Monotone homogeneity of dendrites

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Abstract. Sufficient as well as necessary conditions are studied for a dendrite or a dendroid to be homogeneous with respect to monotone mappings. The obtained results extend ones due to H. Kato and the first named author. A number of open problems are asked.

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All spaces considered in this paper are assumed to be metric and all mappings are continuous. By a *continuum* we mean a compact connected space.

We shall use the notion of *order of a point* in the sense of Menger-Urysohn (see e.g. [14, § 51, I, p. 274]), and we denote by $\operatorname{ord}(p, X)$ the order of the continuum X at a point $p \in X$. Points of order 1 in a continuum X are called *end points* of X; the set of all end points of X is denoted by $E(X)$. Points of order at least 3 are called *ramification points* of X; the set of all ramification points of X is denoted by $R(X)$.

A *dendrite* means a locally connected continuum containing no simple closed curve. We denote by D_3 the *standard universal dendrite of order* 3, i.e. a dendrite X characterized by the following two conditions (see e.g. [3, Section 3, pp. 167–169]):

(1) each ramification point of X is of order 3, and

(2) for each arc $A \subset X$ we have $\operatorname{cl}(A \cap R(X)) = A$.

It is known that if a dendrite X satisfies (1), then it can be embedded into D_3.

A *dendroid* means an arcwise connected and hereditarily unicoherent continuum. A dendroid is locally connected if and only if it is a dendrite. A dendroid X is said to be *smooth at a point* $p \in X$ provided that for every point $x \in X$ and for every sequence $\{x_n\}$ of points converging to x the sequence of arcs px_n converges to the arc px. A dendroid is said to be *smooth* if there is a point at which it is smooth.

Let X and Y be continua. A mapping $f : X \to Y$ is said to be

- *monotone* provided that $f^{-1}(y)$ is connected for each $y \in Y$;
- *light* if $f^{-1}(y)$ has one-point components for each $y \in Y$ (note that if the inverse images of points are compact, this condition is equivalent to the property that they are zero-dimensional);

- *open* if f maps each open set in X onto an open set in Y;
- *confluent* if for each subcontinuum Q of Y and for each component C of $f^{-1}(Q)$ the equality $f(C) = Q$ holds.

Thus each monotone mapping is confluent. Also each open mapping is confluent ([21, (7.5), p. 148]).

Let $\mathcal{M}$ be a class of mappings between continua. A mapping $f : X \to Y$ is said to be *hereditarily* $\mathcal{M}$ provided that for every subcontinuum $Z \subset X$ the partial mapping $f \mid Z : Z \to f(Z) \subset Y$ is in $\mathcal{M}$. The following statement is known (see [17, Theorem 6.7, p. 51 and (6.10), p. 53]).

3. Statement. *Let X and Y be dendroids, and let $f : X \to Y$ be a surjection. Then the following conditions are equivalent:*

(i) *f is monotone;*
(ii) *f is hereditarily monotone;*
(iii) *f is hereditarily confluent.*

Let $\mathcal{M}$ be a class of mappings. A continuum X is said to be *homogeneous with respect to the class* $\mathcal{M}$, or shortly $\mathcal{M}$-*homogeneous*, provided that for every two points p and q of X there exists a surjective mapping $f : X \to X$ such that $f(p) = q$ and $f \in \mathcal{M}$. If $\mathcal{M}$ is a class of homeomorphisms, then X is simply called *homogeneous*. Continua X and Y are said to be $\mathcal{M}$-*equivalent* provided that there are in $\mathcal{M}$ surjective mappings $f : X \to Y$ and $g : Y \to X$. A class $\mathcal{M}$ of mappings is said to be *neat* if all homeomorphisms are in $\mathcal{M}$ and the composition of any two mappings in $\mathcal{M}$ is also in $\mathcal{M}$. Note that monotone, open, confluent and light mappings between continua form neat classes of mappings ([13, Remark, p. 220], [17, Chapter 5, Part A, (5.1) and (5.4), p. 27]).

