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On the cardinality of functionally Hausdorff spaces

Alessandro Fedeli

Abstract. In this paper two new cardinal functions are introduced and investigated. In particular the following two theorems are proved:
(i) If X is a functionally Hausdorff space then $|X| \leq 2^{fs(X)\psi_\tau(X)}$;
(ii) Let X be a functionally Hausdorff space with $fs(X) \leq \kappa$. Then there is a subset S of X such that $|S| \leq 2^\kappa$ and $X = \bigcup\{cl_{\tau\theta}(A) : A \in [S]^{\leq\kappa}\}$.

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A space X is said to be functionally Hausdorff if whenever $x \neq y$ in X there is a continuous real valued function f defined on X such that $f(x) = 0$ and $f(y) = 1$.

In the last years many results involving cardinal functions related to s (spread) have been obtained by several authors (see e.g. [8], [9], [10], [12]).

In this paper we give a result on the bound of the cardinality of functionally Hausdorff spaces using two new cardinal functions fs and ψ_τ related to s and ψ respectively. Moreover we prove, for functionally Hausdorff spaces, a variant of a well-known result on spread due to Shapirovskii ([11, Theorem 3], [4, Theorem 5.1]).

We refer the reader to [1], [4] and [7] for notations and definitions not explicitly given. $\chi(X)$, $s(X)$ and $\psi(X)$ denote respectively the character, the spread and the pseudocharacter of a space X.

Let A be a subset of a space X:

(i) ([5], [6]) The τ-closure of A, denoted by $cl_\tau(A)$, is the set of all points $x \in X$ such that any cozero-set neighbourhood of x intersects A.

(ii) ([2]) The $\tau\theta$-closure of A, denoted by $cl_{\tau\theta}(A)$, is the set of all points $x \in X$ such that $cl_\tau(V) \cap A \neq \emptyset$ for every open neighbourhood V of x.

For every X and every $A \subset X$ we have $\overline{A} \subset cl_{\tau\theta}(A) \subset cl_\tau(A)$. It is clear that if X is completely regular then $\overline{A} = cl_{\tau\theta}(A) = cl_\tau(A)$ for every $A \subset X$.

Definition 1. Let X be a space. The functional spread of X, denoted by $fs(X)$, is the smallest infinite cardinal number κ such that for every open family $\mathcal{U}$ of X and every $A \subset \bigcup\mathcal{U}$ there exist a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ and a $B \in [A]^{\leq\kappa}$ such that $A \subset cl_{\tau\theta}(B) \cup \bigcup\{cl_\tau(V) : V \in \mathcal{V}\}$.

Remark 2. Let $\mathcal{U}$ be an open cover of a space X, let $s(X) \leq \kappa$. By a well-known result of Shapirovskii it follows that there are a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ and a $A \in [X]^{\leq\kappa}$ such that $X = \overline{A} \cup \bigcup \mathcal{V}$. Since $s(Y) \leq s(X)$ for every subspace Y of X it easily follows that $fs(X) \leq s(X)$. However the above inequality can be proper as the following example shows. For every $x \in R$ let $\mathcal{B}_x = \{\{x\} \cup B(x, \frac{1}{n}) \cap Q : n \in N\}$ and let X be R with the topology generated by the neighbourhood system $\{\mathcal{B}_x\}_{x \in R}$. Then X is a functionally Hausdorff space such that $fs(X) = \omega < s(X)$.

Definition 3. Let X be a functionally Hausdorff space and let $x \in X$. A family of open neighbourhoods of x is said to be a τ-pseudobase for x if $\bigcap\{cl_\tau(U) : U \in \mathcal{U}\} = \{x\}$. Let $\psi_\tau(x, X)=$ min $\{|\mathcal{U}| : \mathcal{U}$ is a τ-pseudobase for $x\} + \omega$, the τ-pseudocharacter of X is defined as follows: $\psi_\tau(X)=$ sup $\{\psi_\tau(x, X) : x \in X\}$.

Remark 4. It is obvious that for every Tychonoff space X we have $\psi_\tau(X) = \psi(X)$. Moreover $\psi_\tau(X) \leq \chi(X)$ for every functionally Hausdorff space X. In fact let $x \in X$ and let $\mathcal{B}_x$ be a local base at x, we claim that $\bigcap\{cl_\tau(B) : B \in \mathcal{B}_x\} = \{x\}$. Let us consider a point $y \in X \setminus \{x\}$, since X is functionally Hausdorff there is a continuous mapping $f : X \to I$ such that $f(x) = 0$ and $f(y) = 1$. Let $B \in \mathcal{B}_x$ such that $B \subset f^{-1}([0, \frac{1}{2}))$, then $cl_\tau(B) \subset f^{-1}([0, \frac{1}{2}])$. Hence $y \notin cl_\tau(B)$.

