# Surjectivity results for nonlinear mappings without oddness conditions

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*Abstract.* Surjectivity results of Fredholm alternative type are obtained for nonlinear operator equations of the form $\lambda T(x) - S(x) = f$, where $T$ is invertible, and $T, S$ satisfy various types of homogeneity conditions. We are able to answer some questions left open by Fučík, Nečas, Souček, and Souček. We employ the concept of an $a$-stably-solvable operator, related to nonlinear spectral theory methodology. Applications are given to a nonlinear Sturm-Liouville problem and a three point boundary value problem recently studied by Gupta, Ntouyas and Tsamatos.

*Keywords:* $(K, L, a)$ homeomorphism, $a$-homogeneous operator, $a$-stably solvable map
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## 1. Introduction

The authors of [1] studied various surjectivity results for nonlinear operator equations of the form

$$\lambda T(x) - S(x) = f \tag{1.1}$$

when $T$ is invertible. They considered various types of homogeneity conditions, in particular a key assumption was that $T$ was a so-called $(K, L, a)$-homeomorphism. The precise definition of this and other concepts mentioned in the introduction is given a little later.

In their paper [1], the authors gave theorems of Fredholm alternative type under the assumptions that $T$ is an odd $(K, L, a)$-homeomorphism and $S : X \to Y$ is an odd compact (completely continuous) operator. Furthermore, they established existence of a solution of the equation (1.1) for each $f \in Y$ provided $\lambda \neq 0$ if $T$ is an odd $a$-homogeneous map and $S$ is an odd $b$-strongly quasihomogeneous map with $a > b$. In the case $a < b$ they proved the same assertion in finite dimensional spaces but the infinite-dimensional case was an unsolved problem.

In this paper, we employ different methods which allow us to answer some of their open questions. By introducing the concept of an $a$-stably-solvable operator, we obtain some surjectivity results for $\lambda T - S$ under weaker conditions. One of the theorems generalizes the result of existence of a solution of (1.1) in case $a < b$ to the infinite-dimensional case. These results seem not to be able to be proven by their methods.

It is possible to give simple examples that show that our results are real extensions of the earlier ones, but we prefer to give more substantial applications. We discuss a nonlinear Sturm-Liouville problem on the half line following work by Toland [4]. He studied eigenvalues and asymptotic bifurcation points whereas we obtain surjectivity when $\lambda$ is not one of these eigenvalues.

We also discuss existence of solutions to a three point boundary value problem recently studied by Gupta, Ntouyas and Tsamatos, in [6], [7], [8]. The boundary conditions are of the type $x(0) = 0$, $x(1) = \alpha x(\eta)$. Those authors assume that $\alpha < 1/\eta$ but we suppose only that $\alpha \neq 1/\eta$. We obtain a different criterion for existence which improves on Theorem 4 of [6] in some cases but is less good in others.

## 2. Prerequisites

We will make use of the class of $k$-set contractive maps and of the theory of degree for $I - f$ where $f$ is $k$-set contractive, see for example [5]. We give some notations and definitions that we shall use.

Given any continuous map $f$ from a subset $D(f)$ of a complex Banach space $X$ into a Banach space $Y$ written $f : D(f) \subseteq X \to Y$, let $\alpha(\Omega)$ denote the measure of noncompactness of the bounded set $\Omega$ (see for example [5]), and let

$$\begin{aligned}
&\alpha(f) = \inf\{k \geq 0 : \alpha(f(\Omega)) \leq k\alpha(\Omega) \ \text{ for every bounded } \ \Omega \subset D(f)\},\\
&\omega(f) = \sup\{k \geq 0 : \alpha(f(\Omega)) \geq k\alpha(\Omega) \ \text{ for every bounded } \ \Omega \subset D(f)\},\\
&d(f) = \liminf_{\|x\|\to\infty,\, x\in D(f)} \frac{\|f(x)\|}{\|x\|}, \quad |f| = \limsup_{\|x\|\to\infty,\, x\in D(f)} \frac{\|f(x)\|}{\|x\|}.
\end{aligned}$$

Here $|f|$ is called the quasinorm of $f$ and $f$ is said to be quasibounded if $|f| < \infty$. Maps with $\alpha(f) < 1$ are $k$-set contractive (also condensing) with $k = \alpha(f)$. Note that a map $f$ satisfies $\alpha(f) = 0$ if and only if $f$ is compact, that is, $\overline{f(\Omega)}$ is compact for every bounded set $\Omega$.

We shall also use of some notions employed by Furi, Martelli and Vignoli [2] in their theory of spectrum of nonlinear operators. We recall some of these.

