On the density of the hyperspace of a metric space

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Abstract. We calculate the density of the hyperspace of a metric space, endowed with the Hausdorff or the locally finite topology. To this end, we introduce suitable generalizations of the notions of totally bounded and compact metric space.

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1. Introduction

The behaviour of cardinal functions on hyperspaces has not been as well investigated as that relative to other topological operations. Generally speaking, this kind of investigation often consists of computing the value of a given cardinal function on the hyperspace in terms of its value on the base space, and possibly other topological properties of the latter.

Some recent results showed that it can turn out to be somehow surprising, such as the possibility that a cardinal function decreases by passing from the base space to the hyperspace (see [5]).

In this paper, we study the density of the hyperspace of a metric (or metrizable) space, equipped with the Hausdorff or the locally finite topology. To this end, besides the density of the base space, two suitable generalizations of the notions of totally bounded and compact metric space will play a fundamental role. Such notions turn out to be of some independent interest, and will be studied in detail in the forthcoming paper ([3]).

In the following, the symbol $|A|$ will denote the cardinality of the set A, while $\operatorname{cof}(\nu)$ will be the cofinality of the cardinal ν. Every topological space is assumed to be infinite.

2. Generalized total boundedness and compactness

Definitions. For every $\varepsilon > 0$ let $\mathcal{UD}_\varepsilon$ be the family of all ε-uniformly discrete subsets of X, i.e.

$$\mathcal{UD}_\varepsilon = \{A \subset X \mid \forall\, x, y \in A : (x \neq y \Rightarrow d(x,y) \geq \varepsilon)\}.$$

Let

$$\mathcal{UD} = \bigcup_{\varepsilon>0} \mathcal{UD}_\varepsilon$$

be the family of all uniformly discrete subsets.

Let $\mathcal{UD}_\varepsilon^{\max}$ be the subfamily of $\mathcal{UD}_\varepsilon$ containing all the elements which are maximal with respect to the set-theoretic inclusion.

Remark 1. For every $\varepsilon > 0$, the family $\mathcal{UD}_\varepsilon^{\max}$ is not empty; this fact follows by Zorn's lemma as the collection $\mathcal{UD}_\varepsilon$ is clearly inductive. More precisely, every ε-uniformly discrete subset is contained in a maximal ε-uniformly discrete subset.

Every $U \in \mathcal{UD}_\varepsilon^{\max}$ enjoys, in addition to being ε-uniformly discrete, the following property, which is easily proved using maximality:

(D) for all $x \in X$ there exists $u \in U$ such that $d(x, u) < \varepsilon$.

Definition. A subset U that satisfies condition (D) is said to be ε-dense (see [7]).

In fact, the elements of $\mathcal{UD}_\varepsilon^{\max}$ are exactly the subsets which are ε-uniformly discrete and ε-dense at the same time.

Lemma 2. *Let $(r_n)_{n\in\mathbb{N}}$ (where $\mathbb{N} = \omega \setminus \{0\}$) be a sequence of positive real numbers such that $\lim_{n\to\infty} r_n = 0$, and let U_n be an element of $\mathcal{UD}_{r_n}^{\max}$ for every $n \in \mathbb{N}$. Then*

$$\mathbf{d}X = \sup_{n\in\mathbb{N}} |U_n|.$$

Proof: As every discrete subset has cardinality less than or equal to the density of the space ([7, Theorem 4.1.15]), the inequality

$$\mathbf{d}X \geq \sup_{n\in\mathbb{N}} |U_n|$$

is obviously true. We have to show the opposite inequality.

First of all, $\sup_n |U_n|$ cannot be finite. By contradiction, suppose it is finite, and let n be an index for which the supremum is attained. Take a point $z \in X \setminus U_n$ (such a point exists by our global assumption that X is infinite) and let k be a suitable integer greater than or equal to n such that $r_k < \min\{\frac{1}{2}r_n, \frac{1}{2}d(z, U_n)\}$.

For every $x \in U_n$ there exists by the r_k-density of U_k an element $\phi(x) \in U_k$ such that $d(x, \phi(x)) < r_k$. The map $\phi : U_n \to U_k$ is injective because if $y \in U_n$ satisfies $\phi(x) = \phi(y)$ then

$$d(x, y) \leq d(x, \phi(x)) + d(\phi(y), y) < r_k + r_k < \frac{r_n}{2} + \frac{r_n}{2} = r_n$$

and this implies $x = y$ by the r_n-uniform discreteness of U_n. As $|U_k| \leq |U_n|$ (by the choice of n) and both sets are finite, ϕ is bijective.

Furthermore, r_k-density of U_k also implies the existence of $u \in U_k$ such that $d(z, u) < r_k$. We have

$$2r_k < d(z, U_n) \leq d(z, \phi^{-1}(u)) \leq d(z, u) + d(u, \phi^{-1}(u)) < r_k + r_k = 2r_k,$$

which is a contradiction. We have then proved that $\sup_{n\in\mathbb{N}} |U_n| \geq \aleph_0$.

