# A functional representation of the hyperspace monad

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*Abstract.* A functional representation of the hyperspace monad, based on the semilattice structure of function space, is constructed.

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**0.** All spaces are compact Hausdorff spaces (compacta), all mappings are continuous. A functional representation of the hyperspace functor exp is given in [1]. This representation essentially uses the linear structure on function spaces.

From the algebraic point of view hyperspaces are free Louson semilattices. Hence it would be natural to involve the semi-lattice structure on function spaces into representations of the hyperspace functor. In this paper such a representation is given not only for the hyperspace functor but also for the hyperspace monad.

The paper is organized as follows: in Section 1 we give some necessary definitions and recall a result from [1], in Section 2 we construct the monad $\mathbb{E}$ and in Section 3 we prove that the monad $\mathbb{E}$ and the hyperspace monad are isomorphic.

**1.** By $\mathcal{C}omp$ we denote the category whose objects are compacta (compact Hausdorff spaces) and morphisms are continuous mappings.

For a compactum $X$ by $\exp X$ we denote the set of non-void compact subsets of $X$ provided with the Vietoris topology. A base of this topology consists of the sets of the form $< U_1, \dots, U_n >= \{A \in \exp X \mid A \subset \bigcup_{i=1}^n U_i,\ A \cap U_i \neq \emptyset$ for each $i \in \{1, \dots, n\}\}$ where $U_1, \dots, U_n$ are open in $X$. The space $\exp X$ is called the hyperspace of $X$.

For a continuous mapping $f : X \to Y$ the mapping $\exp f : \exp X \to \exp Y$ is defined by the formula $\exp f(A) = fA \in \exp Y$, $A \in \exp X$. It is easy to see that this defines a functor $\exp : \mathcal{C}omp \to \mathcal{C}omp$ (the hyperspace functor).

Let $F$, $G$ be two functors in the category $\mathcal{C}omp$. We say that a transformation $\varphi : F \to G$ is defined if for every $X \in \mathcal{C}omp$ is defined the mapping $\varphi X : FX \to GX$. The transformation $\varphi = \{\varphi X\}$ is called natural if for every mapping $f : X \to Y$ we have $\varphi Y \circ F(f) = G(f) \circ \varphi X$.

A monad $\mathbb{T} = (T, \eta, \mu)$ in a category $\mathcal{E}$ consists of an endofunctor $T : \mathcal{E} \to \mathcal{E}$ and natural transformations $\eta : \mathrm{Id}_{\mathcal{E}} \to T$ (unity), $\mu : T^2 \to T$ (multiplication) satisfying the relations $\mu \circ T\eta = \mu \circ \eta T = \mathbf{1}_T$ and $\mu \circ \mu T = \mu \circ T\mu$.

A natural transformation $\psi : T \to T'$ is called a morphism from monad $\mathbb{T} = (T, \eta, \mu)$ into monad $\mathbb{T}' = (T', \eta', \mu')$ if $\psi \circ \eta = \eta'$ and $\psi \circ \mu = \mu' \circ \psi T' \circ T\psi$.

A morphism of monads is called an isomorphism, provided each component $\psi X$ is a homeomorphism.

Define the natural transformations $s : \mathrm{Id}_{\mathcal{C}omp} \to \exp$ and $u : \exp^2 \to \exp$ as follows: $sX(x) = \{x\}$ for each $x \in X$; $uX(\mathcal{A}) = \cup\mathcal{A}$, $\mathcal{A} \in \exp^2 X$. Then $\mathbb{H} = (\exp, s, u)$ is monad ([2]).

Now we recall the result from [1]. Let us denote by $C_+(X)$ the space of non-negative continuous functions on $X$ with natural metric, order, linear and multiplicative structures. By $\alpha_X$ we denote the function equal $\alpha$ on $X$.

By $\Phi(X)$ we denote all functionals $\varphi : C_+(X) \to \mathcal{R}_+$ satisfying next conditions:

(1) $\varphi(f+g) \le \varphi(f) + \varphi(g)$,

(2) $\varphi(fg) \le \varphi(f)\varphi(g)$,

(3) $f \le g$ implies $\varphi(f) \le \varphi(g)$,

(4) $\psi(\alpha f) = \alpha\varphi(f)$,

(5) $\psi(f + \alpha_X) = \psi(f) + \alpha$,

(6) $\varphi(\alpha_X) = \alpha$,

where $f, g \in C_+(X)$, $\alpha \in \mathbb{R}_+$.

