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Special almost P-spaces

Alessandro Fedeli

Abstract. Motivated by some examples, we introduce the concept of special almost P-space and show, using the reflection principle, that for every space X of this kind the inequality "$|X| \leq \psi_c(X)^{t(X)}$" holds.

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An almost P-space is a space in which every non-empty G_δ-set has non-empty interior (see e.g. [7]). A subset S of a space X is called d-closed if disjoint closed subsets of S have disjoint closures in X. Obviously every closed subset of a space is d-closed. Moreover observe that:

(i) Every d-closed subset S of a normal space X is C*-embedded in X: let C_1, C_2 be two completely separated sets in S, then C_1 and C_2 have disjoint closures in S. Since S is d-closed in X it follows that $cl_X(C_1) \cap cl_X(C_2) = \emptyset$. By the normality of X it follows that C_1 and C_2 are completely separated in X. Therefore (by the Urysohn's extension theorem) S is C*-embedded in X.

(ii) A C*-embedded subset in a normal space may fail to be d-closed. In fact every d-closed subset of a normal space is normal.

(iii) Every normal C*-embedded subset S in a space X is d-closed: let C_1 and C_2 be two closed disjoint subsets of S, since S is normal it follows that C_1 and C_2 are completely separated in S. Take a continuous function $f : S \to I$ such that $f(C_1) \subset \{0\}$ and $f(C_2) \subset \{1\}$. S is C*-embedded in X so there is a continuous extension $F : X \to I$ of f, therefore C_1 and C_2 have disjoint closures in X and S is d-closed in X.

The purpose of this paper is to show that a good behaviour of the d-closed subsets of an almost P-space X allows us to obtain a bound on the cardinality of X in terms of $t(X)$ and $\psi_c(X)$, where $t(X)$ and $\psi_c(X)$ denote respectively the tightness and the closed pseudocharacter of X (we refer the reader to [3], [5], [6] for notations and terminology not explicitly given).

To this aim let us consider the following examples.

(1) The one-point compactification of an uncountable discrete space is an almost P-space in which every d-closed subset is compact.

(2) The one-point Lindelöfization of an uncountable discrete space is an almost P-space (even a P-space) in which every d-closed subset is Lindelöf.

(3) It is well known that under CH a subspace S of $\beta\omega$ is C*-embedded in $\beta\omega$ if and only if it is weakly Lindelöf (i.e. every open cover of S has a countable subfamily whose union is dense, see e.g. Theorem 1.5.3 in [8]). Therefore $\beta\omega\setminus\omega$ is an almost P-space such that, under CH, every d-closed subset is weakly Lindelöf.

These common aspects of the above examples led us to the following

Definition 1. A special almost P-space X is a Hausdorff almost P-space in which every d-closed subset S is a WL-set in X (i.e. for every open family $\mathcal{U}$ in X such that $S \subset \bigcup\mathcal{U}$ there exists a countable family $\mathcal{V} \subset \mathcal{U}$ such that $S \subset \overline{\bigcup\mathcal{V}}$).

Remark 2. Obviously every weakly Lindelöf subspace of a space X is a WL-set in X. The converse is not true. Let us consider the Katětov H-closed extension $k\omega$ of the discrete space ω, then $k\omega\setminus\omega$ is a discrete WL-set in $k\omega$ of cardinality $2^{\mathfrak{c}}$.

To show our result on the cardinality of special almost P-spaces we need to review some facts on "elementary submodels" (our approach is that of [10], see also [9], [4] and [1], [2]).

Proposition 3 (The reflection principle). *Let $\phi(x, v_0, \dots, v_n)$ be a formula of set-theory with free variables x and the v_i's. If A is any set, then there is a set $\mathcal{M} \supset A$ such that $|\mathcal{M}| \le |A| + \omega$ and, whenever there are $m_0, \dots, m_n \in \mathcal{M}$ such that there is some x such that $\phi(x, m_0, \dots, m_n)$, then there is some $x \in \mathcal{M}$ such that $\phi(x, m_0, \dots, m_n)$ (we say that $\mathcal{M}$ reflects the formula $\exists x\ \phi$). We can also find a single $\mathcal{M}$ which works for finitely many formulas simultaneously.*

Proposition 4. *Let κ, λ be infinite cardinal numbers. If A is a set such that $|A| \le \lambda^\kappa$ and ϕ is a formula of set-theory, then there is a set $\mathcal{M}$ such that $A \subset \mathcal{M}$, $|\mathcal{M}| \le \lambda^\kappa$, $\mathcal{M}$ reflects $\exists x\ \phi$ and moreover $\mathcal{M}$ is closed under κ-sequences (i.e. $[\mathcal{M}]^{\le\kappa} \subset \mathcal{M}$).*

Proposition 5. *Let κ be an infinite cardinal number. Then there are two formulas so that, if $\mathcal{M}$ satisfies Proposition 3 for these two formulas and a set A and if $\kappa \subset A$, $\kappa \in A$, $E \in \mathcal{M}$ and $|E| \le \kappa$, then $E \subset \mathcal{M}$.*

Theorem 6. *If X is a special almost P-space, then $|X| \le \psi_c(X)^{t(X)}$.*

Proof: Let $\kappa = t(X)$, $\lambda = \psi_c(X)$ and let τ be the topology on X. For every $x \in X$ let $\mathcal{B}_x$ be a collection of open neighbourhoods of x with $|\mathcal{B}_x| \le \lambda$ such that $\bigcap\{\overline{B} : B \in \mathcal{B}_x\} = \{x\}$, and let $f : X \to \mathcal{P}(\tau)$ be the map defined by $f(x) = \mathcal{B}_x$ for every $x \in X$.