**Definition 2.1.** Let $f : X \to Y$ be a continuous map from a Banach space $X$ into a Banach space $Y$. The map $f$ is said to be *stably-solvable* if the equation

$$f(x) = h(x)$$

has a solution $x \in X$ for any continuous compact map $h : X \to Y$ with quasinorm $|h| = 0$.
$f$ is said to be *regular* if it is stably-solvable and $d(f)$ and $\omega(f)$ are both positive. When $Y = X$, the resolvent set of $f$ is the set

$$\varrho(f) = \{\lambda \in \mathbb{C}, \lambda I - f \ \text{ is regular}\}$$

and the spectrum of $f$ is $\sigma(f) = \mathbb{C} \setminus \varrho(f)$.

If $f$ is invertible then $\alpha(f^{-1}) = 1/\omega(f)$, so regular invertible maps have $k$-set contractive inverses.

We will consider a generalization of the concept of stably-solvable maps below.

## 3. Surjectivity theorems

We begin with a result which generalizes Theorem 1.2 and Corollary 1.1 in Chapter II of [1]. Those authors studied operators $T$ that are $(K, L, a)$-homeomorphisms, where a (not necessarily linear) map $T : X \to Y$ is said to be a $(K, L, a)$-homeomorphism if

(a) $T$ is a homeomorphism of $X$ onto $Y$, and

(b) there exists real numbers $K > 0, a > 0, L > 0$ such that

$$L\|x\|^a \leq \|T(x)\| \leq K\|x\|^a \quad \text{for each } x \in X.$$

We do not assume so much.

**Theorem 3.1.** *Let $T : D(T) \subseteq X \to Y$ be an operator satisfying the following conditions:*

1. *$T$ is one to one, onto and $T^{-1} : Y \to D(T)$ is continuous;*
2. *there exist real numbers $L > 0, a > 0$ and $b > 0$ such that*

$$\|T(x)\| \geq L\|x\|^a - b \quad \text{for every } x \in D(T);$$

3. *$T$ is bounded, that is, maps bounded sets into bounded sets.*

*Let $S : X \to Y$ be bounded, continuous and suppose that*

$$\limsup_{x \in D(T),\, \|x\| \to \infty} \frac{\|S(x)\|}{\|x\|^a} = A.$$

*Then $\lambda T - S$ maps $D(T)$ onto $Y$ under any one of the following conditions:*

1. *$|\lambda| > \max\{\frac{A}{L}, \frac{\alpha(S)}{\omega(T)}\}$;*
2. *$S$ is compact, and $|\lambda| > \frac{A}{L}$;*
3. *$Y$ is a finite dimensional space, and $|\lambda| > \frac{A}{L}$;*
4. *$S$ is compact, $A = 0$, and $\lambda \neq 0$.*

Proof: Clearly it suffices to prove case 1. Also it is clear that $\lambda T - S$ maps $D(T)$ onto $Y$ if $I - F$ maps $Y$ onto $Y$ where $F : Y \to Y$ is defined by $F(y) = ST^{-1}(y/\lambda)$.

For any bounded set $\Omega \in Y$, we have

$$\begin{aligned} \alpha(F(\Omega)) &= \alpha(ST^{-1}(\Omega/\lambda)) \\ &\leq \alpha(ST^{-1})\alpha(\Omega/\lambda) \\ &\leq \frac{1}{|\lambda|}\frac{\alpha(S)}{\omega(T)}\alpha(\Omega). \end{aligned}$$

Therefore,
$$\alpha(F) \le \frac{1}{|\lambda|}\frac{\alpha(S)}{\omega(T)} < 1.$$
[If $S$ is compact or $Y$ is finite dimensional, then $\alpha(F) = 0$.]

Also we have,
$$\begin{aligned}|F| &= \limsup_{\|y\|\to\infty} \|F(y)\|/\|y\| \\ &= \limsup_{\|y\|\to\infty} \|ST^{-1}(y/\lambda)\|/\|y\|.\end{aligned}$$

Writing $x = T^{-1}(y/\lambda)$, we have $Tx = y/\lambda$, and we obtain
$$\begin{aligned}|F| &= \limsup_{\|Tx\|\to\infty} \frac{\|S(x)\|}{|\lambda|\|Tx\|} \\ &= \limsup_{x\in D(T),\,\|x\|\to\infty} \frac{\|S(x)\|}{|\lambda|\|Tx\|} \\ &\le \limsup_{x\in D(T),\,\|x\|\to\infty} \frac{\|S(x)\|}{|\lambda|(L\|x\|^a - b)} \\ &= \frac{A}{|\lambda|L} < 1.\end{aligned}$$

Hence, by the results of [2], $1 \in \varrho(F)$, in particular $I - F$ maps $Y$ onto $Y$. □

**Remark 3.2.** A result similar to Theorem 3.1 was obtained in [3], where a different method was used.

**Remark 3.3.** Theorem 1.2 of [1] requires that $T$ is a $(K, L, a)$-homeomorphism and that $T, S$ are both odd with $S$ compact, but allow either $|\lambda| > \frac{A}{L}$ or $|\lambda| < \frac{A}{K}$, $\lambda \neq 0$.

