

On a theorem of Fermi

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Abstract. Conformally flat metric \bar{g} is said to be Ricci superosculating with g at the point x_0 if $g_{ij}(x_0) = \bar{g}_{ij}(x_0)$, $\Gamma_{ij}^k(x_0) = \bar{\Gamma}_{ij}^k(x_0)$, $R_{ij}^k(x_0) = \bar{R}_{ij}^k(x_0)$, where R_{ij} is the Ricci tensor. In this paper the following theorem is proved:

If γ is a smooth curve of the Riemannian manifold M (without self-crossing), then there is a neighbourhood of γ and a conformally flat metric \bar{g} which is the Ricci superosculating with g along the curve γ .

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Let $g = g_{ij}dx^i dx^j$ and $\bar{g} = \bar{g}_{ij}dx^i dx^j$ be two metrics on the n -dimensional manifold M . The metric \bar{g} is said to be tangent, respectively osculating with the metric g at the point $x_0 \in M$ ([1]), if

$$g_{ij}(x_0) = \bar{g}_{ij}(x_0), \quad \text{respectively,}$$

$$g_{ij}(x_0) = \bar{g}_{ij}(x_0), \quad \frac{\partial g_{ij}}{\partial x^k}(x_0) = \frac{\partial \bar{g}_{ij}}{\partial x^k}(x_0) \quad \left(\text{or } \Gamma_{ij}^k(x_0) = \bar{\Gamma}_{ij}^k(x_0) \right),$$

where Γ_{ij}^k are the Cristoffel symbols.

If γ is a smooth curve of the manifold M , then there is a local Euclidean metric \bar{g} which is osculating with the g along the curve (this theorem is due to Fermi [1]).

The Riemannian curvature tensor can be computed with the help of the first and second derivatives of the metric tensor g_{ij} ; hence, there exists no local-Euclidean metric which is superosculating of degree two with g . But if we consider a class of conformally flat metrics, there is a conformally flat metric which has some equal combinations of first and second derivatives.

Definition. Conformally flat metric \bar{g} is said to be Ricci superosculating with g at the point x_0 if

$$g_{ij}(x_0) = \bar{g}_{ij}(x_0), \quad \Gamma_{ij}^k(x_0) = \bar{\Gamma}_{ij}^k(x_0), \quad R_{ij}(x_0) = \bar{R}_{ij}(x_0),$$

where R_{ij} is the Ricci tensor.

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Theorem. *If γ is a smooth curve of the Riemannian manifold M (without self-crossing), then there is a neighbourhood of γ and a conformally flat metric \bar{g} which is the Ricci superosculating with g along the curve γ .*

To prove the theorem we use the conformal connection and a conformal development of the curve ([2]–[4]). Let $R^{n+2} = R^{n+1} \times R$ be a pseudo-Euclidean Minkowski space with a Lorentz inner product given by formula ([5]):

$$\langle z_1, z_2 \rangle = (x_1, x_2) - \zeta_1 \zeta_2,$$

where $z_i = (x_i, \zeta_i) \in R^{n+2}$, $i = 1, 2$; (x_1, x_2) is an inner product for the Euclidean space R^{n+1} . Let $C^+ = \{z = (x, \zeta) : \langle z, z \rangle = 0, \zeta > 0\}$ denote the light cone of the Minkowski space. We consider the basis F for the space R^{n+2} as the column of vectors of R^{n+2} . Let $GL(n + 1, 1)$ be a manifold of all the basis of the kind:

$$F^t = \{e_1, \dots, e_n, z, z^*\},$$

where $\{e_1, \dots, e_n\}$ are spacelike vectors (i.e. $\|\bar{g}_{ij}\| = \|\langle e_i, e_i \rangle\|$ is a positively definite matrix),

$$(1) \quad \langle z, e_i \rangle = \langle z^*, e_i \rangle = 0, \quad \langle z, z^* \rangle = -1, \quad \langle z \rangle^2 = \langle z^* \rangle^2 = 0.$$