Now, consider the set $U = \bigcup_{n\in\mathbb{N}} U_n$: clearly U is dense in X, and by $\sup_{n\in\mathbb{N}} |U_n| \geq \aleph_0$ we easily obtain that $|U| = \sup_{n\in\mathbb{N}} |U_n|$. □

Corollary 3. *If* $\operatorname{cof}(\mathbf{d}X) > \aleph_0$ *then there exists* $U \in \mathcal{UD}$ *with* $|U| = \mathbf{d}X$.

Proof: Let us use the notations of the preceding lemma with, for instance, $r_n = \frac{1}{n}$. If for every n it were $|U_n| < \mathbf{d}X$, then by the definition of cofinality and the lemma it would follow that $\operatorname{cof}(\mathbf{d}X) = \aleph_0$. This contradicts the hypothesis and so there exists an n for which $|U_n| = \mathbf{d}X$. $\square$

Definition. A metric space (X, d) is said **totally bounded in the generalized sense** or simply **GTB** iff for every $\varepsilon > 0$ there exists an ε-dense subset $N \subset X$ with $|N| < \mathbf{d}X$.

A totally bounded metric space is a GTB space, as such a space is separable (i.e. has density $\aleph_0$) and so the condition $|N| < \mathbf{d}X$ means that N is finite.

An equivalent definition of generalized total boundedness is given by the following theorem. In practice, it says that uniformly discrete subsets of a GTB space cannot achieve the highest cardinality.

Theorem 4. *A metric space* X *is GTB iff* $U \in \mathcal{UD}$ *implies* $|U| < \mathbf{d}X$.

Proof: Let X be GTB, and $U \in \mathcal{UD}_\varepsilon$ with $\varepsilon > 0$. Consider a $\frac{\varepsilon}{2}$-dense subset N with $|N| < \mathbf{d}X$; then for every $x \in U$ there exists $\phi(x) \in N$ such that $d(x, \phi(x)) < \frac{\varepsilon}{2}$. With computations analogous to those performed in Lemma 2 it can be shown that the application ϕ is injective. Then $|U| \leq |N| < \mathbf{d}X$ and one implication is thus proved. The other implication is easily proved using the fact that any set in $\mathcal{UD}_\varepsilon^{\max}$ is ε-dense. $\square$

Corollary 5. *If* X *is a GTB space then* $\operatorname{cof}(\mathbf{d}X) = \aleph_0$.

Proof: Apply the theorem with Corollary 3. $\square$

Example 6. *If* ν *is a cardinal with* $\operatorname{cof}(\nu) = \aleph_0$ *then there exists a GTB space* X *with* $\mathbf{d}X = \nu$.

Proof: Let ν be a cardinal with countable cofinality. Assume $\nu > \aleph_0$ as the case $\nu = \aleph_0$ is trivial. Then there exists an ascending sequence of infinite cardinals ν_n with $\nu_n < \nu$ and $\nu = \sup_{n\in\mathbb{N}} \nu_n$. For every n let X_n be a set with cardinality ν_n (we can suppose the X_n mutually disjoint). Consider the set $X = \bigcup_{n\in\mathbb{N}} X_n$, and for every $x \in X$ define the "level" $l(x)$ as the unique $n \in \mathbb{N}$ such that $x \in X_n$. Endow X with the metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \frac{1}{n} & \text{if } x \neq y \text{ and } l(x) = l(y) = n, \\ 1 & \text{if } l(x) \neq l(y). \end{cases}$$

This metric induces the discrete topology on X and so $\mathbf{d}X = |X| = \nu$. On the other hand, if U is an ε-uniformly discrete subset of X, then it is easily seen that U splits up into two parts, whose one is contained in some $\bigcup_{n=1}^{N} X_n$ for a large enough N, and the other one has at most countably many elements. Hence $|U| < \nu$, and using Theorem 3 we have that X is a GTB space. $\square$

The notion of GTB space leads us in a natural way to introduce a generalization of compact spaces. We will say that a topological space X is **compact in the generalized sense** (briefly, **GK**) if for every open cover $\mathcal{U}$ of X, there exists a subcover $\mathcal{V}$ such that $|\mathcal{V}| < \mathbf{d}X$. We will consider only metrizable GK spaces.

The next characterization generalizes the well-known result that a metrizable space is compact if and only if it has no closed and discrete subset of cardinality $\aleph_0$, and will prove to be extremely useful in the following.

Theorem 7. *A metrizable space X is GK if and only if it has no closed and discrete subset of cardinality equal to $\mathbf{d}X$.*

Proof: First, suppose that there exists a closed and discrete subset D of X with $|D| = \mathbf{d}X = \nu$. By Hausdorff's extension theorem, there exists a compatible metric d on X which agrees with the $0-1$ metric on D. With that metric, $D \in \mathcal{UD}_1$ and hence we can find $\tilde{D} \in \mathcal{UD}_1^{\max}$ with $D \subseteq \tilde{D}$; clearly, the cardinality of $\tilde{D}$ is still equal to ν. Thus the collection $\{S_d(x,1) \mid x \in \tilde{D}\}$ is a minimal open cover of X of cardinality ν, and X is not GK.

