A note on Möbius inversion over power set lattices

Klaus Dohmen

Abstract. In this paper, we establish a theorem on Möbius inversion over power set lattices which strongly generalizes an early result of Whitney on graph colouring.

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1. Introduction

An important technique in combinatorics is the principle of Möbius inversion over partially ordered sets (see [3, Chapter 25]). For power set lattices, the principle of Möbius inversion states the following:

Proposition. *Let S be a finite set, f and g mappings from the power set of S into an additive group such that $g(X) = \sum_{Y\in[X,S]} f(Y)$ for any $X \subseteq S$, where $[X,S]$ denotes the interval $\{Y \mid X \subseteq Y \subseteq S\}$. Then, for any $X \subseteq S$,*

$$f(X) = \sum_{Y\in[X,S]} (-1)^{|Y\setminus X|}\, g(Y). \tag{1}$$

Proof: By the asserted relation between f and g, the sum in (1) equals

$$\sum_{Y\in[X,S]} (-1)^{|Y\setminus X|} \sum_{Z\in[Y,S]} f(Z) = \sum_{Z\in[X,S]} f(Z) \sum_{Y\in[X,Z]} (-1)^{|Y\setminus X|},$$

and this is $f(X)$ since the inner sum on the right is zero unless $X = Z$. □

2. A modified inversion formula

The following theorem states that under certain conditions not all terms have to be considered when evaluating the sum in (1). It can be thought of as a generalization of Whitney's Broken-Circuits-Theorem on graph colouring.

Theorem. *Let S be a poset and f, g mappings from the power set of S into an additive group such that $g(X) = \sum_{Y\in[X,S]} f(Y)$ for any $X \subseteq S$. For fixed $X \subseteq S$, let $\mathcal{C}$ be a set of non-empty subsets of S such that each $C \in \mathcal{C}$ is bounded* *from below by an element* $\underline{C} \in S \setminus (C \cup X)$ *and* $f(Y) = 0$ *for all* Y *including* $C \cup X$ *and not containing* $\underline{C}$. *Then*

$$f(X) \;=\; \sum_{Y \in [X,S] \cap \mathcal{Y}_0} (-1)^{|Y \setminus X|}\, g(Y), \tag{2}$$

where

$$\mathcal{Y}_0 \;:=\; \{Y \subseteq S \,|\, Y \not\supseteq C \text{ for all } C \in \mathcal{C}\}. \tag{3}$$

Proof: Let $\leq$ denote the partial ordering relation on S and $\leq^*$ one of its linear extensions. For each subset Y of S, $\min^* Y$ denotes the minimum of Y with respect to $\leq^*$. Consider an enumeration $C_1, \dots, C_n$ of $\mathcal{C}$ such that $\min^* C_1 \leq^* \dots \leq^* \min^* C_n$, and define $\mathcal{Y}_m := \{Y \subseteq S \,|\, C_m \subseteq Y,\, C_{m+1} \not\subseteq Y, \dots, C_n \not\subseteq Y\}$ for $m = 1, \dots, n$. Obviously, the power set of S is the disjoint union of $\mathcal{Y}_0, \dots, \mathcal{Y}_n$. The proposition gives

$$f(X) \;=\; \sum_{m=0}^{n} \; \sum_{Y \in [X,S] \cap \mathcal{Y}_m} (-1)^{|Y \setminus X|}\, g(Y).$$