The above inequality can be proper. Let τ be the euclidean topology on R and let X be R with the topology $\sigma = \{V \setminus C : V \in \tau, C \subset R$ and $|C| \leq \omega\}$. Then X is a functionally Hausdorff space such that $\psi_\tau(X) = \omega < \chi(X)$.

A relation between ψ_τ and fs is given in the following

Proposition 5. *If X is a functionally Hausdorff space then $\psi_\tau(X) \leq 2^{fs(X)}$.*

Proof: Let $fs(X) = \kappa$ and $x \in X$. Since X is functionally Hausdorff then for every $y \in X \setminus \{x\}$ there are open sets U_y and V_y such that $x \in U_y$, $y \in V_y$ and $cl_\tau(U_y) \cap cl_\tau(V_y) = \emptyset$. Since $fs(X) = \kappa$ we can find $A, B \in [X \setminus \{x\}]^{\leq\kappa}$ such that $X \setminus \{x\} \subset cl_{\tau\theta}(A) \cup \bigcup\{cl_\tau(V_y) : y \in B\}$. Let $\mathcal{C} = \{C \subset A : x \notin cl_\tau(C)\}$, for every $C \in \mathcal{C}$ take a cozero-set $G(C)$ such that $x \in G(C)$ and $cl_\tau(G(C)) \subset X \setminus cl_\tau(C)$. Set $\mathcal{A} = \{G(C) : C \in \mathcal{C}\}$, $\mathcal{B} = \{U_y : y \in B\}$ and $\mathcal{U} = \mathcal{A} \cup \mathcal{B}$. Clearly $|\mathcal{U}| \leq 2^\kappa$. We claim that the family $\mathcal{U}$ of open neighbourhoods of x is a τ-pseudobase for x. Let us take $z \in X \setminus \{x\}$. If $z \in \bigcup\{cl_\tau(V_y) : y \in B\}$ then there is an $y \in B$ such that $z \notin cl_\tau(U_y) \supset \bigcap\{cl_\tau(U) : U \in \mathcal{U}\}$. If $z \in cl_{\tau\theta}(A)$ let $\mathcal{B}_z = \{B_\lambda : \lambda \in \Lambda\}$ be the family of all open neighbourhoods of z, choose a point $x_\lambda \in cl_\tau(B_\lambda \cap V_z) \cap A$ for every $B_\lambda \in \mathcal{B}_z$ and set $C = \{x_\lambda : \lambda \in \Lambda\}$. Clearly $C \subset A$, $z \in cl_{\tau\theta}(C) \subset cl_\tau(C) \subset cl_\tau(V_z)$ and $x \notin cl_\tau(C)$. Therefore $C \in \mathcal{C}$ and $z \notin cl_\tau(G(C)) \supset \bigcap\{cl_\tau(U) : U \in \mathcal{U}\}$. Hence $\bigcap\{cl_\tau(U) : U \in \mathcal{U}\} = \{x\}$. □

Theorem 6. *If X is a functionally Hausdorff space then $|X| \leq 2^{fs(X)\psi_\tau(X)}$.*

Proof: Let $fs(X)\psi_\tau(X) = \kappa$, and for each $x \in X$ let $\mathcal{V}_x$ be a τ-pseudobase for x with $|\mathcal{V}_x| \leq \kappa$. Construct a sequence $\{A_\alpha : \alpha < \kappa^+\}$ of subsets of X and a sequence of open collections $\{\mathcal{V}_\alpha : 0 < \alpha < \kappa^+\}$ such that:

(i) $|A_\alpha| \le 2^\kappa$ for every $\alpha < \kappa^+$;

(ii) $\mathcal{V}_\alpha = \{V : V \in \mathcal{V}_x, x \in \bigcup_{\beta<\alpha} A_\beta\}$, $0 < \alpha < \kappa^+$;

(iii) If $\mathcal{W}$ is a family of $\le \kappa$ elements of $\mathcal{V}_\alpha$ and K_λ, $\lambda < \kappa$, are subsets of $\bigcup_{\beta<\alpha} A_\beta$ with $|K_\lambda| \le \kappa$ and $X \neq \bigcup_{\lambda<\kappa} cl_{\tau\theta}(K_\lambda) \cup \bigcup\{cl_\tau(W) : W \in \mathcal{W}\}$ then $A_\alpha \setminus (\bigcup cl_{\tau\theta}(K_\lambda) \cup \bigcup\{cl_\tau(W) : W \in \mathcal{W}\}) \neq \emptyset$.

Let $A = \bigcup_{\alpha<\kappa^+} A_\alpha$. It is enough to show that $A = X$. Suppose not and let $z \in X \setminus A$. Let $\mathcal{V}_z = \{V_\lambda : \lambda \in \Lambda\}$, $|\Lambda| \le \kappa$, since $\{z\} = \bigcap\{cl_\tau(V_\lambda) : \lambda \in \Lambda\}$ it follows that $X \setminus \{z\} = \bigcup\{X \setminus cl_\tau(V) : \lambda \in \Lambda\}$.