Conversely, suppose that X is not GK. Then there exists an open cover $\mathcal{U}$ of X which does not admit any subcover of cardinality less than $\mathbf{d}X$. By paracompactness we may assume that $\mathcal{U}$ is locally finite.

Let $\mathbf{d}X = \nu$, and define simultaneously by transfinite induction a ν-sequence $(x_\alpha)_{\alpha\in\nu}$ of elements of X and a ν-sequence $(V_\alpha)_{\alpha\in\nu}$ of elements of $\mathcal{U}$, such that both $\alpha \mapsto x_\alpha$ and $\alpha \mapsto V_\alpha$ are one-to-one, and $x_\alpha \in V_\alpha$ for every $\alpha \in \nu$; this is possible as for every $\alpha \in \nu$, $\bigcup_{\beta<\alpha} V_\beta \neq X$. Now, the local finiteness of $\mathcal{U}$ implies that the set $D = \{x_\alpha \mid \alpha \in \nu\}$ is closed and discrete. □

As an easy consequence of the preceding result, we obtain the following theorem which generalizes the property that a metrizable space is compact if and only if every compatible metric on it is totally bounded.

Theorem 8. *A metrizable space X is GK if and only if every compatible metric on X is GTB.*

Proof: If X is GK with $\mathbf{d}X = \nu$ and d is a compatible metric on X, then given any $\varepsilon > 0$, the open cover $\{S_d(x,\varepsilon) \mid x \in X\}$ (where $S_d(x,\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}$) admits a subcover $\mathcal{V}$ having cardinality less than ν; taking the central point from each element of $\mathcal{V}$ gives an ε-dense subset of X, whose cardinality is less than ν.

Conversely, suppose that X is not GK. Then there exists a set $D = \{x_\alpha \mid \alpha \in \nu\}$ which is closed and discrete in X. Again, by Hausdorff's extension theorem there exists a compatible metric d on X which agrees with the $0-1$ metric on D. It is clear that d is not GTB, since no $\frac{1}{2}$-dense subset with cardinality less than ν can be found in X. □

Example 9. *If ν is a cardinal number with $\operatorname{cof}(\nu) = \aleph_0$, then there exists a GK metrizable space X with $\mathbf{d}X = \nu$.*

PROOF: Consider the topological space constructed in Example 5, and add a point (say "∞") provided with the fundamental system of neighbourhoods $\{V_n \mid n \in \mathbb{N}\}$ where $V_n = \{\infty\} \cup (\bigcup_{n' \geq n} X_{n'})$ for every $n \in \mathbb{N}$.

It is easily checked that $\tilde{X} = X \cup \{\infty\}$ is a metrizable space whose density is ν. Furthermore, given any open cover $\mathcal{U}$ of X, there exists $\tilde{A} \in \mathcal{U}$ such that $\infty \in \tilde{A}$, and hence we have that $\bigcup_{n' \geq n} X_{n'} \subseteq \tilde{A}$ for a suitable $n \in \mathbb{N}$; thus it is clear that we can obtain a subcover with cardinality at most $\nu_{n-1} < \nu$.

Notice that the metrizability of X can be easily checked by observing, for instance, that X has a σ-discrete base. Nevertheless, we give here an explicit compatible metric $\tilde{d}$ for X. Put:

$$\tilde{l}(x) = \begin{cases} l(x) & \text{if } x \in X, \\ +\infty & \text{if } x = \infty, \end{cases}$$

(where $l(x)$ is the function defined in Example 6) and

$$\tilde{d}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \frac{1}{\min\{\tilde{l}(x), \tilde{l}(y)\}} & \text{if } x \neq y. \end{cases}$$

The metric d on X does not coincide on X with the metric d of Example 5. Indeed, the latter cannot be extended to the whole of $\tilde{X}$. $\square$

With a slight modification to the preceding example, it is possible to construct a GK space which is also connected (see [7, Exc 4.1.H(b)]).

Let us investigate more deeply the structure of a GK space. Such spaces present a sort of "core" of points with high local density which happens to be non-empty, compact (in the classical sense).

