Another note on countable Boolean algebras

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Abstract. We prove that a Boolean algebra is countable iff its subalgebra lattice admits a continuous complementation.

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The title refers to [4], where T. Jech proved that the subalgebra lattice $Sub\,A$ of a countable Boolean algebra A is complemented, i.e., for each $B \leq A$ there exists $B^* \leq A$ such that $B \cap B^* = \{0, 1\}$ and $B \cup B^*$ generates A. Independently this and stronger results were proved at about the same time by J.B. Remmel [6] and S. Todorčević (unpublished, his proof and many more facts on $Sub\,A$ are given in the survey [1]).

In Section 1 we describe a construction of complements B^* with the additional feature that for each $a \in A$, whether or not a belongs to B^* depends only on the intersection of B with a finite subalgebra of A. In other words, the mapping $B \mapsto B^*$ is continuous with respect to the natural topology on $Sub\,A$, a subbase of which, by definition, consists of all sets

$$\{C \in Sub\,A : a \in C\} \text{ and } \{C \in Sub\,A : a \notin C\},$$

where a runs through A. Notice that all sets $\{C \in Sub\,A : B \cap F = C \cap F\}$ constitute a base at the point $B \in Sub\,A$, when F runs through all finite subalgebras of A.

In Section 2 we prove that countability is necessary for continuous complementation. Thus

Theorem 1. *A Boolean algebra is countable iff its subalgebra lattice admits a continuous complementation.*

Our notation is in accordance with [5], with the exception that we use $\vee$, $\wedge$, and $-$ for the lattice-theoretic Boolean operations of join, meet and complementation and reserve $+$ for symmetric difference: $a + b = (a \wedge -b) \vee (b \wedge -a)$. In connection with $+$, meets are sometimes called products and denoted by $\cdot$ instead of $\wedge$. Recall that each Boolean algebra is a ring with unit under this addition and multiplication. Moreover, $a + a = 0$ for all a.

1. The construction of complements

Let A be the given countable Boolean algebra. We can assume that A is infinite, for, in the finite case continuity is for free and the existence of complements guaranteed by the above mentioned results. We use the well-known fact (cf. 15.10 in [5]) that A has an ordered base, i.e., a set of generators K, say, which is a chain under the Boolean partial order. We can and will assume that $0, 1 \notin K$. Then there is the following normal form assertion, where $\langle M \rangle$ denotes the subalgebra of A generated by M.

(1) If $L \subseteq K$, then each non-zero element of $\langle L \rangle$ can be uniquely written as $l_1 + l_2 + \cdots + l_q$, where $q \geq 1$ and $l_1 < l_2 < \cdots < l_q$ all belong to $L \cup \{1\}$.

It is well-known that, for arbitrary $M \subseteq A$, the subalgebra $\langle M \rangle$ consists of 0 and all finite sums of products (= meets) of elements of $M \cup \{1\}$. Being a chain, $L \cup \{1\}$ is closed under products, which yields the existence of the desired representations.

Assuming that the element a has two different representations $a = l_1' + \cdots + l_p' = l_1'' + \cdots + l_q''$ we get $0 = a + a = l_1' + \cdots + l_p' + l_1'' + \cdots + l_q''$ and, after cancellation of possible pairs $l_i' = l_j''$ and rearrangement, $0 = l_1 + \cdots + l_r$, with $1 \leq r \leq p + q$ and $l_1 < l_2 < \cdots < l_r$. The number of terms is at least two, for $0 \notin L \cup \{1\}$. But then

$$l_r = l_1 + \cdots + l_{r-1} \leq l_1 \vee \cdots \vee l_{r-1} = l_{r-1} < l_r,$$

which is a contradiction.