The set $\Phi(X)$ provides a topology with a base consisting of sets of the form $(\mu; \varphi_1, \dots, \varphi_n; \varepsilon) = \{\mu' \in \Phi(X) \mid |\mu'(\varphi_i) - \mu(\varphi_i)| < \varepsilon$ for each $i \in \{1, \dots, k\}\}$ where $\varphi_1, \dots, \varphi_k \in C_+(X)$, $\varepsilon > 0$. It is shown in [1] that the functor $\Phi(X)$ is naturally isomorphic to the hyperspace functor.

**2.** Let us consider the space $C(X;[0;1])$ equipped by sup-metric and semilattice operation – pointwise minimum.

Let $\mu \in C(C(X,[0;1]),[0;1])$. We say that $\mu$ is normed if $\mu(c_X) = c$ for each $c \in [0;1]$.

We say that $\mu$ is supported on a closed set $A \subset X$ if for each functions $g_1, g_2 : X \to [0;1]$ such that $g_1|A = g_2|A$ we have $\mu(g_1) = \mu(g_2)$.

The minimal closed set $A$ on which $\mu$ is supported will be called the support of $\mu$ (briefly $A = supp(\mu)$). So, we can consider $\mu$ as an element from $C(C(A,[0;1]),[0;1])$ if $A = supp(\mu)$.

We say that $\mu$ is symmetric on its support if for each $\varphi \in C(A,[0;1])$ for each $h \in \mathrm{Auth}(\varphi(A))$ we have $\mu(\varphi) = \mu(h \circ \varphi)$. Finally we say that $\mu$ preserve the semilattice operation if $\mu(\min\{\varphi_1, \varphi_2\}) = \min\{\mu(\varphi_1), \mu(\varphi_2)\}$ for each $\varphi_1, \varphi_2 \in C(X,[0;1])$.

Let us define the space $E(X)$ consisting of $\mu \in C(C(X,[0;1]),[0;1])$ which are normed, symmetric on its support and preserve the semilattice operation. The space $E(X)$ provides a topology with a base consisting of sets of the form $(\mu; \varphi_1, \dots, \varphi_n; \varepsilon) = \{\mu' \in \Phi(X) \mid |\mu'(\varphi_i) - \mu(\varphi_i)| < \varepsilon$ for each $i \in \{1, \dots, k\}\}$ where $\varphi_1, \dots, \varphi_k \in C(X,[0;1])$, $\varepsilon > 0$.

Let $f : X \to Y$ be a mapping. Define the mapping $E(f) : E(X) \to E(Y)$ by the formula $E(f)(\mu)(\varphi) = \mu(f^*(\varphi))$ where $\mu \in E(X)$, $\varphi \in C(Y,[0;1])$ and the mapping $f^* : C(Y,[0;1]) \to C(X,[0;1])$ can be defined by the formula $f^*(\varphi)(x) = \varphi(f(x))$, $x \in X$, $\varphi \in C(Y,[0;1])$.

Later we shall prove that $E(X)$ is a compactum. So, $E$ is an endofunctor on the category $\mathcal{C}omp$.

Let us define the mapping $\eta X : X \to E(X)$ by the formula $\eta X(x)(\varphi) = \varphi(x)$ and the mapping $\mu X : E^2(X) \to E(X)$ by the formula $\mu X(\alpha)(g) = \alpha(\tilde{g})$ where $\alpha \in E^2(X)$, $g \in C(X, [0;1])$ and the mapping $\tilde{g} : E(X) \to [0;1]$ is defined by the formula $\tilde{g}(\mu) = \mu(g)$, $\mu \in E(X)$.

It is easy to check that $\eta X$ and $\mu X$ are the components of natural transformations $\eta$ and $\mu$ such that $\mathbb{E} = (E, \eta, \mu)$ is the monad on the category $\mathcal{C}omp$.

**3.** In this section we prove that the monad $\mathbb{E}$ is isomorphic to the monad $\mathbb{H} = (\exp, s, u)$.

Let us consider the mapping $tX : \exp X \to E(X)$ defined by the formula $t(A)(f) = \inf f(A)$, $f \in C(X, [0;1])$.