In the University of Houston Mathematics Problem Book H. Cook posed the following problem (see [7, Problem 150, p. 388]).

Let $\mathcal{M}$ be a neat class of mappings. It is evident that

(4) *if a continuum is $\mathcal{M}$-equivalent to a homogeneous continuum, then it is $\mathcal{M}$-homogeneous.*

For what classes $\mathcal{M}$ of mappings the converse implication holds true, i.e.

5. Problem (Cook)**.** If a continuum X is $\mathcal{M}$-homogeneous, is there a homogeneous continuum Y such that X is $\mathcal{M}$-equivalent to Y?

In Section 2 of [3] H. Kato has given a negative answer to this problem for the classes $\mathcal{M}$ of monotone mappings (as well as for any class of mappings that comprises monotone ones, e.g. of confluent ones — see [17, Sections 3 and 4, pp. 12–28], in particular Table II on p. 28) by showing the following proposition ([13, Proposition 2.4, p. 223] and [12, Example 2.4, p. 59]; compare also Remark 2.8 of [12], p. 62).

6. Proposition (Kato)**.** *The standard universal dendrite D_3 of order 3 is homogeneous with respect to monotone mappings.*

The answer follows because each continuum which is monotone equivalent to (in particular which is a monotone image of) a dendrite is a dendrite (compare e.g. [4, Proposition 4.19, p. 11]), while no dendrite, being locally connected and planar ([21, Chapter 4, (7.32), p. 77]) is homogeneous (since the only locally connected planar homogeneous continuum is the simple closed curve, [18]). In the light of the above remarks it is clear that

7. Proposition. *Every monotone homogeneous dendrite can be taken as a counterexample to the implication mentioned in Problem* 5 *for the class* $\mathcal{M}$ *of monotone mappings between continua.*

Therefore it would be interesting to answer the following question (compare [3, Question 7.2, p. 186]).

8. Question. What dendrites are monotone homogeneous?

The next result is closely related to the above question.

9. Theorem. *A dendrite is confluently homogeneous if and only if it is monotone homogeneous.*

Proof: Let X be a dendrite. Since each monotone mapping is confluent, one implication is obvious. Assume X is confluently homogeneous. Let $p, q \in X$ and let a confluent mapping $f : X \to X$ be given with $f(p) = q$. Then there is a unique factorization $f = f_2 \circ f_1$ into confluent mappings such that $f_1 : X \to f_1(X)$ is monotone and $f_2 : f_1(X) \to X$ is open and light ([4, Lemma 5.4, p. 14]). Note that the intermediate space $f_1(X)$ is a dendrite ([4, Proposition 4.19, p. 11]). Since f_2 is open and light, it follows from Whyburn's theorem on the lifting of dendrites under light open mappings ([21, Theorem 2.4, p. 188]; compare [4, Lemma 5.5, p. 14] and [9, Theorem I.3, p. 410]) that there exists a dendrite $A \subset f_1(X)$ such that $f_1(p) \in A$ and $f_2 \,|\, A : A \to f_2(A) = X$ is a homeomorphism. Let $B = f_1^{-1}(A) \subset X$. Thus B is a dendrite, and $p \in B$. Since by Statement 3 any monotone mapping on a dendrite is hereditarily monotone, the partial mapping $f_1 \,|\, B : B \to f_1(B) = A \subset f_1(X)$ is monotone. Hence

$$f \,|\, B = (f_2 \,|\, A) \circ (f_1 \,|\, B) : B \to f(B) = f_2(f_1(B)) = X$$

is monotone. Further, since every subcontinuum of a dendrite is its monotone retract ([15, Theorem 2.1, p. 332]; compare [11, Theorem, p. 157]), there exists a monotone retraction $r : X \to B$. Then the composite $g = (f \,|\, B) \circ r : X \to X$ is a monotone surjection. Finally $g(p) = f(r(p)) = f(p) = q$. The proof is complete. $\square$