Definition. If x is a point of a topological space X, we call local density of x in X the cardinal number:

$$\mathbf{ld}(x, X) := \min\{\mathbf{d}V \mid V \text{ is an (open) nbhd of } x \text{ in } X\}.$$

Define the "core" of X as:

$$\Delta(X) = \{x \in X \mid \mathbf{ld}(x, X) = \nu\}.$$

Lemma 10. *Let X be an infinite GK metrizable space, and let $\mathbf{d}X = \nu$. Then $\Delta(X)$ is compact.*

PROOF: By contradiction, suppose $\Delta(X)$ is not compact. Then there exists a closed discrete subset M of X with $|M| = \aleph_0$. Write M as $\{x_n \,|\, n \in \mathbb{N}\}$, where $n \mapsto x_n$ is one-to-one; then fix a compatible metric d on X, and for every $n \in \mathbb{N}$ put $\varepsilon_n = d(x_n, M \setminus \{x_n\})$. It follows that the family $\mathcal{M} = \{\overline{S}_d(x_n, \frac{\varepsilon_n}{4}) \,|\, n \in \mathbb{N}\}$ is discrete. Indeed, given any $x \in X$, if x is some $x_{\bar{n}}$ then it is easily shown that $S_d(x_{\bar{n}}, \frac{3}{4}\varepsilon_{\bar{n}}) \cap \overline{S}_D(x_n, \frac{\varepsilon_n}{4}) = \emptyset$ for $n \neq \bar{n}$. If, on the contrary, $x \neq x_n$ for every

$n \in \mathbb{N}$, put $\delta = d(x, M)$: the open ball $S_d(x, \frac{\delta}{2})$ misses all but at most one element of $\mathcal{M}$.

As $\mathrm{cof}\,(\nu) = \aleph_0$, there exists a strictly increasing sequence $(\nu_n)_{n\in\mathbb{N}}$ of cardinals whose supremum is ν. For every $n \in \mathbb{N}$, we have that $\mathbf{d}\overline{S}_d(x_n, \varepsilon_n) = \nu > \nu_n$, and hence there exists a closed and discrete subset D_n of $\overline{S}_d(x_n, \varepsilon_n)$ such that $|D_n| = \nu_n$; clearly, each D_n is closed in X too, and the family $\{D_n \,|\, n \in \mathbb{N}\}$ is discrete. It follows easily that $D = \bigcup_{n\in\mathbb{N}} D_n$ is a closed and discrete subset of X. This is impossible, as $|D| = \nu$ and X is GK. $\square$

Lemma 11. *If X is an infinite GK metrizable space, then $\Delta(X) \neq \emptyset$.*

Proof: Let $\nu = \mathbf{d}X$ and $(\nu_n)_{n\in\mathbb{N}}$ be as in the preceding lemma. By contradiction, suppose $\mathbf{ld}(x, X) < \nu$ for every $x \in X$: let us associate to every $x \in X$ two open nbhds V_x and W_x of x such that $\mathbf{d}V_x = \mathbf{ld}(x, X)$, and $\overline{W}_x \subseteq V_x$. By paracompactness, there exists a locally finite open cover $\mathcal{A}$ of X which refines $\{W_x \,|\, x \in X\}$. For every $A \in \mathcal{A}$, the set $\overline{A}$ is contained in some $\overline{W}_x$, which is in turn contained in V_x : therefore $\mathbf{d}\overline{A} \leq \mathbf{d}V_x = \mathbf{ld}(x, X) < \nu$. We have two possible cases.

<u>1st case</u>: there exists $\bar{n} \in \mathbb{N}$ such that $\forall\, A \in \mathcal{A} : \mathbf{d}A \leq \mathbf{d}\overline{A} \leq \nu_{\bar{n}}$.

As X is GK, there exists a subcover $\mathcal{A}'$ of $\mathcal{A}$ such that $|\mathcal{A}'| < \nu$. For every $A \in \mathcal{A}'$, fix a dense subset D_A of A with $|D_A| = \mathbf{d}A \leq \nu_{\bar{n}}$, and put $D = \bigcup_{A\in\mathcal{A}'} D_A$: it is easily shown that D is dense in X, what is impossible, as clearly $|D| \leq \nu_{\bar{n}} \cdot |\mathcal{A}'| < \nu$.

<u>2nd case</u>: for every $n \in \mathbb{N}$, there exists $A_n \in \mathcal{A}$ such that $\mathbf{d}\overline{A}_n > \nu_n$. This implies that for every $n \in \mathbb{N}$, there exists a closed and discrete subset D_n of $\overline{A}_n$, with $|D_n| = \nu_n$.

Observe that, for every $A \in \mathcal{A}$, the set $\{n \in \mathbb{N} \,|\, A_n = A\}$ must be finite (as every $\overline{A}$ with $A \in \mathcal{A}$ has density less that ν, but the density of the sets $\overline{A}_n$ tends to ν); this implies — as $\mathcal{A}$ is locally finite — that the indexed family $\{\overline{A}_n\}_{n\in\mathbb{N}}$ is in turn locally finite (in the sense that every $x \in X$ has a nbhd V such that the set $\{n \in \mathbb{N} \,|\, V \cap \overline{A}_n \neq \emptyset\}$ is finite), and the same is true for the indexed family $\{D_n\}_{n\in\mathbb{N}}$. Since every D_n is closed in $\overline{A}_n$ — and hence in X — and discrete, we easily obtain that the set $D = \bigcup_{n\in\mathbb{N}} D_n$ is in turn closed and discrete in X; this is impossible because $|D| = \nu$ and X is GK. $\square$

Remark 12. If X is a metrizable GK space with $\mathbf{d}X = \nu > \aleph_0$, then the subset $\Delta(X)$ of X has empty interior. Indeed, let x be any point of $\Delta(X)$ and V any (open) nbhd of x : we have that $\mathbf{d}V \geq \mathbf{ld}(x, X) = \nu > \aleph_0$. Thus V cannot be contained in $\Delta(X)$, as $\Delta(X)$ is compact and hence $\mathbf{d}\Delta(X) \leq \aleph_0$.