For every $\lambda \in \Lambda$ let $S_\lambda = A \cap (X \setminus cl_\tau(V_\lambda))$, and for every $y \in S_\lambda$ let $U_y \in \mathcal{V}_y$ such that $z \notin cl_\tau(U_y)$. Since $fs(X) \le \kappa$ there are $B_\lambda, C_\lambda \in [S_\lambda]^{\le\kappa}$ such that $S_\lambda \subset cl_{\tau\theta}(C_\lambda) \cup \bigcup\{cl_\tau(U_y) : y \in B_\lambda\}$.

Let $B = \bigcup\{B_\lambda : \lambda \in \Lambda\}$, hence $A = \bigcup\{S_\lambda : \lambda \in \Lambda\} \subset \bigcup\{cl_{\tau\theta}(C_\lambda) : \lambda \in \Lambda\} \cup \bigcup\{cl_\tau(U_y) : y \in B\}$ and $z \notin \bigcup\{cl_{\tau\theta}(C_\lambda) : \lambda \in \Lambda\} \cup \bigcup\{cl_\tau(U_y) : y \in B\}$ (clearly $z \notin \bigcup\{cl_\tau(U_y) : y \in B\}$, moreover for every $\lambda \in \Lambda$ V_λ is an open neighbourhood of z such that $cl_\tau(V_\lambda) \cap C_\lambda = \emptyset$, so $z \notin \bigcup\{cl_{\tau\theta}(C_\lambda) : \lambda \in \Lambda\}$).

Choose $\alpha \in \kappa^+$ such that $B \cup \bigcup\{C_\lambda : \lambda \in \Lambda\} \subset \bigcup\{A_\beta : \beta \in \alpha\}$. Now $X \neq \bigcup\{cl_{\tau\theta}(C_\lambda) : \lambda \in \Lambda\} \cup \bigcup\{cl_\tau(U_y) : y \in B\}$, so by (iii) $A_\alpha \setminus (\bigcup\{cl_{\tau\theta}(C_\lambda) : \lambda \in \Lambda\} \cup \bigcup\{cl_\tau(U_y) : y \in B\}) \neq \emptyset$. Since $A \subset \bigcup\{cl_{\tau\theta}(C_\lambda) : \lambda \in \Lambda\} \cup \bigcup\{cl_\tau(U_y) : y \in B\}$ we have a contradiction. $\square$

Remark 7. The above theorem can be proved using elementary submodels (our approach follows that of [13], [14], [3]). Let $\kappa = fs(X)\psi_\tau(X)$ and let τ and $\mathcal{G}$ be the topology on X and the family of all cozero sets of X respectively. For every $x \in X$ let $\mathcal{B}_x$ be a τ-pseudobase for x such that $|\mathcal{B}_x| \le \kappa$ and let $\psi : X \to \mathcal{P}(\tau)$ be the map defined by $\psi(x) = \mathcal{B}_x$ for every $x \in X$. Let $\mathcal{M}$ be an elementary submodel such that $|\mathcal{M}| = 2^\kappa$, $X, \tau, \mathcal{G}, \psi \in \mathcal{M}$ and $\mathcal{M}$ is closed under κ-sequences. Observe that for every $x \in X \cap \mathcal{M}$ it follows that $\mathcal{B}_x \subset \mathcal{M}$. We claim that $X \subset \mathcal{M}$ (and hence $|X| \le 2^\kappa$). Suppose not, choose a point $z \in X \setminus \mathcal{M}$ and let $\mathcal{B}_z = \{B_\lambda : \lambda \in \Lambda\}$, $|\Lambda| \le \kappa$. Since $\{z\} = \bigcap\{cl_\tau(B_\lambda) : \lambda \in \Lambda\}$ it follows that $X \setminus \{z\} = \bigcup\{X \setminus cl_\tau(B_\lambda) : \lambda \in \Lambda\}$. Let $S_\lambda = X \cap \mathcal{M} \cap (X \setminus cl_\tau(B_\lambda))$ for every $\lambda \in \Lambda$. For every $y \in S_\lambda$ let $U_y \in \mathcal{M}$ such that $y \in U_y$ and $z \notin cl_\tau(U_y)$. $\{U_y : y \in S_\lambda\}$ is a family of open subsets of X such that $S_\lambda \subset \bigcup\{U_y : y \in S_\lambda\}$. Since $fs(X) \le \kappa$ there are $A_\lambda \in [S_\lambda]^{\le\kappa}$ and $\mathcal{V}_\lambda \in [\mathcal{U}_\lambda]^{\le\kappa}$ such that $S_\lambda \subset cl_{\tau\theta}(A_\lambda) \cup \bigcup\{cl_\tau(V) : V \in \mathcal{V}_\lambda\}$. Let $V_\lambda = \bigcup\{cl_\tau(V) : V \in \mathcal{V}_\lambda\}$, observe that $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}, \mathcal{A} = \{cl_{\tau\theta}(A_\lambda) : \lambda \in \Lambda\} \subset \mathcal{M}$ and $\mathcal{M}$ is closed under κ-sequences so $\mathcal{V}, \mathcal{A} \in \mathcal{M}$. Set $V = \bigcup \mathcal{V}$ and $A = \bigcup \mathcal{A}$, by elementarity it follows that $A \cup V \in \mathcal{M}$. Now $z \in X \setminus (A \cup V)$ so by elementarity there is some $x \in X \cap \mathcal{M}$ such that $x \notin A \cup V$. Since $X \cap \mathcal{M} \subset A \cup V$ we have a contradiction.