As A is countable, we can fix an injective enumeration $(k_n)_{n<\omega}$ of K. For a given subalgebra B of A we define subsets L_n^B of $\{k_i : i < n\}$ in the following inductive way.

$$L_0^B = \emptyset \quad \text{and} \quad L_{n+1}^B = \begin{cases} L_n^B, & \text{if there are } l_1, \ldots, l_p \in L_n^B \ (p \geq 0!) \\ & \text{such that } k_n + l_1 + \cdots + l_p \in B. \\ L_n^B \cup \{k_n\}, & \text{otherwise.} \end{cases}$$

We are now going to show that by letting B^* be the subalgebra generated by $L^B = \bigcup_{n<\omega} L_n^B$ we get the desired continuous complementation. Notice first that, by construction,

(2) $k_n \in L^B \iff k_n \in L_{n+1}^B$.

To prove $B^* \cap B = \{0, 1\}$, we assume the contrary and consider some alleged $b \in B^* \cap B \setminus \{0, 1\}$. Passing to $1 + b$ if necessary, claim (1) yields a representation $b = k_{n_1} + \cdots + k_{n_q}$ with all $k_{n_i} \in L^B$. Let n_t be maximal among the n_i. Then, by construction, $k_{n_t} \notin L_{n_t+1}^B$, hence, by (2), $k_{n_t} \notin L^B$, a contradiction.

To prove that $B \cup B^*$ generates A, it is clearly sufficient to express each k_n in the form $b + b^*$. This is trivial if $k_n \in L^B$. But otherwise $k_n \notin L_{n+1}^B$ and there is some finite (possibly zero) sum $f \in \langle L_n^B \rangle \subseteq B^*$ such that $k_n + f \in B$. So, $k_n = (k_n + f) + f$ is the desired representation.

It remains to check continuity. If a equals 0 or 1, every B^* contains it. Otherwise, according to claim (1), a can be uniquely written as

$$a = k_{n_1} + \cdots + k_{n_q} \quad \text{or} \quad a = k_{n_1} + \cdots + k_{n_q} + 1,$$

with $k_{n_1}, \ldots k_{n_q} \in K$. By uniqueness, $a \in B^* \Leftrightarrow k_{n_1}, \ldots, k_{n_q} \in L^B$. By construction, whether or not k_n belongs to L^B depends only on the intersection of B with $\langle \{k_i : i \leq n\} \rangle$. So, whether or not a belongs to B^* depends only on the intersection of B with the finite subalgebra $\langle \{k_i : i \leq \max\{n_1, \ldots, n_q\}\} \rangle$ of A.

2. The converse

In order to keep the argument short, we use Stone duality and conceive the given Boolean algebra A as $Clop\,X$, the algebra of all clopen (= closed and open) subsets of some compact and zero-dimensional topological space X. Accordingly, we use the set-theoretic notation for the Boolean operations. We assume that $B \mapsto B^*$ is a continuous complementation $Sub\,A \to Sub\,A$. Our aim is to show that X is metrizable.

For $x, y \in X$ we let $B(x, y)$ denote the subalgebra $\{a \in A : x \in a \Leftrightarrow y \in a\}$ of $A = Clop\,X$ and use $B^*(x, y)$ for its complement. Obviously

(3) $B(x, x) = A$, hence $B^*(x, x) = \{\emptyset, X\}$.

Let x and y be distinct now. If $a \in B^*(x, y)$ does not separate x and y, then it belongs to $B^*(x, y) \cap B(x, y) = \{\emptyset, X\}$ and equals $\emptyset$ or X. Repeated application of this observation yields that

(4) if $x \neq y$, then $B^*(x, y)$ is a four-element subalgebra of A.

Indeed, consider $a, b \in B^*(x, y) \setminus \{\emptyset, X\}$. Then both a and b must separate x and y, so their symmetric difference does not and equals, therefore, $\emptyset$ or X. So $a = b$ or $a = X \setminus b$. This shows that $B^*(x, y)$ has at most four elements. But $B^*(x, y)$ cannot be the two-element subalgebra, for, otherwise, $B(x, y) \cup B^*(x, y) = B(x, y)$ could not generate the whole of A.

Next we observe that

(5) the assignment $(x, y) \mapsto B(x, y)$ defines a continuous mapping $X^2 \to Sub\,A$.