The following simple example shows that for $0 \neq \lambda \in \mathbb{C}$, even when $T$ is a $(K, L, a)$-homeomorphism there is no result similar to Theorem 3.1 in the case $|\lambda| < \frac{A}{K}$ without some extra hypothesis (such as oddness of the maps).

**Example 3.4.** Let $T$ and $S : \mathbb{C} \to \mathbb{C}$ be defined by
$$T(z) = z,\ S(x + iy) = |x| + iy,$$
and let $\lambda = 1/2$. Then $a = 1$, $K = L = 1$, $T$ is odd, $S$ is not odd. Also $A = 1$, $S, T$ are compact maps, but $\lambda T - S$ is not onto since $z/2 - S(z) = 1$ has no solution.

We recall the following concepts from [1].

**Definition 3.5.** Suppose that $a > 0$.

(a) A map $F_0 : X \to Y$ is called *$a$-homogeneous* if $F_0(tu) = t^a F_0(u)$ for every $t \geq 0$ and $u \in X$.

(b) $F : X \to Y$ is said to be *$a$-quasihomogeneous* relative to $F_0$ if $F_0 : X \to Y$ is $a$-homogeneous and

$$t_n \searrow 0,\ u_n \rightharpoonup u_0,\ t_n^a F(u_n/t_n) \to g \in Y$$

together imply that $g = F_0(u_0)$. [Here $u_n \rightharpoonup u_0$ denotes weak convergence.]

(c) $F : X \to Y$ is said to be *$a$-strongly quasihomogeneous* relative to $F_0$ if

$$t_n \searrow 0,\ u_n \rightharpoonup u_0 \ \text{ imply that } \ t_n^a F(u_n/t_n) \to F_0(u_0) \in Y.$$

It is known ([1]) that in case (c) $F_0$ is $a$-homogeneous and also must be strongly continuous, that is $u_n \rightharpoonup u_0$ implies $F_0 u_n \to F_0 u_0$.

By applying Theorem 3.1 instead of Corollary 1.1 of [1], we obtain the following generalization of Theorem 4.1 of [1], where we can dispense with the assumption that $T, S$ are odd maps.

**Theorem 3.6.** *Let $X$ be reflexive and let $T$ satisfy the conditions of Theorem 3.1. Let $S : X \to Y$ be a compact $b$-strongly quasihomogeneous operator relative to $S_0$ and suppose that $a > b$. Then for $\lambda \neq 0$, $\lambda T - S$ maps $D(T)$ onto $Y$.*

Proof: By Theorem 3.1, part 4, it suffices to show that

$$\lim_{\|x\| \to \infty,\, x \in D(T)} \frac{\|S(x)\|}{\|x\|^a} = 0.$$

This was proved in Theorem 4.1 of [1] but we include the proof for completeness. If this is false, there is a sequence $\{x_n\}$ with $\|x_n\| \to \infty$ and $\varepsilon > 0$ such that $\|Sx_n\|/\|x_n\|^a \geq \varepsilon$, for all sufficiently large $n$. Letting $u_n = x_n/\|x_n\|$ and $t_n = 1/\|x_n\|$ we have, for a subsequence, that

$$S(x_n)/\|x_n\|^b \to S_0(u_0).$$

Since $a > b$ this gives $S(x_n)/\|x_n\|^a \to 0$, a contradiction. □

**Remark 3.7.** The authors of [1] say that the case $a < b$ seems to be unsolved in the infinite dimensional case. We shall give an answer below, see Theorem 3.12.

We introduce the following extension of the concept of stably solvable maps which is appropriate to our needs.