Let $A = \lambda^\kappa \cup \{X, \tau, \lambda^\kappa, f\}$ and apply Propositions 3–5 to obtain a set $\mathcal{M}$ such that $\mathcal{M} \supset A$, $|\mathcal{M}| = \lambda^\kappa$ and which reflects enough formulas to carry out the argument at hand. More precisely we ask that $\mathcal{M}$ reflects enough formulas so that the following conditions are satisfied:

(i) $\mathcal{M}$ is closed under κ-sequences;

(ii) $\mathcal{B}_x \in \mathcal{M}$ for every $x \in X \cap \mathcal{M}$;

(iii) if $B \subset X$ and $B \in \mathcal{M}$, then $\overline{B} \in \mathcal{M}$;

(iv) if $\mathcal{A} \in \mathcal{M}$, then $\bigcup \mathcal{A} \in \mathcal{M}$;
(v) if B is a subset of X such that $X \cap \mathcal{M} \subset B$ and $B \in \mathcal{M}$, then $X = B$;
(vi) if $E \in \mathcal{M}$ and $|E| \leq \lambda^\kappa$, then $E \subset \mathcal{M}$;
(vii) if $A, B \in \mathcal{M}$, then $A \cap B \in \mathcal{M}$;
(viii) if A is non-empty and $A \in \mathcal{M}$, then $A \cap \mathcal{M} \neq \emptyset$.

For example if $\phi(x, v_0, v_1)$ is the formula $(x \in v_0 \wedge x \notin v_1)$ and $\mathcal{M}$ reflects the formula $\exists x\ \phi$, then (v) is satisfied. In fact let B be a subset of X such that $B \in \mathcal{M}$. Set $m_0 = X$ and $m_1 = B$, if $B \neq X$ (i.e. if there is some x such that $\phi(x, m_0, m_1)$), then there is some $x \in \mathcal{M}$ such that $\phi(x, m_0, m_1)$, i.e. $X \cap \mathcal{M} \not\subset B$.

Observe that by (ii) and (vi) $\mathcal{B}_y \subset \mathcal{M}$ for every $y \in X \cap \mathcal{M}$.

First we check that $X \cap \mathcal{M}$ is d-closed in X. Let F_1, F_2 be two closed disjoint subsets of $X \cap \mathcal{M}$, we claim that they have disjoint closures in X. Suppose there is a point $x \in \overline{F}_1 \cap \overline{F}_2$, since $t(X) = \kappa$ it follows that there are $A \in [F_1]^{\leq\kappa}$ and $B \in [F_2]^{\leq\kappa}$ such that $x \in \overline{A} \cap \overline{B}$. Since $A, B \in \mathcal{M}$ ($A, B \subset \mathcal{M}$ and $\mathcal{M}$ is closed under κ-sequences) it follows that $\overline{A} \cap \overline{B} \in \mathcal{M}$ (by (iii) and (vii)). Therefore by (viii) there is some $x \in \overline{A} \cap \overline{B} \cap \mathcal{M}$, so $x \in cl_{X\cap\mathcal{M}}(A) \cap cl_{X\cap\mathcal{M}}(B) \subset F_1 \cap F_2$, which is a contradiction.

Now let us show that $X \subset \mathcal{M}$ (and hence $|X| \leq \psi_c(X)^{t(X)}$). Suppose not and take a point $x \in X \setminus \mathcal{M}$. For every $y \in X \cap \mathcal{M}$ let $B_y \in \mathcal{B}_y$ such that $x \notin \overline{B}_y$ and observe that $B_y \in \mathcal{M}$. Since $X \cap \mathcal{M} \subset \bigcup\{B_y : y \in X \cap \mathcal{M}\}$ and $X \cap \mathcal{M}$ is a WL-set in X there is some $S \in [X \cap \mathcal{M}]^{\leq\omega}$ such that $X \cap \mathcal{M} \subset \overline{\bigcup_{y\in S} B_y}$. Now set $G = Int(\bigcap\{X \setminus \overline{B}_y : y \in S\})$, since $\bigcap\{X \setminus \overline{B}_y : y \in S\}$ is a non-empty G_δ-set and X is an almost P-space it follows that $G \neq \emptyset$. Moreover $G \cap \overline{\bigcup_{y\in S} B_y} = \emptyset$.

Since $\{B_y : y \in S\} \subset \mathcal{M}$ and $\mathcal{M}$ is closed under κ-sequences, it follows that $\{B_y : y \in S\} \in \mathcal{M}$. Hence, by (iv), $\bigcup\{B_y : y \in S\} \in \mathcal{M}$.

Now observe that, by (iii), $\overline{\bigcup_{y\in S} B_y} \in \mathcal{M}$ and therefore, by (v), $\overline{\bigcup_{y\in S} B_y} = X$, a contradiction. $\square$

Remark 7. Obviously every special almost P-space is weakly Lindelöf. Moreover it is well-known that $|X| \leq 2^{\chi(X)}$ for every weakly Lindelöf T_4-space X (see [5, Theorem 4.13]), so it is natural to compare this estimation with the one given in Theorem 6. To this end let us consider the one-point compactification X of the discrete space of cardinality $2^{\aleph_0}$. It is easily seen that $|X| = \psi_c(X)^{t(X)} < 2^{\chi(X)}$.

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Dipartimento di Matematica Pura ed Applicata, Università, 67100 L'Aquila, Italy

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