10. Remarks. A mapping $f : X \to Y$ between continua X and Y is said to be *semi-confluent* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$. It is obvious that any confluent mapping is semi-confluent. One can ask if Theorem 9

can be generalized to the equivalence between monotone homogeneity and semi-confluent homogeneity of dendrites.
(1) First, note that the mapping $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 2x & \text{for } x \in [0, 1/2] \\ -x + 3/2 & \text{for } x \in [1/2, 1] \end{cases}$$

is semi-confluent and maps an inner point of $[0,1]$ to an end point (and vice versa), whence it follows that $[0,1]$ is semi-confluently homogeneous, while it is not monotone homogeneous (because any monotone self-mapping of $[0,1]$ maps ends to ends, see [21, Chapter 9, (1.1), p. 165]). Thus one cannot put "semi-confluently" in place of "confluently" in Theorem 9.
(2) Second, note also that the class of semi-confluent mappings is not neat, because the composite of two semi-confluent mappings need not be semi-confluent, see [16, Example 3.4, p. 254]; cf. [17, Example 5.10, p. 31].

11. Question. Is the equivalence of confluent and monotone homogeneities true not only for dendrites (Theorem 9) but also for (a) smooth dendroids, (b) all dendroids?

The following easy result, which is a strengthened form of (4), seems to play an important role in answering Question 8.

12. Statement. *Let $\mathcal{M}$ be a neat class of mappings. If a continuum X is $\mathcal{M}$-equivalent to an $\mathcal{M}$-homogeneous continuum Y, then X is $\mathcal{M}$-homogeneous.*

Indeed, let $p, q \in X$ and let surjections $f : X \to Y$ and $g : Y \to X$ be in $\mathcal{M}$. Since for any $y \in g^{-1}(q)$ there is in $\mathcal{M}$ a surjection $h : Y \to Y$ such that $h(f(p)) = y$, the composite $g \circ h \circ f : X \to X$ is in $\mathcal{M}$ and maps p into q.
As a corollary to Proposition 6 and Statement 12 we get the following.

13. Proposition. *If a dendrite X is monotone equivalent to the standard universal dendrite D_3 of order 3, then X is monotone homogeneous.*

However, in the monotone equivalence between X and D_3 in (7) only one mapping is essential, because for every dendrite X there is a monotone surjective mapping of D_3 onto X ([3, Corollary 6.5, p. 180]). Thus we have the next corollary.

14. Corollary. *If there exists a monotone mapping of a dendrite X onto D_3, then X is monotone homogeneous.*

As an application of Corollary 14 we obtain the following assertion.

15. Proposition. *If for a dendrite X the set $R(X)$ of its ramification points is a dense subset of X, then X is monotone homogeneous.*

Proof: In fact, it is shown in Proposition 3.2 of [3], p. 169, that if for a dendrite X the condition $\operatorname{cl} R(X) = X$ is satisfied, then there exists a homeomorphism $h : D_3 \to h(D_3) \subset X$. Since every subcontinuum of a dendrite is a monotone retract of X (see [15, Theorem 2.1, p. 332]; compare also [11, Theorem, p. 157]), there exists a monotone retraction $r : X \to h(D_3)$. Then the composite $h^{-1} \circ r : X \to D_3$ is a monotone surjection, and the conclusion is a consequence of Corollary 14. □

As a generalization of the standard universal dendrite D_m of a fixed order $m \in \{3, 4, \ldots, \omega\}$ (see e.g. [3, Section 3, pp. 167–169]) a concept of a standard universal dendrite D_S of orders in $S \subset \{3, 4, \ldots, \omega\}$ has been introduced in [6, Definition 6.3, p. 230] as a dendrite X such that

(16) if $p \in R(X)$, then $\mathrm{ord}(p, X) \in S$;

(17) for each arc A contained in X and for every $m \in S$ there is in A a point p with $\mathrm{ord}(p, X) = m$.