**Definition 3.8.** A continuous map $f : D(f) \subseteq X \to Y$ is said to be *$a$-stably-solvable* for some $a > 0$ if the equation

$$f(x) = h(x)$$

has a solution $x \in D(f)$ for any continuous compact map $h : X \to Y$ with

$$|h|_a := \limsup_{\|x\|\to\infty} \frac{\|h(x)\|}{\|x\|^a} = 0.$$

**Lemma 3.9.** *Suppose $T : D(T) \subseteq X \to Y$ is as in Theorem 3.1. Then $T$ is $a$-stably-solvable.*

Proof: Let $h : X \to Y$ be a compact map with $|h|_a = 0$. Then $\alpha(T^{-1}h) = 0$, and

$$\begin{aligned}\limsup_{\|x\|\to\infty} \frac{\|T^{-1}h(x)\|}{\|x\|} &= \limsup_{\|x\|\to\infty} \frac{\|T^{-1}h(x)\|}{\|h(x)\|^{\frac{1}{a}}} \left(\frac{\|h(x)\|}{\|x\|^a}\right)^{\frac{1}{a}} \\ &\leq \limsup_{\|x\|\to\infty} \left(\frac{1}{L}\right)^{\frac{1}{a}} \left(\frac{\|h(x)\|}{\|x\|^a}\right)^{\frac{1}{a}} \to 0.\end{aligned}$$

Therefore, $|T^{-1}h| = 0$. This implies that $1 \in \varrho(T^{-1}h)$, so that $I - T^{-1}h$ is onto, that is, there exists $x \in D(T)$ such that $x = T^{-1}h(x)$, that is, $Tx = hx$. $\square$

**Lemma 3.10** (The Continuation Principle for $a$-stably-solvable maps)**.**
*Let $f : D(f) \subseteq X \to Y$ be $a$-stably-solvable, $h : X \times [0,1] \to Y$ be continuous, compact and such that $h(x,0) = 0$ for all $x \in D(f)$. Let*

$$U = \{x \in D(f),\ f(x) = h(x,t) \ \text{ for some } \ t \in [0,1]\}.$$

*Then, if $f(U)$ is bounded, the equation*

$$f(x) = h(x,1)$$

*has a solution.*

Proof: Let $B_r = \{y \in Y,\ \|y\| < r\}$, and let $r > 0$ be chosen so that $\overline{f(U)} \subset B_r$. Let $\varphi : X \to [0,1]$ be continuous and such that

$$\varphi(y) = \begin{cases} 1, & \text{for } y \in \overline{f(U)}, \\ 0, & \text{for } \|y\| \geq r, \end{cases}$$

and let $\pi$ be the radial retraction of $Y$ onto $\overline{B_r}$. Then the equation

$$f(x) = \pi h(x, \varphi(f(x)))$$

has a solution $x_0 \in D(f)$ since $\pi h$ is compact and

$$|\pi h|_a = \lim_{\|x\|\to\infty} \frac{\|(\pi h)(x)\|}{\|x\|^a} = 0.$$

If $\|f(x_0)\| = r$, then $\varphi(f(x_0)) = 0$, and $f(x_0) = \pi h(x_0, 0) = 0$, a contradiction. Thus $\|f(x_0)\| < r$, and $f(x_0) = h(x_0, \varphi(f(x_0)))$, which shows that $x_0 \in U$ and therefore $f(x_0) = h(x_0, 1)$. □

Theorem 3.1 of [1] gave theorems of Fredholm alternative type for the couple $(T, S)$ when $T, S$ were both odd. Recall that $\lambda$ is said to be an eigenvalue for the couple $T_0, S_0$ if there is $x_0 \neq 0$ such that $\lambda T_0 x_0 - S_0 x_0 = 0$. Using Lemmas 3.9 and 3.10 we can give the following result when neither $T$ nor $S$ is odd.

**Theorem 3.11.** *Let $X$ be a reflexive Banach space, and let $T$ be as in Theorem 3.1 with $D(T) = X$ and also $a$-quasihomogeneous relative to $T_0$. Let $S : X \to Y$ be a compact $a$-strongly-quasihomogeneous operator relative to $S_0$. If $\lambda \neq 0$, and for every $t \in (0, 1]$, $\lambda/t$ is not an eigenvalue for the couple $(T_0, S_0)$, then $\lambda T - S$ maps $X$ onto $Y$.*

Proof: For arbitrary $y \in Y$, let

$$U = \{x \in X,\ \lambda T(x) = h(x,t) = t[S(x) + y],\ t \in [0,1]\}.$$

We show that $U$ is bounded. If not, there exists $x_n \subset U$, $\|x_n\| \to \infty$, such that

$$\lambda T(x_n) = t_n[S(x_n) + y], \quad t_n \in [0, 1],$$

so that

$$\begin{aligned} \frac{\lambda T(x_n)}{\|x_n\|^a} &= t_n \left( \frac{S(x_n)}{\|x_n\|^a} + \frac{y}{\|x_n\|^a} \right) \\ &= t_n \frac{1}{\|x_n\|^a} S\left( \frac{x_n/\|x_n\|}{1/\|x_n\|} \right) + t_n \frac{y}{\|x_n\|^a}. \end{aligned}$$