Remark 13. If X is a metrizable GK space with $\mathbf{d}X = \nu > \aleph_0$, X is not homogeneous (recall that a topological space X is said to be homogeneous if for every $x, y \in X$ there exists an auto-homeomorphism $\phi : X \to X$ such that $\phi(x) = y$). As a matter of fact, since the local density of a point is invariant under homeomorphisms, and since $\Delta(X)$ is nonempty by Lemma 11, if X were homogeneous then we would have $\Delta(X) = X$. The latter would imply that X is compact by Lemma 10, and so $\mathbf{d}X = \aleph_0$.

3. The density of the Hausdorff and the locally finite hypertopologies

Let (X, d) be a metric space, the hyperspace $c_0(X)$ is the set of all closed and non-empty subsets of X. On $c_0(X)$ the well known Hausdorff (extended) metric H_d is defined as

$$\mathrm{H}_d(A, B) := \max(\mathrm{e}_d(A, B),\ \mathrm{e}_d(B, A))$$

where

$$\mathrm{e}_d(A, B) := \sup_{a \in A} d(a, B).$$

The topology $\mathbf{H}_d$ generated by H_d on $c_0(X)$ is called the *Hausdorff metric hypertopology.* The purposes of this section is to compute the density of the hyperspace $c_0(X)$ endowed with $\mathbf{H}_d$ in terms of the density of X and, possibly, some additional hypotheses on the metric d. To avoid trivialities, we will always assume that X is infinite. The result we are looking for is already known in the separable case [1], [2]:

Theorem 14. *If X is a separable metric space then:*

$$\mathbf{d}(c_0(X), \mathbf{H}_d) = \begin{cases} \aleph_0 & \text{if } X \text{ is totally bounded,} \\ 2^{\aleph_0} & \text{if } X \text{ is not totally bounded.} \end{cases}$$

To deal with the general case, we first state two cardinal inequalities. The next lemma gives an upper bound for the density of any topology on $c_0(X)$, as it only depends on set-theoretical considerations about the hyperspace.

Lemma 15 (Upper bound)**.** *For every metric space (X, d) and every topology τ on $c_0(X)$, we have that $\mathbf{d}(c_0(X), \tau) \leq 2^{\mathbf{d}X}$.*

PROOF: As X is a metric space, its density equals its weight. Thus X admits a base with $\mathbf{d}X$ elements, and then at most $2^{\mathbf{d}X}$ subsets of X are closed. Hence $\mathbf{d}(c_0(X), \tau) \leq |c_0(X)| \leq 2^{\mathbf{d}X}$. □

Lemma 16. *If $U \in \mathcal{UD}$, then $\mathbf{d}(c_0(X), \mathbf{H}_d) \geq 2^{|U|}$.*

PROOF: $U \in \mathcal{UD}_\varepsilon$ for some positive real ε. If A and B are distinct subsets of U, then it is easy to see that $\mathrm{H}_d(A, B) \geq \varepsilon$. So the power set $\wp(U)$ is an (ε-uniformly) discrete subset of $c_0(X)$. As the density of a space cannot be strictly less than the cardinality of a discrete subset (see [7, Theorem 4.1.15]) we have the estimate. □

As is usual in set theory, we denote with the symbol $2^{<\nu}$ the quantity $\sup_{\xi<\nu} 2^\xi$ for every cardinal ν.

Theorem 17. *If X is a metric space then*

$$\mathbf{d}(c_0(X),\mathbf{H}_d)=\begin{cases}2^{<\mathbf{d}X} & \text{if } X \text{ is GTB,}\\ 2^{\mathbf{d}X} & \text{if } X \text{ is not GTB.}\end{cases}$$

PROOF: Let us consider first the case when X is not GTB. By Theorem 4 there exists a ε-uniformly discrete subset U of X with $|U|=\mathbf{d}X$. Using both Lemmas 15 and 16 we have

$$2^{\mathbf{d}X}=2^{|U|}\le \mathbf{d}(c_0(X),\mathbf{H}_d)\le 2^{\mathbf{d}X}$$

and so equality holds.