To check this, it is sufficient to consider the preimages of subbasic sets in the space $Sub\,A$. Well, $\{(x, y) : a \in B(x, y)\} = a^2 \;\cup\; (X \setminus a)^2$ and $\{(x, y) : a \notin B(x, y)\} = a \times (X \setminus a) \;\cup\; (X \setminus a) \times a$ are both clopen for each $a \in A$.

It follows that the mapping $(x, y) \mapsto B^*(x, y)$ is also continuous. Finally, we need the following claim.

(6) For every four-element subalgebra B of A the set

$$W_B = \{(x, y) \in X^2 : B^*(x, y) = B\}$$

is clopen in X^2 and does not intersect the diagonal $\Delta = \{(x, x) : x \in X\}$.

To see this, we fix $b \in B \setminus \{\emptyset, X\}$. We already know that $B^*(x,y)$ has at most four elements. Therefore,

$$B^*(x,y) = B \iff b \in B^*(x,y).$$

It follows that W_B is clopen, being the preimage of the subbasic clopen set $\{C \in Sub\,A : b \in C\}$ under the continuous mapping B^*. By (3), $B^*(x,x)$ is always two-element, so W_B cannot intersect Δ.

Assertions (4) and (6) yield a representation of $X^2 \setminus \Delta$ as the disjoint union $\bigcup_B W_B$ of clopen (hence compact) subsets of X^2. It follows that $X^2 \setminus \Delta$ is paracompact. To end our argument it remains to apply the taylormade metrization theorem of G. Gruenhage (2.6 in [3]): *The compact space X is metrizable iff $X^2 \setminus \Delta$ is paracompact.*

3. The topological version

Recall (cf. [2, 2.7.20]) that the exponential or hyperspace $exp\,X$ of a topological space X is the set of all non-empty closed subsets of X equipped with the Vietoris topology. A subbase of this topology consists of all

$$\{F \in exp\,X : F \subseteq U\} \quad \text{and} \quad \{F \in exp\,X : F \cap U \neq \emptyset\},$$

where U runs through all open subsets of X.

Theorem 2. *A compact and zero-dimensional space X is metrizable iff there is a continuous mapping $f : X \times exp\,X \to X$ such that*

$$f(x,F) \in F \quad \textit{and} \quad f(x,F) = x \textit{ if } x \in F.$$

We just sketch the proof. If $F \subseteq X$, then $B(F) = \{b \in Clop\,X : F \subseteq b \text{ or } b \cap F = \emptyset\}$ is a subalgebra of $Clop\,X$. Assuming that X is metrizable, the Boolean algebra $Clop\,X$ is countable so Theorem 1 applies and each $B(F)$ has a continuously chosen complement $B^*(F)$. A somewhat lengthy verification then shows that the desired mapping $f : X \times exp\,X \to X$ can be defined by

$$\{f(x,F)\} = \bigcap\{b^* \triangle b : x \in b^* \in B^*(F);\ b \cap F = \emptyset\}.$$

Here $\triangle$ means symmetric difference, the set-theoretic version of $+$. The reader who wants to fill in the details is advised to prove the following claim first.

(7) Every $a \in Clop\,X$ can be uniquely written in the form $b^* \triangle b$, where $b \in B^*(F)$ and $b \cap F = \emptyset$.

For the other direction, one can mimic the proof in Section 2. Let the mapping $f : X \times exp\,X \to X$ be given. For $x, y \in X$ we let $b(x,y)$ denote the clopen set $\{z \in X : f(z, \{x,y\}) = x\}$. Then $b(x,x) = X$ and $\emptyset \neq b(x,y) \neq X$ for $x \neq y$. A routine but tedious verification shows that $W_b = \{(x,y) \in X^2 : b(x,y) = b\}$ is clopen for all $b \in Clop\,X \setminus \{\emptyset, X\}$. Then the decomposition

$$X^2 \setminus \Delta = \bigcup\{W_b : b \neq X\}$$

witnesses the paracompactness of $X^2 \setminus \Delta$ and Gruenhage's theorem can be applied as before.

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