Without loss of generality we assume that $x_n/\|x_n\| \rightharpoonup x_0$, $t_n \to t_0 \in [0, 1]$. Then there exists a subsequence $\{x_{n_k}\}$ such that

$$t_{n_k} \frac{1}{\|x_{n_k}\|^a} S\left( \frac{x_{n_k}/\|x_{n_k}\|}{1/\|x_{n_k}\|} \right) \to t_0 S_0(x_0),$$

$$\lim_{n\to\infty} \frac{\lambda T(x_{n_k})}{\|x_{n_k}\|^a} = t_0 S_0(x_0).$$

Since $T$ is $a$-quasihomogeneous relative to $T_0$, we obtain

$$\lambda T_0(x_0) = t_0 S_0(x_0).$$

However,
$$\frac{\|\lambda T(x_{n_k})\|}{\|x_{n_k}\|^a} \geq |\lambda|L - \frac{|\lambda|b}{\|x_{n_k}\|^a} > |\lambda|L/2,$$
for $n_k$ sufficiently large so that $\|t_0 S_0(x_0)\| > 0$. Hence $t_0 \neq 0$, and $S_0(x_0) \neq 0$.

From the definition of $a$-strongly-quasihomogeneous operator it is easy to show that $S_0(0) = 0$. Thus $x_0 \neq 0$, and $\lambda/t_0$ is an eigenvalue of $(T_0, S_0)$, a contradiction. Thus $U$ is bounded. By Lemma 3.9, $\lambda T : X \to Y$ is $a$-stably-solvable. So by Lemma 3.10, the equation $\lambda T(x) = S(x) + y$ has a solution $x \in X$, that is $\lambda T - S$ is onto. □

The next two results extend Theorem 4.2 of [1] to the infinite dimensional case.

**Theorem 3.12.** *Let $X$ be a reflexive Banach space. Let $T$ be a bounded, odd mapping satisfying the following conditions:*

1. *$T : D(T) \subseteq X \to Y$ is one to one, onto and $T^{-1} : Y \to D(T)$ is continuous;*
2. *there exist real numbers $K > 0, a > 0$ and $q$ such that*
$$\|T(x)\| \leq K\|x\|^a + q \quad \text{for every} \quad x \in D(T).$$

*Suppose that $S$ is odd, continuous and $b$-strongly quasihomogeneous relative to $S_0$, and that $\inf_{\{\|x\|=1\}}\|S_0(x)\| > 0$. If $a < b$, then for every $\lambda$ with $|\lambda| > \alpha(S)/\omega(T)$, $\lambda T - S$ is $a$-stably-solvable.*

Proof: First we show that there exists $R > 0$ such that $\lambda x - T^{-1}Sx \neq 0$ whenever $\|x\| \geq R$. If there exists $\{x_n\} \subset X$, $\|x_n\| \to \infty$ such that
$$\lambda x_n - T^{-1}S(x_n) = 0$$
we may assume that $\frac{x_n}{\|x_n\|} \rightharpoonup x_0$. Then we have
$$\frac{\|S(x_n)\|}{\|x_n\|^b} = \frac{\lambda T(x_n)}{\|x_n\|^b} \leq \frac{|\lambda|K\|x_n\|^a + q}{\|x_n\|^b} \to 0.$$
Since $S$ is $b$-strongly quasihomogeneous relative to $S_0$, we have
$$\frac{1}{\|x_n\|^b}S(x_n) = \frac{1}{\|x_n\|^b}S\left(\frac{x_n/\|x_n\|}{1/\|x_n\|}\right) \to S_0(x_0).$$
As $S_0$ is strongly continuous we also have $S_0\left(\frac{x_n}{\|x_n\|}\right) \to S_0(x_0)$. Since $\inf_{\|x\|=1}\|S_0(x)\| > 0$ it follows that $S_0(x_0) \neq 0$, this contradicts the above. Let $B_r(0) = \{x \in X, \|x\| < r\}$, where $r > R$. Then $\alpha(T^{-1}S) < |\lambda|$ and the topological degree $d\left(I - T^{-1}S/\lambda,\ B_r(0),\ 0\right)$ is odd, hence nonzero (see, for example, [5]). For a compact operator $h : X \to Y$ with $h = 0$ for $\|x\| = r$,
$$d\left(I - T^{-1}S/\lambda - T^{-1}h/\lambda,\ B_r(0),\ 0\right) \neq 0$$

because of boundary value dependence of degree.