It is known that if two dendrites satisfy (16) and (17) with the same set S, then they are homeomorphic ([6, Theorem 6.2, p. 229]). Since condition (17) implies that $\mathrm{cl}(A \cap R(X)) = A$ for each arc $A \subset X$, which is equivalent to $\mathrm{cl}\, R(X) = X$ (see [3, Theorem 2.4, p. 167]), we get, as a consequence of Proposition 15, the following result which generalizes Proposition 6 and its extension to all standard universal dendrites D_m ([3, Theorem 7.1, p. 186]).

18. Proposition. *For each nonempty set $S \subset \{3, 4, \ldots, \omega\}$ the standard universal dendrite D_S of orders in S is monotone homogeneous.*

The converse to Proposition 15 is not true in general and, moreover, it can be seen that the condition $\mathrm{cl}\, R(X) = X$ is far from being necessary for a dendrite X to be monotone homogeneous. Namely a monotone homogeneous dendrite L_0 is known having the set $R(L_0)$ of its ramification points discrete (thus nowhere dense in L_0). We recall its construction after [3, Example 6.9, p. 182], for the reader's convenience and for further purposes.

We start with the unit interval L_1 in the plane. Divide it in three equal parts, and in the middle of them, M, locate a thrice diminished copy C of the Cantor ternary set. At the mid point of each contiguous interval K to C (i.e. of a component K of $M \setminus C$) we erect perpendicularly to L_1 a straight line segment whose length equals length of K. Denote by L_2 the union of L_1 and of all erected segments (there are countably many of them). We perform the same construction on each of the added segments: divide such a segment into three equal subsegments, locate in the middle subsegment M a copy C of the Cantor set properly diminished, and at the mid point of any component K of $M \setminus C$ construct a perpendicular to K segment as long as K is, and denote by L_3 the union of L_2 and of all attached segments. Continuing in this manner we get an increasing sequence of dendrites L_n. Then

$$L_0 = \mathrm{cl}\left(\bigcup\{L_n : n \in \mathbb{N}\}\right). \tag{19}$$

It is evident that is a dendrite having a discrete set $R(L_0)$ (i.e. each point $p \in R(L_0)$ has a neighborhood U such that $U \cap R(L_0) = \{p\}$). Decompose L_0 into maximal free arcs (i.e. such arcs $A = ab$ that $A \setminus \{a, b\}$ is an open subset of L_0 and that no arc containing A properly has this property) and into singletons. Then the natural projection for this decomposition is a monotone mapping of L_0 onto the standard universal dendrite D_3 of order 3. According to Corollary 14 the dendrite L_0 is monotone homogeneous.

It is shown in [3, Theorem 6.14, p. 185] that a dendrite is monotone equivalent to D_3 if and only if it contains a homeomorphic copy of L_0. Therefore, Proposition 13 can be restated as follows.

20. Proposition. *If a dendrite contains a homeomorphic copy of the dendrite L_0 defined by* (19), *then it is monotone homogeneous.*

It would be interesting to know whether the converses to Proposition 20 (or to Corollary 14) are true. In other words, we have the following question.

21. Question. Does every monotone homogeneous dendrite contain a homeomorphic copy of the dendrite L_0 (equivalently, does it admit any monotone mapping onto D_3)?

Note that if an answer to Question 21 were yes, then containing a copy of L_0 would be a characteristic property of monotone homogeneous dendrites.

By the *Gehman dendrite* G we mean a dendrite having the Cantor ternary set in $[0, 1]$ as the set $E(G)$ of its end points, such that all ramification points of G are of order 3 and are situated in G in such a way that $E(G) = \operatorname{cl} R(G) \setminus R(G)$ (see [19, pp. 422–423] for a detailed description, and [20, Figure 1, p. 203] for a picture). The following two properties of the Gehman dendrite, which are consequences of its definition, will be needed.