Remark 8. The w-compactness degree of a space X, denoted by $wcd(X)$, is the smallest infinite cardinal κ such that for every open cover $\mathcal{U}$ of X there is a $\mathcal{V} \in [\mathcal{U}]^{\le\kappa}$ such that $X = \bigcup\{cl_\tau(V) : V \in \mathcal{V}\}$. In [2] it is shown that $|X| \le 2^{wcd(X)\chi(X)}$ for every functionally Hausdorff space X. It is worth noting that Theorem 6 can give a better bound than the above result. The space X

in Remark 4 is a functionally Hausdorff space such that $|X| = 2^{fs(X)\psi_\tau(X)} < 2^{wcd(X)\chi(X)}$.

A fundamental result of Shapirovskii on spread says that if X is a Hausdorff space with $s(X) \leq \kappa$ then there is a subset S of X such that $|S| \leq 2^\kappa$ and $X = \bigcup\{\overline{A} : A \in [S]^{\leq\kappa}\}$.

We conclude this paper with the following

Theorem 9. *Let X be a functionally Hausdorff space with $fs(X) \leq \kappa$. Then there is a subset S of X such that $|S| \leq 2^\kappa$ and $X = \bigcup\{cl_{\tau\theta}(A) : A \in [S]^{\leq\kappa}\}$.*

Proof: By Proposition 5 it follows that $\psi_\tau(X) \leq 2^\kappa$, so for every $x \in X$ there is a τ-pseudobase $\mathcal{B}_x$ for x such that $\mathcal{B}_x \leq 2^\kappa$. Let τ and $\mathcal{G}$ be the topology on X and the family of all cozero-sets of X respectively. Moreover let $\psi : X \to \mathcal{P}(\tau)$ be the map defined by $\psi(x) = \mathcal{B}_x$ for every $x \in X$. Take an elementary submodel $\mathcal{M}$ of cardinality 2^κ such that $X, \tau, \mathcal{G}, \psi \in \mathcal{M}$ and which is closed under κ-sequences. $X \cap \mathcal{M}$ is a subset of X with the required properties. Let $x \in X$, we may assume that $x \notin X \cap \mathcal{M}$. We claim that there is a subset A of X such that $|A| \leq \kappa$ and $x \in cl_{\tau\theta}(A)$. Observe that $\mathcal{B}_y \subset \mathcal{M}$ for every $y \in X \cap \mathcal{M}$. Now for every $y \in X \cap \mathcal{M}$ take a $B_y \in \mathcal{B}_y$ (so $B_y \in \mathcal{M}$) such that $x \notin cl_\tau(B_y)$. Since $fs(X) \leq \kappa$ it follows that there are $A, B \in [X \cap \mathcal{M}]^{\leq\kappa}$ such that $X \cap \mathcal{M} \subset cl_{\tau\theta}(A) \cup \bigcup\{cl_\tau(B_y) : y \in B\}$. Since $A \in [\mathcal{M}]^{\leq\kappa}$ and $\mathcal{M}$ is closed under κ-sequences it follows that $A \in \mathcal{M}$ and hence $cl_{\tau\theta}(A) \in \mathcal{M}$. Moreover $\{cl_\tau(B_y) : y \in B\} \in [\mathcal{M}]^{\leq\kappa}$ and again $\{cl_\tau(B_y) : y \in B\} \in \mathcal{M}$. Therefore $cl_{\tau\theta}(A) \cup \bigcup\{cl_\tau(B_y) : y \in B\} \in \mathcal{M}$, hence $X = cl_{\tau\theta}(A) \cup \bigcup\{cl_\tau(B_y) : y \in B\}$ and $x \in cl_{\tau\theta}(A)$. □

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Dipartimento di Matematica Pura ed Applicata, Università, 67100 L'Aquila, Italy

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