(1) every Suslin algebra has a regular subalgebra which is isomorphic to a regular open algebra of some Suslin tree,
(1) every nontrivial quotient of a Suslin algebra is a Suslin algebra, so
(2) there is an increasing sequence $\{\mathcal{S}_\xi \mid \xi < \omega_2\}$ of regular subalgebras of $\mathcal{S}$ such that the quotient $\mathcal{S}_{\xi+1}/\mathcal{S}_\xi$ is isomorphic to a regular open algebra of a Suslin tree for all $\xi < \omega_2$.

Observe that the proof of Lemma 4 also shows that ccc-ness of $\mathcal{S}$ is preserved after forcing OCA and (**). Also, observe that it is impossible to make nonisomorphic sets of reals isomorphic without adding reals. So forcing with $\mathcal{S}$ adds to universe a sequence of $\aleph_2$ many nonisomorphic $\aleph_1$-dense sets of reals (corresponding to Suslin trees as in (3)), and it does not destroy OCA and (**).

References

[1] Abraham U., Rubin M., Shelah S., *On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types*, Ann. of Pure and Appl. Logic **29** (1985), 123–206.

[2] Baumgartner J., *All $\aleph_1$-dense sets of reals can be isomorphic*, Fundamenta Mathematicae **79** (1973), 100–106.

[3] Devlin K., Johnsbråten H., *The Souslin Problem*, Springer Lecture Notes in Mathematics, # 405, 1974.

[4] Dordal P.L., *Towers in $[\omega]^\omega$ and ${}^\omega\omega$*, Ann. of Pure and Appl. Logic **45.3** (1989), 247–277.

[5] Fremlin D., *Consequences of Martin's Axiom*, Cambridge University Press, 1984.

[6] Gruenhage G., *Cosmicity of cometrizable spaces*, Trans. AMS **313** (1989), 301–315.

[7] Todorčević S., *Partition Problems in Topology*, AMS, Providence, Rhode Island, 1989.

[8] Todorčević S., *Oscillations of sets of integers*, to appear.

[9] Veličković B., *OCA and automorphisms of $\mathcal{P}(\omega)/fin$*, Topology Appl. **49** (1993), 1–13.

[10] Weese M., *personal communication.*

Department of Mathematics, University of Toronto, Toronto, Ontario M5S 3G3, Canada
and
Matematicki Institut, Kneza Mihaila 35, Beograd, Yugoslavia

(*Received* October 16, 1995)