For each $n \in \mathbb{N}$ let $\sigma_n$ be continuous and such that

$$\sigma_n(x) = \begin{cases} 1 & \text{for } \|x\| \leq n, \\ 0 & \text{for } \|x\| \geq 2n. \end{cases}$$

Then, if $h : X \to Y$ is a compact operator, with $|h|_a = 0$, for every $n > R/2$, the equation

$$\lambda T(x) - S(x) = \sigma_n(x) h(x)$$

has a solution $x_n \in D(T)$. If for all $n$, we have $\|x_n\| > n$, then

$$\frac{\lambda T(x_n) - S(x_n)}{\|x_n\|^b} = \frac{\sigma_n(x_n) h(x_n)}{\|x_n\|^b}.$$

Assume that $x_n/\|x_n\| \rightharpoonup x_0$. Then from

$$\frac{\lambda T(x_n) - S(x_n)}{\|x_n\|^b} \to -S_0(x_0) \neq 0 \quad (n \to \infty),$$

and

$$\frac{\sigma_n(x_n) h(x_n)}{\|x_n\|^b} = \sigma_n(x_n) \frac{h(x_n)}{\|x_n\|^a} \frac{\|x_n\|^a}{\|x_n\|^b} \to 0 \quad (n \to \infty),$$

we reach a contradiction. Hence there exists $n$, such that $\|x_n\| \leq n$, and then

$$\lambda T(x_n) - S(x_n) = h(x_n),$$

and we are done. $\square$

**Theorem 3.13.** *Let $X$ be a reflexive Banach space, $T, T_1 : D(T) \to Y$ and $S, S_1 : X \to Y$ be of the form $T = T_1 + R$, $S = S_1 + R'$, where $T_1$ satisfies the same conditions as $T$ in Theorem 3.12, $S_1$ is odd, continuous and $b$-strongly quasihomogeneous relative to $S_0$, and $R, R' : X \to Y$ are compact operators with $|R|_a = |R'|_a = 0$. Suppose that $a < b$, and that $\inf_{\{\|x\|=1\}} \|S_0(x)\| > 0$. Then $\lambda T - S$ maps $D(T)$ onto $Y$ for every $\lambda$ with $|\lambda| > \alpha(S)/\omega(T)$.*

Proof: For $y \in Y$, let $h(x) = -\lambda R(x) + R'(x) + y$, so that $h$ is compact and $|h|_a = 0$. By Theorem 3.12, the equation

$$\lambda T_1(x) - S_1(x) = h(x)$$

has a solution $x_0 \in D(T)$. Hence

$$\lambda T(x_0) - S(x_0) = y,$$

that is $\lambda T - S$ is onto. $\square$

## 4. Applications

The following applications are examples of situations that can be settled by the above theorems but apparently cannot be handled by the results in [1].

**Example 4.1.** We consider a nonlinear Sturm-Liouville problem on an unbounded domain, namely the following nonlinear differential equation:

$$\begin{gathered} -(p(x)u'(x))' + q(x)u(x) = \lambda\{u(x) + g(x)f(u(x))\}, \\ \text{for} \quad x \in (0,\infty), \quad \text{and} \quad u(0) = 0. \end{gathered} \tag{4.1}$$

In [4] it was shown that certain eigenvalues $\lambda$ are asymptotic bifurcation points. Under the same assumptions we will show that if $v$ is continuous, the equation

$$\begin{gathered} -(p(x)u'(x))' + q(x)u(x) = \lambda\{u(x) + g(x)f(u(x))\} + v(x) \\ \text{for} \quad x \in (0,\infty), \quad \text{and} \quad u(0) = 0 \end{gathered} \tag{4.2}$$

has a solution when $\lambda$ is not one of these eigenvalues.

We recall the assumptions made in [4].

1. $p: \ [0,\infty) \to \mathbb{R}$ is continuous and continuously differentiable on $(0,\infty)$, with $p'$ bounded and $0 < P_1 \le p(x) \le P_2 < \infty$ for all $x \in [0,\infty)$.
2. $q: \ [0,\infty) \to \mathbb{R}$ is continuous with $0 < Q_1 \le q(x) \le Q_2 < \infty$ for all $x \in [0,\infty)$.
3. $f$ is a continuously differentiable function from $\mathbb{R}$ into itself, and there exist positive real numbers $P$ and $K$ such that $|f(p)| \le K|p|^r$ for all $p \ge P$, for some $r < 1$.
4. $g \in H_0^1(0,\infty)$.

For $u : [0,\infty) \to \mathbb{R}$ and $x \in [0,\infty)$ let $H$ be defined by $(Hu)(x) = g(x)f(u(x))$. Let $A : H_0^1 \bigcap W^{2,2} \to L^2$ be the self-adjoint extension of the operator $A_0$ defined by $A_0 u = -(p(x)u'(x))' + q(x)u(x)$ with domain the set of twice continuously differentiable functions with compact support in $(0,\infty)$. Then, ([4]), $A$ is a positive self-adjoint operator in $L^2$ and its positive square root $A^{\frac{1}{2}}$ is a linear homeomorphism of $H_0^1$ onto $L^2$, where $H_0^1$ is the closure of $C_0^\infty$ in $W^{1,2}$ and $C_0^\infty$ is the linear space of all infinitely differentiable, real-valued functions with compact support in $(0,\infty)$.