We claim that the inner sum on the right-hand side is zero for $m = 1, \dots, n$. The assertions force $\underline{C_m} < c$ and hence $\underline{C_m} <^* c$ for every $c \in C_m$. From the latter we conclude $\underline{C_m} <^* \min^* C_m \leq^* \min^* C_k$ and therefore $\underline{C_m} \notin C_k$ for $k = m, \dots, n$. For such k, $C_k \subseteq Y$ if and only if $C_k \subseteq Y_m$ where $Y_m := (Y \setminus \{\underline{C_m}\}) \cup (\{\underline{C_m}\} \setminus Y)$. By this, $Y \in \mathcal{Y}_m$ if and only if $Y_m \in \mathcal{Y}_m$. In addition, $X \subseteq Y$ if and only if $X \subseteq Y_m$. Hence,

$$\sum_{Y \in [X,S] \cap \mathcal{Y}_m} (-1)^{|Y \setminus X|}\, g(Y) = \frac{1}{2} \sum_{Y \in [X,S] \cap \mathcal{Y}_m} \Big((-1)^{|Y \setminus X|}\, g(Y) + (-1)^{|Y_m \setminus X|}\, g(Y_m) \Big).$$

Since $|Y \setminus X| \not\equiv |Y_m \setminus X| \pmod 2$, it suffices to check $g(Y) = g(Y_m)$ for all $Y \in [X,S] \cap \mathcal{Y}_m$. By the asserted relation between f and g,

$$g(Y) \;=\; \sum_{\substack{Z \in [Y,S], \\ \underline{C_m} \notin Z}} f(Z) + \sum_{\substack{Z \in [Y,S], \\ \underline{C_m} \in Z}} f(Z).$$

It is easy to see that the right sum remains unchanged when Y is replaced by Y_m. The same holds for the left sum which, by the assertions of the theorem, equals zero. $\square$

Remark. To compare the number of terms in (1) and (2), we define $\chi := |\mathcal{Y}_0 \cap [X,S]|/|[X,S]|$. Obviously, $0 \le \chi \le 1$. By the well-known principle of inclusion and exclusion (which is a particular case of the next corollary),

$$\chi = \sum_{\mathcal{C}' \subseteq \mathcal{C}} (-1)^{|\mathcal{C}'|} 2^{|X| - |X \cup \bigcup_{C \in \mathcal{C}'} C|} . \tag{4}$$

Hence, if $\mathcal{C}$ contains n pairwise disjoint sets of cardinality m ($n \in \mathbb{N}_0$, $m \in \mathbb{N}$) all of them being disjoint with X, then $\chi \le (1 - 2^{-m})^n$, and this tends to zero as $n \to \infty$. □

Corollary. *Let $\mathcal{A}$ be a boolean algebra of sets, P a mapping from $\mathcal{A}$ into an additive group such that $P(\emptyset) = 0$ and $P(A \cup B) = P(A) + P(B)$ for all disjoint pairs $A, B \in \mathcal{A}$, S a finite poset, $\{A_s\}_{s \in S} \subseteq \mathcal{A}$, $X \subseteq S$ and $\mathcal{C}$ a set of non-empty subsets of S such that each $C \in \mathcal{C}$ is bounded from below by an element $\underline{C} \in S \setminus (C \cup X)$ and $\bigcap_{c \in C} A_c \subseteq A_{\underline{C}}$. Then*

$$P\left(\bigcap_{x \in X} A_x \cap \bigcap_{s \in S \setminus X} \complement A_s\right) = \sum_{Y \in [X,S] \cap \mathcal{Y}_0} (-1)^{|Y \setminus X|} P\left(\bigcap_{y \in Y} A_y\right),$$

where $\mathcal{Y}_0$ is defined as in (3) *and $\complement A_s$ denotes the complement of A_s in $\mathcal{A}$.*

Proof: For $Y \subseteq S$ define $f(Y) := P(\bigcap_{y \in Y} A_y \cap \bigcap_{s \in S \setminus Y} \complement A_s)$, $g(Y) := P(\bigcap_{y \in Y} A_y)$. For Y including C and not containing $\underline{C}$ there is some $B \in \mathcal{A}$ such that $f(Y) = P(\bigcap_{c \in C} A_c \cap \complement A_{\underline{C}} \cap B)$, and hence $f(Y) = 0$. Therefore, the theorem can be applied. □