Let us fix the basis $F_e \in GL(n + 1, 1)$, then any other basis $F \in GL(n + 1, 1)$ can be identified with the linear transformation \mathcal{F} such that $F = \mathcal{F} \cdot F_e$. The matrix-valued 1-form Φ is defined by

$$\Phi = d\mathcal{F} \cdot \mathcal{F}^{-1} = \|\omega_i^j\|, \quad i, j = 1, \dots, n + 2,$$

or

$$(2) \quad dF = d(\mathcal{F} \cdot F_e) = d\mathcal{F} \cdot \mathcal{F}^{-1} \cdot \mathcal{F} \cdot F_e = \Phi \cdot F.$$

Now (1) implies

$$\begin{aligned} d\bar{g}_{ij} &= \omega_i^k \bar{g}_{kj} + \omega_j^k \bar{g}_{ki}, \\ \omega_{n+1}^{n+2} &= \omega_{n+2}^{n+1} = 0, \quad \omega_{n+1}^{n+1} + \omega_{n+2}^{n+2} = 0, \\ \omega_{n+1}^k \bar{g}_{ki} - \omega_i^{n+2} &= 0, \quad \omega_{n+2}^k \bar{g}_{ki} - \omega_i^{n+1} = 0. \end{aligned}$$

Let us set

$$\omega_{n+1}^i = \omega^i, \quad \omega_i^{n+2} = \omega_i, \quad \omega_{n+2}^i = \omega^{*i}, \quad \omega_i^{n+1} = \omega_i^*, \quad \alpha = \omega_{n+1}^{n+1}.$$

The equation of Cartan-Maurera $d\Phi + \frac{1}{2}[\Phi, \Phi] = 0$ for Φ is written

$$\begin{aligned} d\omega_i^j &= \sum_{k=1}^n \omega_i^k \wedge \omega_k^j + \omega_i^* \wedge \omega^j + \omega_i \wedge \omega^{*j}, \quad d\alpha = \omega_i \wedge \omega^{*i}, \\ d\omega^i &= \omega^k \wedge \omega_k^i + \alpha \wedge \omega^i, \quad d\omega^{*i} = \omega^{*k} \wedge \omega_k^i - \alpha \wedge \omega^{*i}. \end{aligned}$$

We now give a construction of the conformal Cartan connection. Let $g = g_{ij}dt^i dt^j$ be the Riemannian metric on the manifold M . Let Φ be the differential matrix 1-forms defined by the formulas:

$$\omega^i = dt^i, \quad \omega_{i\cdot}^j = \Gamma_{ik}^j dt^k, \quad \omega_i^* = \frac{1}{n-2} \left(R_{ik} dt^k - \frac{Rg_{ik} dt^k}{2(n-1)} \right) = A_{ik} dt^k, \quad \alpha = 0,$$

where Γ_{ik}^j are the Cristoffel symbols, R_{ik} is the Ricci tensor, A_{ik} is the reduced Ricci tensor, R is the scalar curvature. The form Φ is called the normal conformal Cartan connection of the Riemannian manifold M ([2]-[4]). We denote this matrix of differential 1-forms by $\Phi = \{\omega^i, \omega_{i\cdot}^j, \omega_i^*, 0\}$.

Definition. Let $\{M, g\}$ be the Riemannian manifold; and let α be a differential 1-form on M . The extended conformal Cartan connection matrix $\tilde{\Phi} = \{\omega^i, \tilde{\omega}_{i\cdot}^j, \tilde{\omega}_i^*, \alpha\}$ on the manifold M , associated with α , is the matrix defined by the formulas:

$$\begin{aligned} \tilde{\omega}_{i\cdot}^j &= \omega_{i\cdot}^j - \alpha_i \omega^j + \alpha^j \omega_i = \left(\Gamma_{ik}^j - \alpha_i \delta_k^j + \alpha^j g_{ik} \right) dt^k, \\ \tilde{\omega}_i^* &= \omega_i^* - \alpha_i \alpha - D\alpha_i + \frac{|\alpha|^2}{2} \omega_i = \left(A_{ij} - \alpha_i \alpha_j - \alpha_{i,j} + \frac{|\alpha|^2}{2} g_{ij} \right) dt^j, \end{aligned}$$

where $\alpha = \alpha^i \omega_i = \alpha_i \omega^i$, $D\alpha_i$ is the covariant derivative, $|\alpha|^2 = \alpha^i \alpha_i$.