(22) *The set $R(G)$ of ramification points of the Gehman dendrite G is discrete.*

(23) *Every convergent sequence of distinct ramification points of the Gehman dendrite G has an end point of G as its limit.*

24. Proposition. *For the Gehman dendrite G there is no monotone surjection $f : G \to G$ which maps a ramification of G point to any of its end points.*

Proof: Assume there are $x \in R(G)$, $y \in E(G)$ and a monotone surjection $f : G \to G$ with $f(x) = y$. Then $f^{-1}(y)$ is a continuum and, since $G \setminus \{y\}$ is connected ([21, Chapter 5, (1.1), (iv), p. 88]), the set $f^{-1}(G \setminus \{y\})$ is connected, too ([21, Chapter 8, (2.2), p. 138]). Thus the continuum $f^{-1}(y)$ does not disconnect G, whence it follows that $\operatorname{bd} f^{-1}(y)$ is a singleton, say $\{b\}$, and

$$\operatorname{ord}(b, G) = 2 \tag{25}$$

([21, Chapter 5, (1.1), (iv), p. 88]). Let $a \in G \setminus f^{-1}(y)$. Since by Statement 3 the partial mapping $f \mid ab$ is monotone, we infer that $f(ab)$ is an arc from $f(a)$ to $f(b) = y$ ([21, Chapter 9, (1.1), p. 165]). Take a sequence of points $y_n \in xy \cap R(G)$ which has y as its limit. By the ramification point covering property (see [9, Theorem I.1, p. 410]) there is a sequence of points $x_n \in R(G)$ such that $f(x_n) = y_n$

for each $n \in \mathbb{N}$. Since $y_n \neq y$, we have $x_n \in G \setminus f^{-1}(y)$. By compactness of G we may assume that $\{x_n\}$ converges to a point p. Now (22) and (23) imply that $p \in E(G)$. By continuity of f we infer that $f(p) = y$, so $p \in f^{-1}(y)$. Thus $p \in \mathrm{bd}\ f^{-1}(y)$, i.e. $p = b$, a contradiction to (25) by (23). The proof is complete. $\square$

26. Corollary. *The Gehman dendrite G is not monotone homogeneous.*

If we enlarge the considered classes of continua from dendrites to dendroids, and of mappings from monotone to confluent, then we get the following two analogs of Question 8:

27. Question. What dendroids are monotone homogeneous?

28. Question. What dendroids are confluently homogeneous?

Only a partial answer to Question 28 is known that concerns open mappings (see [2, Theorem, p. 409]).

(29) *No dendroid is openly homogeneous.*

To get more results connected with Questions 27 and 28 we recall some concepts related to the structure of dendroids. Recall that an *end point* in a dendroid X means a point $p \in X$ being an end point of any arc A such that $p \in A \subset X$, and that by a *ramification point* of a dendroid X we understand a point p being the vertex of a simple triod contained in X. If X is a dendrite, then these concepts coincide with the previous ones (i.e. end points are exactly points of order 1, and vertices of triods contained in X are exactly points of order at least 3). The set of all end points and of all ramification points in a dendroid X will be denoted again by $E(X)$ and $R(X)$, respectively.

Given a dendroid X, we denote by $\Delta(X)$ the subdendroid of X which is irreducible about $R(X)$, i.e. such that $R(X) \subset \Delta(X)$ and no proper subdendroid of $\Delta(X)$ contains $R(X)$. Recall that $\Delta(X)$ is uniquely determined (see [10, Theorem 1, p. 3]).

30. Theorem. *If a dendroid X is monotone homogeneous, then the subdendroid $\Delta(X)$ has infinitely many end points*

Proof: Suppose on the contrary that the set $E(\Delta(X))$ of end points of $\Delta(X)$ is finite (whence it follows that $\Delta(X)$ is a dendrite), and consider two cases.

Case 1. $R(X) \cap (\Delta(X) \setminus E(\Delta(X))) \neq \emptyset$.