Remark. Let X be empty and $S_{\min}$ resp. $S_{\max}$ denote the set of minimal resp. maximal elements of S. If the mapping $s \mapsto A_s$ is antitone, then it can be achieved that $\mathcal{Y}_0$ is the power set of $S_{\min}$ (Proof: Set $\mathcal{C} := \{\{s\} \mid s \in S \setminus S_{\min}\}$, and for each $C \in \mathcal{C}$ choose a lower bound $\underline{C} \in S \setminus C$.). By the duality principle for posets, 'below' can be replaced by 'above' both in the theorem and in the corollary. By this, if $s \mapsto A_s$ is isotone, then it can be achieved that $\mathcal{Y}_0$ becomes the power set of $S_{\max}$. □

Example 1. In (4), $\mathcal{C}$ can be replaced by the set of its minimal elements with respect to set inclusion. This is an immediate consequence of the corollary and the preceding remark since $C \mapsto [C,S]$ is an antitone mapping. □

Example 2. A *hypergraph* is a set S of non-empty sets whose union $\bigcup S$ is finite. The elements of S resp. $\bigcup S$ are the *edges* resp. *vertices* of the hypergraph; their number is denoted by $m(S)$ resp. $n(S)$. Define $m^*(S) := \sum_{s \in S}(|s| - 1)$. For $k \in \mathbb{N}$, let $S^{(k)}$ consist of all k-element edges of S. The edges of $S^{(1)}$ are called *loops*. The subsets of S are called *partial hypergraphs of S*. A *cycle in S* is a sequence $(v_1, s_1, \dots, v_k, s_k)$ where $k > 1$ and $v_1, \dots, v_k$ resp. $s_1, \dots, s_k$ are distinct vertices resp. edges, $v_i, v_{i+1} \in s_i$ for $i = 1, \dots, k-1$ and $v_k, v_1 \in s_k$. With respect to a linear ordering relation on S, a *broken circuit of* S is obtained from the edge-set of a cycle in S by removing the smallest edge. For any $\lambda \in \mathbb{N}$, a λ-colouring of S is a mapping $f : \bigcup S \longrightarrow \{1, \dots, \lambda\}$ (the set of *colours*). For $X \subseteq S$, $P_{S,X}(\lambda)$ stands for the number of λ-colourings of S such that X is the set of monochromatic edges. We now establish the following statement:

Let S be a loop-free, linearly ordered hypergraph, and let X be a partial hypergraph of S such that $S^{(2)} \setminus X$ is an initial segment of S and each cycle in S has an edge of $S^{(2)} \setminus X$. Then $P_{S,X}(\lambda) = \sum_{i,j} \rho_{ij} \lambda^{n(S)-i}$ where ρ_{ij} equals $(-1)^{j-|X|}$ times the number of partial hypergraphs Y of S including X but no broken circuits of S and satisfying $m^*(Y) = i$ and $m(Y) = j$.

Proof: For $s \in S$ define A_s as the set of λ-colourings of S such that s is monochromatic. For any broken circuit C of S let $\underline{C}$ be the unique edge such that $C \cup \{\underline{C}\}$ is the edge-set of a cycle in S. The assertions force $\underline{C} \in S^{(2)} \setminus (C \cup X)$. Obviously, $\underline{C} \in S^{(2)}$ entrains $\bigcap_{c \in C} A_c \subseteq A_{\underline{C}}$. By the corollary, $P_{S,X}(\lambda) = \sum_Y (-1)^{|Y \setminus X|} |\bigcap_{y \in Y} A_y|$ where the summation is extended over all partial hypergraphs Y of S including X but no broken circuits of S. By [1, Proposition], $|\bigcap_{y \in Y} A_y| = \lambda^{n(S)-m^*(Y)}$. The result now follows. $\square$

Note. A particular case of the previous example, namely where X is empty, is published in [2]. For simple graphs and empty X, the above statement is due to Whitney (see [4]) and called *Whitney's Broken-Circuits-Theorem.* $\square$

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Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany

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