The curvature matrix of the extended Cartan connection is defined by the formula:

$$d\tilde{\Phi} + \frac{1}{2}[\tilde{\Phi}, \tilde{\Phi}] = \begin{pmatrix} \widetilde{W}_{i\cdot}^j & \tilde{S}_i & 0 \\ \hline 0 & 0 & 0 \\ \hline \tilde{S}^j & 0 & 0 \end{pmatrix},$$

where $\widetilde{W}_{i\cdot}^j = W_{i\cdot}^j$ are the forms of the Weyl tensor (it is independent of the choice of α), $\tilde{S}^i = S^i + \alpha^s W_{s\cdot}^j$ are the forms of the Schouten-Weyl tensor ([6]).

Definition. Let $\gamma = \{x^i(t) : a \leq t \leq b\}$ be a smooth curve of the manifold $\{M, g, \alpha\}$ and $F_0 = \{e_1^0, \dots, e_n^0, z^0, z^{0*}\} \in GL(n+1, 1)$ be a starting basis such that:

$$g_{ij}(\gamma(a)) = \langle e_i^0, e_j^0 \rangle.$$

The lifting $F : [a, b] \rightarrow GL(n+1, 1)$ is the solution of the differential matrix equation $\dot{F} = \Phi(\dot{\gamma})F, \quad F(a) = F_0$. The vector function $z : [a, b] \rightarrow C^+$ (the component of F) is called the conformal development of γ .

Remark 1. The development z does not depend on the choice of the local basis $\{e_1, \dots, e_n\}$ and the form α on the manifold M (for the proof see [3]).

Remark 2. In the case of the continuous curve γ , the lifting of the curve, for an arbitrary connection, was defined in [7].

Lemma. Let $Z : D \rightarrow C^+$ be an immersion of the domain $D \subset R^n$ into the light cone $C^+ \subset R^{n+2}$ of the Minkowski space R^{n+2} , and let $\bar{g} = \langle dZ, dZ \rangle = \bar{g}_{ij} dx^i dx^j$ denote the conformally flat metric ([5]), then

$$d^2 Z = Z_{ij} dx^i dx^j = Z_k \bar{\Gamma}_{ij}^k dx^i dx^j + Z^* \bar{g}_{ij} dx^i dx^j + Z \bar{A}_{ij} dx^i dx^j,$$

where $\bar{\Gamma}_{ij}^k$ are the Cristoffel symbols, \bar{A}_{ik} is the reduced Ricci tensor of the metric \bar{g} , the vector Z^* is defined by the formulas:

$$\langle Z^*, Z_i \rangle = 0, \quad \langle Z, Z^* \rangle = -1, \quad \langle Z^*, Z^* \rangle = 0,$$

where $Z_i = \frac{dZ}{dx^i}$, $Z_{ij} = \frac{\partial^2 Z}{\partial x^i \partial x^j}$.

PROOF: The vectors $\{Z_1, \dots, Z_n, Z, Z^*\}$ form a basis, hence

$$(3) \quad Z_{ij} = Z_k M_{ij}^k + Z^* N_{ij} + Z P_{ij}.$$

It is easy to see, from (1), that

$$M_{ij}^k = \bar{\Gamma}_{ij}^k, \quad N_{ij} = \bar{g}_{ij}.$$

If we differentiate (3) with respect to x^s and equate the mixed partial derivatives Z_{ijs} and Z_{isj} , we obtain $P_{ij} = \bar{A}_{ij}$. □

Remark. The tensors \bar{g}_{ij} and \bar{A}_{ij} completely define the second fundamental form of the surface $\{Z(x)\}$ in the Minkowski space.

PROOF OF THE THEOREM: Let $\{x^1, x^2, \dots, x^n\}$ be the Fermi co-ordinate system ([8]) in the tube about the curve

$$\gamma = \left\{ x : x^2 = \dots = x^n = 0, a \leq x^1 \leq b \right\},$$

that is:

- (a) the tangent vectors $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ form the orthonormal frame field along the curve γ ,
- (b) the curves $\{x^1 = const, x^i = c^i s : s > 0, i \geq 2\}$ are geodesics, where $\{c^i\}_{i=2}^n$ are constants.