Let $c \in R(X) \cap (\Delta(X) \setminus E(\Delta(X)))$. Then there is a point $d \in X$ such that $cd \cap (\Delta(X) \setminus E(\Delta(X))) = \{c\}$. Fix a point $e \in E(\Delta(X))$. Since X is monotone homogeneous, there is a monotone surjection $f : X \to X$ such that $f(e) = d$. Since f is monotone, it has the ramification point covering property (see [9, Theorem I.1, p. 410]). Further, since $f(X) = X$ and $R(X) \subset \Delta(X)$, we obtain $R(X) \subset f(R(X)) \subset f(\Delta(X))$, so the continuum $f(\Delta(X))$ contains $R(X)$. Since $\Delta(X)$ is the minimal continuum containing $R(X)$ (by its definition), we get

$\Delta(X) \subset f(\Delta(X))$. Now $e \in \Delta(X)$ implies $d = f(e) \in f(\Delta(X))$, thus $\Delta(X) \cup \{d\} \subset f(\Delta(X))$. Therefore we infer that

$$\text{card}\, E(\Delta(X)) < \text{card}\, E(f(\Delta(X))). \tag{31}$$

However, since by Statement 3 every monotone mapping on a dendrite is hereditarily monotone, the partial mapping $f \,|\, \Delta(X) : \Delta(X) \to f(\Delta(X))$ is monotone, and thereby it has the end point covering property, that is,

$$E(f(\Delta(X))) \subset f(E(\Delta(X)))$$

(see [4, Proposition 4.20, p. 11]), whence it follows that

$$\text{card}\, E(f(\Delta(X))) \le \text{card}\, f(E(\Delta(X))) \le \text{card}\, E(\Delta(X)),$$

contrary to (31).

Case 2. $R(X) \cap (\Delta(X) \setminus E(\Delta(X))) = \emptyset$.

Then each ramification point of X is an end point of $\Delta(X)$, and thus $R(X)$ is finite. This contradicts monotone homogeneity of X by Proposition 2.2 of [12, p. 59] saying that if the set $R(X)$ of ramification points of a dendroid X is finite, then X is not confluently (hence not monotone) homogeneous. The proof is complete. □

32. Remark. Proposition 5.45 of [4, p. 18] says that if a surjective mapping $f : X \to Y$ between dendrites X and Y is monotone and if the set $R(X)$ is contained in an arc, then the set $R(Y)$ is also contained in an arc, whence it follows that if for a dendrite X the continuum $\Delta(X)$ is an arc, then X is not monotone homogeneous. Theorem 30 above can be seen as a strong generalization of this result, as well as a generalization of the above quoted result of Kato (Proposition 2.2 of [12, p. 59]; see the final part of the proof of Theorem 30) in its part related to monotone mappings.

33. Remark. Note that the converse to Theorem 30 is not true. In fact, for the Gehman dendrite G we have $\Delta(G) = G$ by construction, whence $E(\Delta(G)) = E(G)$ is the Cantor set, while G is not monotone homogeneous according to Corollary 26.

34. Question. In the light of Statement 3 one can substitute either "hereditarily monotone" or "hereditarily confluent" for "monotone" in Theorem 30. Can one substitute "confluent" for "monotone" in Theorem 30 as well?

A more specific results concerning Question 27 can be obtained if we additionally assume that the dendroid X under consideration is smooth. To this aim we recall the following proposition (see [5, Corollary 10, p. 309]).

35. Proposition. *If a dendroid X is smooth at a point p and a surjective mapping $f : X \to Y$ is monotone, then Y is a dendroid that is smooth at a point $f(p)$.*

Thus, if a smooth dendroid is monotone homogeneous, then it is smooth at each of its points, and therefore it is a dendrite ([5, Corollary 5, p. 299]). So, we have the following assertion.

36. Proposition. *If a smooth dendroid is monotone homogeneous, then it is a dendrite.*

37. Question. Is smoothness an essential assumption in Proposition 36?

Recall that a continuum X has the *property of Kelley* provided that for each point $x \in X$, for each sequence of points $x_n \in X$ converging to X and for each continuum K in X containing the point x there is in X a sequence of continua K_n with $x_n \in K_n$ converging to K. Since each dendroid having the property of Kelley is smooth ([8]), we get the following corollary to Proposition 36.