**Lemma.** *The mapping $tX$ is homeomorphism from* $\exp X$ *onto* $E(X)$.

Proof: Let $tX(A) = \mu \in E(X)$ and $(\mu; \varphi_1, \dots, \varphi_n; \varepsilon)$ be a neighborhood of $\mu$. Choose an open covering $O_1, \dots, O_k$ of $A$ such that $\operatorname{diam} \varphi_i(O_j) < \varepsilon$ for each $i \in \{1, \dots, n\}$, $j \in \{1, \dots, k\}$. Then we have $tX(< O_1, \dots, O_k >) \subset (\mu; \varphi_1, \dots, \varphi_n; \varepsilon)$ and hence the mapping $tX$ is continuous.

Consider $A_1, A_2 \in \exp X$ such that $A_1 \neq A_2$. We may assume that there exists a point $a \in A_1$ such that $a \notin A_2$. Choose a function $f \in C(X, [0;1])$ such that $f(a) = 0$ and $f(A_2) = 1$. Then we have $tX(A_1)(f) = 0 < 1 = tX(A_2)(f)$. Hence the mapping $tX$ is injective.

Let us prove that the mapping $tX$ is surjective. Consider $\nu \in E(X)$. We can assume that $supp(\nu) = X$. So, we must show that $\nu(f) = \inf f(X)$. We can assume that $\inf f(X) = 0$.

Suppose $\nu(f) = a > 0$. Since $\nu$ is normed, there exists a point $x \in X$ such that $f(x) \geq a$. We can assume that $f(x) \leq a$ for each $x \in X$ (in the opposite case we can consider the function $\inf\{f, a_X\}$).

Let us consider two cases:

(1) $f(X) = [0;a]$. Define a homeomorphism $h : [0;a] \to [0;a]$ by the formula $h(t) = a - t$, $t \in [0;a]$. It follows from the symmetry condition that $\nu(h \circ f) = \nu(f) = a$. Put $g = \inf\{h \circ f, f\}$. Then we have $\nu(g) = \inf\{\nu(h \circ f), \nu(f)\} = a$ but $g(x) \leq \frac{1}{2}a$ for each $x \in X$. Hence $\nu(g) \leq \frac{1}{2}a$ and we obtain the contradiction.

(2) There exists a point $b \in (0;a)$ such that $b \notin f(X)$. Consider the function $f_1 : X \to \{b;a\}$ defined by the formula:

$$f_1(x) = \begin{cases} a, & f(x) > b \\ b, & f(x) < b. \end{cases}$$

Since $f \leq f_1 \leq a_X$, we have $\nu(f_1) = a$. Define the function

$$f_2(x) = \begin{cases} b, & f(x) > b \\ a, & f(x) < b. \end{cases}$$

It follows from the symmetry condition that $\nu(f_2) = \nu(f_1) = a$. But $\inf\{f_1, f_2\} = b_X$ and $\nu(\inf\{f_1, f_2\}) = b$. We obtain the contradiction again.

Thus the mapping $tX$ is homeomorphism and the lemma is proved. □

It follows from Lemma that $E(X)$ is compactum.

**Theorem.** *A transformation $t = \{tX\}$ is the isomorphism from monad $\mathbb{H} = (\exp, s, u)$ into monad $\mathbb{E} = (E, \eta, \mu)$.*

PROOF: Let us show that $t$ is natural transformation. Let $f : X \to Y$ be a mapping and $A \in \exp X$. Then we have $tY \circ \exp f(A) = \inf \varphi(f(A))$, $\varphi \in C(Y, [0;1])$ and $E(f) \circ tX(A)(\varphi) = \inf f^*(\varphi)(A) = \inf \varphi(f(A))$.

Now let us show that $t$ is the morphism of monads $\mathbb{H}$ and $\mathbb{E}$. The identity $t \circ s = \eta$ is obvious. Let us check the identity $t \circ u = \mu \circ tE \circ \exp t$. Take any $\mathcal{A} \in \exp^2 X$ and $\varphi \in C(X, [0;1])$. Then we have $t \circ u(\mathcal{A})(\varphi) = \inf \varphi(\cup\mathcal{A})$ and $\mu \circ tE \circ \exp t(\mathcal{A})(\varphi) = tE \circ \exp t(\mathcal{A})(\tilde{\varphi}) = \inf\{\inf \varphi(A) \mid A \in \mathcal{A}\} = \inf \varphi(\cup\mathcal{A})$.

Now the statement of the theorem follows from Lemma. □

## References

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