Let X be GTB and for every positive integer n choose $U_n\in\mathcal{UD}^{\max}_{1/n}$. If $\xi<\mathbf{d}X$, then by Lemma 2 there exists an integer n_0 such that $\xi\le|U_{n_0}|$. Then Lemma 16 above gives

$$2^{\xi}\le 2^{|U_{n_0}|}\le \mathbf{d}(c_0(X),\mathbf{H}_d).$$

As ξ is arbitrary, we proved that $2^{<\mathbf{d}X}\le \mathbf{d}(c_0(X),\mathbf{H}_d)$.

Consider the subset of $(c_0(X),\mathbf{H}_d)$ given by

$$\mathcal{U}=\bigcup_{n\in\mathbb{N}}(\wp(U_n)\setminus\{\emptyset\}).$$

Let us compute $|\mathcal{U}|$:

$$\begin{aligned}|\mathcal{U}|&\le\sum_{n\in\mathbb{N}}|\wp(U_n)|=\sum_{n\in\mathbb{N}}2^{|U_n|}\\&\le\aleph_0\cdot\sup_{n\in\mathbb{N}}2^{|U_n|}\\&\le\aleph_0\cdot 2^{<\mathbf{d}X}=2^{<\mathbf{d}X}.\end{aligned}$$

As the other inequality is an immediate consequence of Lemma 2, we have that $|\mathcal{U}|=2^{<\mathbf{d}X}$. The thesis will now follow if we show that $\mathcal{U}$ is dense in $c_0(X)$.

Let $F\in c_0(X)$ and $\delta>0$, choose n_0 such that $\frac{1}{n_0}<\delta$ and consider the set

$$H=\Big\{x\in U_{n_0}\mid d(x,F)<\frac{1}{n_0}\Big\}.$$

Clearly $H\in\mathcal{U}$ and by construction, $\mathrm{e}_d(H,F)\le\frac{1}{n_0}$. For every $x\in F$ there exists, by maximality of U_{n_0}, a $u\in U_{n_0}$ such that $d(x,u)<\frac{1}{n_0}$. But necessarily $u\in H$ and so $d(x,H)<\frac{1}{n_0}$; passing to the supremum, we have that $\mathrm{e}_d(F,H)\le\frac{1}{n_0}$. Then $\mathrm{H}_d(H,F)\le\frac{1}{n_0}<\delta$ and so $\mathcal{U}$ is dense in $(c_0(X),\mathbf{H}_d)$. □

The interpretation of this result raises an interesting set-theoretic question. The key point is to decide when, in the relation

$$2^{<\nu}\le 2^{\nu}\tag{1}$$

which is always true under **ZFC**, we can distinguish the equality from the strict inequality.

We have the following result:

Theorem 18. *The following statements are equivalent in* **ZFC***:*

(i) $2^{<\nu} < 2^{\nu}$ *for every cardinal* ν*;*

(ii) $\xi < \nu$ *implies* $2^{\xi} < 2^{\nu}$ *for every couple of cardinals* ξ *and* ν*.*

Proof: Suppose that (ii) holds, then, by definition of cofinality

$$\operatorname{cof}(2^{<\nu}) = \operatorname{cof}(\sup_{\xi<\nu} 2^{\xi}) = \operatorname{cof}(\nu) \leq \nu.$$

But in **ZFC** we always have that $\operatorname{cof}(2^{\nu}) > \nu$ (see [8, Corollary 10.41]) and so (i) necessarily holds.

Suppose that (ii) does not hold, so there exist ξ and ν with $\xi < \nu$ and $2^{\xi} = 2^{\nu}$. Then

$$2^{\nu} = 2^{\xi} \leq 2^{<\nu} \leq 2^{\nu}$$

then $2^{<\nu} = 2^{\nu}$ and (i) does not hold. □

Let us say that $\mathcal{A}$ holds if one (hence both) of the statements of Theorem 18 holds. It is known that both $\mathcal{A}$ and $\neg\mathcal{A}$ are consistent with **ZFC**, and $\mathcal{A}$ is implied by GCH.

(**ZFC**) If $\nu = \aleph_0$, then $2^{<\nu} = \aleph_0$. We already noticed that GTB coincides with total boundedness in the separable case, so we have old Theorem 14 without additional set-theoretic axioms.

(**ZFC** $+\mathcal{A}$) In this case the two conditions

(d1) $\mathbf{d}(c_0(X), \mathbf{H}_d) = 2^{<\mathbf{d}X}$;

(d2) $\mathbf{d}(c_0(X), \mathbf{H}_d) = 2^{\mathbf{d}X}$;

are always mutually exclusive. Then they give necessary and sufficient conditions for the base space X to be GTB or not.