We claim (and will show below) that for $0 < |\lambda| < Q := \liminf_{x\to\infty} q(x)$, and $\lambda$ not an eigenvalue of $A$, the operator

$$u \mapsto u - \lambda A^{-1}u + \lambda A^{-1/2} H A^{-1/2} u$$

from $L^2 \to L^2$ is onto. Assuming this, it follows that the equation

$$Au = \lambda u + \lambda H u + v$$

has a solution $u \in H_0^1 \bigcap W^{2,2}$ for any $v \in L^2$ ([4, Lemma 4.18]). Hence if $v$ is continuous, using the same arguments as in Lemma 4.20 of [4] it follows that the equation (4.2) has a solution.

We now establish the claim made above. Let $\mu = 1/\lambda$, and let $T, S : L^2 \to L^2$ be defined by

$$Tu = \mu u - A^{-1}u, \quad Su = A^{-1/2}HA^{-1/2}u.$$

Suppose that $|\mu| > \alpha(A^{-1}) = 1/Q$ ([4, Theorem 4.23]), and that $\mu$ is not an eigenvalue of $A^{-1}$. Then $T$ is a bounded linear operator, which is one to one, onto, and has a continuous inverse. So it is a $(K, L, 1)$-homeomorphism of $L^2$ onto $L^2$. Furthermore, $T$ is 1-quasihomogeneous relative to $T$ since it has continuous inverse. It has been shown that $S$ is a compact operator and the quasinorm $|S| = 0$ in the space $L^2$ ([4, Lemma 4.17]). Assume that there exist $u_n \in L^2$ with $u_n \rightharpoonup u_0$, $t_n \searrow 0$ such that

$$t_n S\left(u_n/t_n\right) > \varepsilon_0 > 0.$$

Then $\{\|u_n/t_n\|_2\}$ is unbounded. If $\|u_{n_k}/t_{n_k}\|_2 \to \infty$, $(n_k \to \infty)$, then we have

$$\|t_{n_k} S\left(u_{n_k}/t_{n_k}\right)\|_2 = \frac{\|S\left(u_{n_k}/t_{n_k}\right)\|_2}{\|u_{n_k}\|_2/t_{n_k}} \|u_{n_k}\|_2 \to 0,$$

a contradiction. Thus we have shown that $S$ is a 1-strongly quasihomogeneous operator relative to $S_0 = 0$ in the space $L^2$. For any $t \in (0, 1]$,

$$(1/t)(\mu I - A^{-1})(u) = 0 \Longrightarrow u = 0,$$

so $1/t$ is not an eigenvalue of the couple $(T, 0)$. By Theorem 3.11, $T - S$ maps $L^2$ onto $L^2$. Thus we have reached the conclusion.

The following second-order $m$-point nonlinear boundary value problem (BVP) has been studied recently by Gupta, Ntouyas and Tsamatos ([6], [7], [8]):

$$\begin{gathered} x''(t) = f(t, x(t), x'(t)) + e(t) \quad 0 < t < 1, \\ x(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i). \end{gathered} \tag{4.3}$$

It was shown that the problem of existence of a solution for the BVP (4.3) can be studied via the three boundary value problem

$$\begin{gathered} x''(t) = f(t, x(t), x'(t)) + e(t) \quad 0 < t < 1, \\ x(0) = 0, \quad x(1) = \alpha x(\eta), \end{gathered} \tag{4.4}$$

where $\eta \in (0, 1)$ and $\alpha \in \mathbb{R}$.

Some conditions for the existence of a solution for the BVP (4.4) were obtained in [6] using the Leray-Schauder continuation theorem. Their results suppose that $\alpha < 1/\eta$. By using Theorem 3.1, we obtain the following result which gives a different condition for the existence of a solution for (4.4) under the more general hypothesis $\alpha \neq 1/\eta$.