38. Corollary. *If a dendroid having the property of Kelley is monotone homogeneous, then it is a dendrite.*

It is known that if a continuum is openly homogeneous, then it has the property of Kelley ([1, Statement, p. 380]), while confluent homogeneity does not imply the property of Kelley even for curves (i.e. one-dimensional continua), as it has ben shown by H. Kato in [12, pp. 55–58]. However, Kato's example ([12, Figure 2, p. 57]) is very far from being either a dendroid or a planar curve. Thus we have the following two questions.

39. Questions. Does confluently homogeneity imply the property of Kelley for (a) dendroids, (b) planar curves?

We close the paper with a question that was a starting point of our study presented above.

40. Question. Does monotone homogeneity of continua imply the property of Kelley?

References

[1] Charatonik J.J., *The property of Kelley and confluent mappings*, Bull. Polish Acad. Sci. Math. **31** (1983), 375–380.

[2] Charatonik J.J., *Generalized homogeneity of curves and a question of H. Kato*, Bull. Polish Acad. Sci. Math. **36** (1988), 409–411.

[3] Charatonik J.J., *Monotone mappings of universal dendrites*, Topology Appl. **38** (1991), 163–187.

[4] Charatonik J.J., Charatonik W.J., Prajs J.R., *Mapping hierarchy for dendrites*, Dissertationes Math. (Rozprawy Mat.) **333** (1994), 1–52.

[5] Charatonik J.J., Eberhart C., *On smooth dendroids*, Fund. Math. **67** (1970), 297–322.

[6] Charatonik W.J., Dilks A., *On self-homeomorphic spaces*, Topology Appl. **55** (1994), 215–238.

[7] Cook H., Ingram W.T., Kuperberg K.T., Lelek A., Minc P. (editors), *Continua, with the Houston problem book*, M. Dekker, 1995.
[8] Czuba S.T., *On dendroids with Kelley's property*, Proc. Amer. Math. Soc. **102** (1988), 728–730.
[9] Eberhart C., Fugate J.B., Gordh G.R., Jr., *Branchpoint covering theorems for confluent and weakly confluent maps*, Proc. Amer. Math. Soc. **55** (1976), 409–415.
[10] Goodkyoontz J.T., Jr., *Some functions on hyperspaces of hereditarily unicoherent continua*, Fund. Math. **95** (1977), 1–10.
[11] Gordh G.R., Jr., Lum L., *Monotone retracts and some characterizations of dendrites*, Proc. Amer. Math. Soc. **59** (1976), 156–158.
[12] Kato H., *Generalized homogeneity of continua and a question of J.J. Charatonik*, Houston J. Math. **13** (1987), 51–63.
[13] Kato H., *On problems of H. Cook*, Topology Appl. **26** (1987), 219–228.
[14] Kuratowski K., *Topology, vol. II*, Academic Press and PWN, 1968.
[15] Lum L., *A characterization of local connectivity in dendroids*, in: Studies in Topology, Academic Press, 1975, pp. 331–338.
[16] Maćkowiak T., *Semi-confluent mappings and their invariants*, Fund. Math. **69** (1973), 251–264.
[17] Maćkowiak T., *Continuous mappings on continua*, Dissertationes Math. (Rozprawy Mat.) **158** (1979), 1–91.
[18] Mazurkiewicz S., *Sur les continus homogènes*, Fund. Math. **5** (1924), 137–146.
[19] Nikiel J., *A characterization of dendroids with uncountably many endpoints in the classical sense*, Houston J. Math. **9** (1983), 421–432.
[20] Nikiel J., *On Gehman dendroids*, Glasnik Mat. **20** (40) (1985), 203–214.
[21] Whyburn G.T., *Analytic topology*, Amer. Math. Soc., 1942.

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