Let $B_\varepsilon(\gamma) = \{x : \sum_{i=2}^n (x^i)^2 < \varepsilon^2, a \leq x^1 \leq b\}$ denote the (solid) tube of the radius ε about γ . We will determine the immersion

$$Z : B_\varepsilon(\gamma) \rightarrow C^+,$$

such that $\bar{g} = \langle dZ, dZ \rangle$ will be the required conformally flat metric.

Let $x_0 = (a, 0, \dots, 0) \in B_\varepsilon(\gamma)$ be the starting point. Any point $x = (x^1, \dots, x^n) \in B_\varepsilon(\gamma)$ can be connected with x_0 by the curve (open polygon) l_x consisting of the two arcs. The first arc is a line segment along the axis of $B_\varepsilon(\gamma)$:

$$x(t) = \{a(1-t) + tx^1, 0, \dots, 0\}, \quad 0 \leq t \leq 1,$$

the second arc is the geodesic segment which is orthogonal to the axis

$$x(t) = \{x^1, (t-1)x^2, \dots, (t-1)x^n\}, \quad 1 \leq t \leq 2.$$

The curve $l_x : [0, 2] \rightarrow B_\varepsilon(\gamma)$ depends smoothly on the terminal point x . Let \mathcal{L} denote the family of all such curves l_x .

Let us fix the basis $F_0 \in GL(n+1, 1)$ and associate l_x with the lifting

$$F_x(t) = \{e_1(t)_x, \dots, e_n(t)_x, z(t)_x, z^*(t)_x\},$$

into the group $GL(n+1, 1)$. Then we may consider the map

$$\mathbf{F} : x \in B_\varepsilon(\gamma) \rightarrow F_x(2) \in GL(n+1, 1).$$

Let $\bar{\Phi} = \{\bar{\omega}^i, \bar{\omega}_i^j, \bar{\omega}_i^*, \alpha\}$ be a matrix 1-forms corresponding to the map \mathbf{F} by the formula (2). Let $Z(x) = z(2)_x$ be a component of the map \mathbf{F} , that is the terminal point of the development l_x .

Since $\bar{\Phi} = \Phi|_{\mathcal{L}}$, then the forms $\bar{\Phi}$ and Φ are equal along the axis $B_\varepsilon(\gamma)$ (*on the tangent space of the manifold M*). We have

$$\begin{aligned} dZ &= Z_i \bar{\omega}^i = Z_i dx^i, \\ dZ_i &= \bar{\omega}_i^j Z_j + \bar{\omega}_i Z^* + \bar{\omega}_i^* Z, \end{aligned}$$

for any point of γ . Then

$$d^2 Z = \bar{\omega}_i^j dx^i Z_j + \bar{\omega}_i dx^i Z^* + \bar{\omega}_i^* dx^i Z = \omega_i^j dx^i Z_j + \omega_i dx^i Z^* + \omega_i^* dx^i Z,$$

for any point of γ . The application of the Lemma gives us the required result.

Other applications of the conformal development curves can be find in [3]. \square

REFERENCES

- [1] Cartan E., *Leçons sur la Géométrie des Espaces de Riemann*, Gauthier-Villars, Paris, 1928, p. 242.
- [2] Cartan E., *Les espaces à connection conforme*, Ann. Soc. Po. Math. **2** (1923), 171–221.
- [3] Slavskii V.V., *Conformal development of the curve on the Riemannian manifold in the Minkowski space*, Siberian Math. Journal **37** (1996), no. 3, 676–699.
- [4] Akivis M.A., Konnov V.V., *Sense local aspects of the theory of conformal structure*, Russian Math. Surveys **48** (1993), 3–40.

- [5] Slavskii V.V., *Conformally flat metrics and the geometry of the pseudo-Euclidean space*, Siberian Math. Jour. **35** (1994), no. 3, 674–682.
- [6] Besse A.L., *Einstein Manifolds*, Erg. Math. Grenzgeb. 10, Berlin-Heidelberg-New York, 1987.
- [7] Reshetnyak Yu.G., *On the lifting of the non-regular path in the bundle manifold and its applications*, Siberian Math. Jour. **16** (1975), no. 3, 588–598.
- [8] Gray A., Vonhecke L., *The volumes of tubes about curves in a Riemannian manifold*, Proc. London Math. Soc. **44** (1982), no. 2, 215–243.

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