**Theorem 4.2.** *Let $f\colon [0,1]\times\mathbb{R}^2\to\mathbb{R}$ be a function satisfying Carathéodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, $r(t)$ in $L^1(0,1)$ such that*

$$|f(t,x_1,x_2)|\le p(t)|x_1|+q(t)|x_2|+r(t)$$

*for a.e. $t\in[0,1]$ and all $(x_1,x_2)\in\mathbb{R}^2$. Also let $\eta\in(0,1)$, $\alpha\ge 0$, $\alpha\ne 1/\eta$ be given. Then for any given $e\in L^1(0,1)$ the boundary value problem (4.4) has at least one solution in $C^1[0,1]$ provided that*

$$\|p\|_1+\|q\|_1<\begin{cases}(1-\alpha\eta)/2, & \text{if} \quad \alpha\eta<1,\\ (\alpha\eta-1)/2\alpha\eta, & \text{if} \quad \alpha\eta>1.\end{cases} \tag{4.5}$$

Proof: Let $X$ denote the Banach space $C^1[0,1]$ with the norm

$$\|x\|=\max\{\|x\|_\infty,\|x'\|_\infty\}.$$

Let $Y$ denote the Banach space $L^1(0,1)$ with its usual norm.

The linear operator $L: D(L)\subset X\to Y$ is defined by setting

$$D(L)=\{x\in W^{2,1}(0,1):\ x(0)=0,\ x(1)=\alpha x(\eta)\},$$

and for $x\in D(L)$,

$$Lx=x''.$$

For $x\in X$, let

$$(Nx)(t)=f(t,x(t),x'(t)),\quad t\in[0,1].$$

Then $N$ is a bounded map from $X$ into $Y$. It can be shown that $L: D(L)\subset X\to Y$ is one to one and onto when $\alpha\ne 1/\eta$. In fact, $L^{-1}=K$, where $K$ is the linear operator $K: Y\to D(L)\subset X$ defined by

$$(Ky)(t)=\int_0^t(t-s)y(s)\,ds+\frac{\alpha t}{1-\alpha\eta}\int_0^\eta(\eta-s)y(s)\,ds-\frac{t}{1-\alpha\eta}\int_0^1(1-s)y(s)\,ds.$$

For $y\in Y$, we have

$$\|Ky\|_\infty\le\left(1+\frac{\alpha\eta+1}{|1-\alpha\eta|}\right)\|y\|_1,$$

where $\|y\|_1$ is the norm of $y$ in the space $L^1(0,1)$. Also

$$\|(Ky)'\|_\infty\le\left(1+\frac{\alpha\eta+1}{|1-\alpha\eta|}\right)\|y\|_1.$$

Thus we have

$$\|Ky\|\le\left(1+\frac{\alpha\eta+1}{|1-\alpha\eta|}\right)\|y\|_1.$$

Let $T = I$ and $S = KN$. Then $\alpha(S) = 0$ by the Arzela-Ascoli theorem. Also we have

$$\begin{aligned}
A &= \limsup_{\|x\|\to\infty} \frac{\|S(x)\|}{\|x\|} \\
&= \limsup_{\|x\|\to\infty} \frac{\|KN(x)\|}{\|x\|} \\
&\le \limsup_{x\to\infty} \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right) \frac{\|N(x)\|_1}{\|x\|} \\
&\le \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right) \limsup_{\|x\|\to\infty} \frac{\|p\|_1\|x\|_\infty + \|q\|_1\|x'\|_\infty + \|r\|_1}{\|x\|} \\
&\le \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right) \limsup_{\|x\|\to\infty} \frac{(\|p\|_1 + \|q\|_1)\|x\| + \|r\|_1}{\|x\|} \\
&= \left(1 + \frac{\alpha\eta + 1}{|1-\alpha\eta|}\right) (\|p\|_1 + \|q\|_1) \\
&= \begin{cases} \frac{2}{1-\alpha\eta}(\|p\|_1 + \|q\|_1) & \text{for } \alpha\eta < 1 \\ \frac{2\alpha\eta}{\alpha\eta-1}(\|p\|_1 + \|q\|_1) & \text{for } \alpha\eta > 1. \end{cases}
\end{aligned}$$

By the assumption (4.5) we see that $A < 1$. Hence, from Theorem 3.1, the operator $I - S = I - KN$ maps $X$ onto $X$.

Hence, given any $e \in L^1(0,1)$, there exits $x \in C^1[0,1]$ such that

$$x(t) - (KNx)(t) = Ke(t).$$

Thus $x = KNx + Ke \in D(L)$ and

$$Lx - Nx = e.$$

This proves that the BVP (4.4) has at least one solution in $C^1[0,1]$. □

**Remark 4.3.** When $\alpha\eta < 1$, the condition (4.5) gives a better result than Theorem 4 of [6] in case $\alpha(1-\eta) > 2$ since their condition demands $\|p\|_1 + \|q\|_1 < \frac{1-\alpha\eta}{\alpha(1-\eta)}$, but is worse in the case $\alpha(1-\eta) < 2$. Also our result can apply when $\alpha\eta > 1$.

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