Furthermore, we have the additional result:

Theorem 19. *If* (X, d) *is GTB then* $(c_0(X), \mathrm{H}_d)$ *is GTB.*

Proof: For every $\varepsilon > 0$ consider in X an $\frac{\varepsilon}{2}$-dense subset N with $|N| < \mathbf{d}X$. As $\operatorname{cof}(\mathbf{d}X) = \aleph_0$ by Corollary 5, $\mathbf{d}X$ is a limit cardinal and so there exists a cardinal ξ with $|N| < \xi < \mathbf{d}X$. We want to show that the family

$$\mathcal{N} = \{\operatorname{Cl} A \mid \emptyset \neq A \subseteq N\}$$

is ε-dense in $(c_0(X), \mathrm{H}_d)$. To do that, choose an $F \in c_0(X)$ and call H the closure of the set $\{x \in N \mid d(x, F) < \frac{\varepsilon}{2}\}$. By the same calculations of Theorem 17 it can be shown that $\mathrm{H}_d(F, H) \leq \frac{\varepsilon}{2} < \varepsilon$ that establishes our claim.

It remains to prove that $\mathcal{N}$ has not the highest cardinality, but using axiom $\mathcal{A}$ we have

$$|\mathcal{N}| \leq |\wp(N)| = 2^{|N|} < 2^{\xi} \leq 2^{<\mathbf{d}X}$$

as required. $\square$

(**ZFC** $+\neg\mathcal{A}$) By the assumption, there exists a cardinal ν with $2^{<\nu} = 2^{\nu}$. Then conditions (d1) and (d2) coincide for every space with density ν.

Furthermore, we may assume that ν is regular, as if it were singular then we could replace it by $\aleph_{\alpha+1}$, where $\aleph_\alpha$ is any cardinal less than ν with $2^{\aleph_\alpha} = 2^{\nu}$. Let X be a metric space with density ν (e.g. the $0-1$ space with cardinality ν); X cannot be a GTB space by Corollary 5 and the regularity of ν (of course $\nu > \aleph_0$), nevertheless $(c_0(X), \mathbf{H}_d)$ has density $2^{<\nu}$. Thus we have a case where condition (d1) is not sufficient to prove that X is GTB.

To obtain a case where condition (d2) is not sufficient to infer that X is not a GTB space, we need to assume something more than $\neg\mathcal{A}$, i.e. that there exists a cardinal μ with both $2^{<\mu} = 2^{\mu}$ and $\operatorname{cof}(\mu) = \aleph_0$. In that case the GTB space given by Example 6 will do the job. The additional assumption is consistent with **ZFC** as, for example, we can pose (see [6, Theorem 1]) for every ordinal α

$$2^{\aleph_\alpha} = \begin{cases} \aleph_{\omega+1} & \text{if } \alpha \le \omega \\ \aleph_{\alpha+1} & \text{if } \alpha > \omega \end{cases}$$

and choose $\mu = \aleph_\omega$: then $2^{<\aleph_\omega} = \aleph_{\omega+1} = 2^{\aleph_\omega}$. In this framework, Theorem 19 is false, as if X is GTB with density μ then

$$\operatorname{cof}\bigl(\mathbf{d}(c_0(X), \mathbf{H}_d)\bigr) = \operatorname{cof}(2^{\mu}) > \mu > \aleph_0$$

and so $c_0(X)$ cannot be GTB.

Definitions. Let X be a topological space and M any subset of X; we put

$$M^+ := \{C \in c_0(X) \mid C \subseteq M\} \ \text{ and } \ M^- := \{C \in c_0(X) \mid C \cap M \neq \emptyset\}.$$

Also, if $\mathcal{F}$ is a collection of subsets of X, put

$$\mathcal{F}^- := \bigcap_{F \in \mathcal{F}} F^- = \{C \in c_0(X) \mid \forall\, F \in \mathcal{F} : C \cap F \neq \emptyset\}.$$

The locally finite topology $\mathbf{LF}$ on $c_0(X)$ is that generated by the subbase

$$\begin{aligned}\Delta_{\mathbf{LF}} := \{A^+ \mid A \ \text{ is open in } X\} \cup \\ \cup \{\mathcal{F}^- \mid \mathcal{F} \ \text{ is a locally finite collection of open subsets of } X\}.\end{aligned}$$

For every $C \in c_0(X)$, a generic basic $\mathbf{LF}$-neighbourhood of C is of the form $A^+ \cap \mathcal{F}^-$, where A is open in X with $C \subseteq A$, and $\mathcal{F}$ is a locally finite collection of open subsets of X such that $\forall\, F \in \mathcal{F} : F \cap C \neq \emptyset$. It is easily checked that we are allowed to consider only basic $\mathbf{LF}$-neighbourhoods of the form $\mathcal{F}^- \cap A^+$, where $\forall\, F \in \mathcal{F} : F \subseteq A$.

Lemma 20. *Let (X,d) be a GK metrizable space with $\mathbf{d}X = \nu > \aleph_0$, and let $\mathcal{F}$ be a locally finite collection of open nonempty subsets of X: then there exists a suitable $m \in \mathbb{N}$ such that for every $F \in \mathcal{F}$, we have that $F \setminus S_d(\Delta(X), \frac{1}{m}) \neq \emptyset$.*

Proof: By the above remark, no element of $\mathcal{F}$ can be contained in $\Delta(X)$: thus we can associate to every $F \in \mathcal{F}$ a point $x_F \in F \setminus \Delta(X)$. As $\mathcal{F}$ is locally finite, the set $C = \{x_F \,|\, F \in \mathcal{F}\}$ is closed. Put $r = D_d(C, \Delta(X))$: then $r > 0$ (as $\Delta(X)$ is compact), and choosing $m \in \mathbb{N}$ with $\frac{1}{m} < r$, we have that $C \cap S_d(\Delta(X), \frac{1}{m}) = \emptyset$. Therefore, such an m satisfies the thesis. □

Lemma 21. *If X is an infinite GK metrizable space, with $\mathbf{d}X = \nu$, then $\mathbf{d}(c_0(X), \mathbf{LF}) \leq 2^{<\nu}$.*

Proof: The case $\nu = \aleph_0$ is immediate, as if X is compact, then the locally finite topology on $c_0(X)$ coincides with the Hausdorff topology relative to any compatible metric on X. Therefore, suppose $\nu > \aleph_0$. Observe that, for every $m \in \mathbb{N}$, the set $Y_m = X \setminus S_d(\Delta(X), \frac{1}{m})$ has density less than ν: indeed, if there exists $m \in \mathbb{N}$ such that $\mathbf{d}Y_m = \nu$, then Y_m — as a closed subset of X — is GK, and hence by Lemma 11 there exists $\bar{y} \in Y_m$ such that $\nu = \mathbf{ld}(\bar{y}, Y_m) \leq \mathbf{ld}(\bar{y}, X)$, whence $\bar{y} \in \Delta(X)$, what is impossible by the definition of Y_m.

For every $m \in \mathbb{N}$, take a dense subset D_m of Y_m with $|D_m| = \mathbf{d}Y_m$ and put $\mathcal{D}_m = \{D' \subseteq D_m \,|\, D' \text{ is closed in } X\}$; also, let $\mathcal{D} = \bigcup_{m\in\mathbb{N}} \mathcal{D}_m$. Clearly, $|\mathcal{D}_m| \leq 2^{|D_m|} \leq 2^{<\nu}$. We will complete the proof showing that the collection $\mathcal{D}$ is dense in $(c_0(X), \mathbf{LF})$.

Let $\mathcal{F}^- \cap A^+$ be a basic nonempty $\mathbf{LF}$-open set, where A is open in X and $\mathcal{F}$ is a locally finite collection of open subsets of X; as already observed, we may also suppose that $\forall F \in \mathcal{F} : F \subseteq A$. By Lemma 20, there exists an $m \in \mathbb{N}$ such that $\forall F \in \mathcal{F} : F \setminus S_d(\Delta(X), \frac{1}{\bar{m}}) \neq \emptyset$, that is $\forall F \in \mathcal{F} : F \cap Y_{\bar{m}} \neq \emptyset$. For every $F \in \mathcal{F}$, choose a point $x_F \in F \cap D_{\bar{m}}$: as $\mathcal{F}$ is locally finite, the set $D = \{x_F \,|\, F \in \mathcal{F}\}$ is closed, and hence $D \in \mathcal{D}_{\bar{m}} \subseteq \mathcal{D}$. On the other hand, it is clear that $D \in \mathcal{F}^-$, and by the property that $\forall F \in \mathcal{F} : F \subseteq A$ we obtain as well that $D \in A^+$. Therefore $(\mathcal{F}^- \cap A^+) \cap \mathcal{D} \neq \emptyset$. □

We can now get the desired result about the density of the locally finite hypertopology, whose symmetry with Theorem 17 is apparent.

Theorem 22. *If X is a metric space then*

$$\mathbf{d}(c_0(X), \mathbf{LF}) = \begin{cases} 2^{<\mathbf{d}X} & \text{if } X \text{ is GK,} \\ 2^{\mathbf{d}X} & \text{if } X \text{ is not GK.} \end{cases}$$

Proof: Suppose first that X is not GK. Then there exists a compatible metric d on X such that (X,d) is not GTB; as $\mathbf{d}(c_0(X), \mathbf{H}_d) = 2^{\mathbf{d}X}$ and $\mathbf{H}_d \leq \mathbf{LF}$ (more exactly $\mathbf{LF}$ is the supremum of all topologies $\mathbf{H}_\rho$, as ρ varies among the compatible metrics on X [4, Theorem 3.3.12]), we have that $\mathbf{d}(c_0(X), \mathbf{LF}) \geq 2^{\mathbf{d}X}$. The opposite inequality is the upper bound proved in Lemma 15.

Suppose now X is GK. The inequality $\mathbf{d}(c_0(X), \mathbf{LF}) \leq 2^{<\mathbf{d}X}$ follows from Lemma 21, and the opposite inequality is easily obtained fixing any compatible metric d on X and observing that $\mathbf{d}(c_0(X), \mathbf{LF}) \geq \mathbf{d}(c_0(X), \mathbf{H}_d) = 2^{<\mathbf{d}X}$ by Theorem 17. $\square$

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