# On double-ruled hypersurfaces in $\boldsymbol{R}^4$

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*Abstract.* We classify locally the induced Riemannian metrics of all irreducible double-ruled hypersurfaces in $\boldsymbol{R}^4$.

*Keywords:* Riemannian manifold, ruled hypersurface
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## 1. Introduction

Ruled hypersurfaces in higher dimensions have been studied by many authors. From the recent time, let us mention [1], [2] and [4].

In accordance with [4], by a *ruled hypersurface* $M \subset \boldsymbol{R}^{n+1}$ we understand a hypersurface admitting a codimension one foliation $\mathcal{F}$ whose leaves (called rulings) are open parts of $(n-1)$-dimensional affine subspaces of $\boldsymbol{R}^{n+1}$. For $n = 2$ the topic is classical and well-known. So, only in the higher dimensions, the local theory is still of some interest.

In [3, Theorem 10.19], the *intrinsic* classification of irreducible ruled hypersurfaces of $\boldsymbol{R}^4$ has been given. Here, four distinct types of possible induced metrics are explicitly described.

The purpose of this article is to classify intrinsically all *double-ruled hypersurfaces* in $\boldsymbol{R}^4$, i.e., those which admit exactly two distinct ruled foliations.

From the corresponding theory in [3] it follows that, if $M \subset \boldsymbol{R}^4$ is a double-ruled hypersurface, then the induced metric of $M$ must be locally of one of the following types (in convenient local coordinates $w, x, y$):

$$g = y^2[(\cos(2x + B(w)) + G(w))dw^2 + dx^2] + dy^2, \tag{1}$$

where $B(w)$ and $G(w)$ are arbitrary functions and $G(w) > 1$, or

$$g = y^2(dw^2 + dx^2) + dy^2. \tag{2}$$

As a part of Theorem 10.19 in [3], it was proved that the metric (2) is the only one which can be realized on a *minimal* double-ruled hypersurface (which is a cone over the Clifford torus). This metric can be also realized on a non-minimal double-ruled hypersurface.

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## 2. The main theorem

We are going to prove the following

**Theorem.** *A metric $g$ of the form (1) can be locally realized on a double-ruled hypersurface in $\boldsymbol{R}^4$ if and only if the functions $B(w)$ and $G(w)$ satisfy one of the following conditions:*

I. *$G(w)$ is constant and $B(w)$ is linear.*

II. *$G(w)$ is non-constant and $G(w), B(w)$ satisfy the system of ordinary differential equations*

$$G'' = 4(G^2-1) + \frac{5}{2}\,\frac{G(G')^2}{G^2-1} - \frac{(G^2-1)B'B''}{G'} + \frac{3}{2}G(B')^2, \tag{3}$$

$$\begin{aligned} B''' = &-\frac{(G^2-1)B'(B'')^2}{(G')^2} + \frac{3}{2}\frac{G(B')^2B''}{G'} + 4\frac{(G^2-1)B''}{G'} \\ &+ \frac{9}{2}\frac{GG'B''}{G^2-1} - \frac{5}{4}\frac{(G')^2B'}{G^2-1} - \frac{15}{4}\frac{(G')^2B'}{(G^2-1)^2}\,. \end{aligned} \tag{4}$$

We see that the system (3), (4) is written in the Cauchy normal form and hence its general solution depends on 5 arbitrary parameters. Yet, from the geometrical point of view, only four parameters are essential because $B(w)$ is of the form $B(w) = B_0(w) + \mathit{constant}$, and the last constant can be eliminated by the new choice of the $x$-coordinate in (1).

*Remark.* A particular family of solutions of (3) and (4) can be written in the "inverse explicit form". Namely, assuming $B(w) = \mathit{constant}$, the equation (4) is satisfied identically and the equation (3) is reduced to the form

$$G'' = 4(G^2-1) + \frac{5}{2}\,\frac{G(G')^2}{G^2-1}. \tag{5}$$

The standard procedure (in which we use the substitutions $G'(w) = p$, $G''(w) = p\;dp/dG$) gives

$$w = \int (G^2-1)^{-1}(k\sqrt{G^2-1} - 8G)^{-1/2}dG \tag{6}$$

where $k$ is an arbitrary constant.

## 3. The proof

We shall recall some basic facts and formulas from [3]. First, the metric (1) can be written in the form $g = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2$ through the orthonormal coframe $(\omega^1, \omega^2, \omega^3)$ given by

$$\begin{cases} \omega^1 &= t(w,x)\; y\; dw, \\ \omega^2 &= y\; dx, \\ \omega^3 &= dy, \end{cases} \tag{7}$$

where

$$t = \sqrt{\cos(2x + B(w)) + G(w)}, \; G(w) > 1. \tag{8}$$

Let $(E_1, E_2, E_3)$ denote the orthonormal frame which is dual to $(\omega^1, \omega^2, \omega^3)$. Suppose that the metric $g$ is realized on a ruled hypersurface $M \subset \boldsymbol{R}^4$, and let $\omega^1 = 0$, i.e., ($w = constant$) determine a ruled foliation. Then the corresponding shape operator $S$ of $M$ takes on the triangular form

$$\begin{cases} SE_1 &= aE_1 + bE_2, \\ SE_2 &= bE_1, \\ SE_3 &= 0, \end{cases} \tag{9}$$

where

$$a = \frac{\overline{a}(w,x)}{y}, \quad b = \frac{\overline{b}(w,x)}{y}. \tag{10}$$

Further, the functions $\overline{a}$, $\overline{b}$ must satisfy

$$\overline{b} = \pm\frac{\sqrt{G^2 - 1}}{t^2}, \tag{11}$$

$$(\overline{b})'_w = (t\bar{a})'_x. \tag{12}$$

(Cf. (10.16) and (10.15) in [3].)

Further, if a second ruled foliation on $M$ exists, then the corresponding tangent distribution is given by an equation $\cos\varphi \,.\, \omega^1 - \sin\varphi \,.\, \omega^2 = 0$, where $\varphi$ is a function of $w$ and $x$ only, $\sin\varphi \neq 0$. Moreover, we have

$$\overline{a} = -2\overline{b}\cot\varphi, \tag{13}$$

$$(t.\sin\varphi)'_x = (\cos\varphi)'_w \tag{14}$$

(see formulas (10.93) and (10.95) in [3]).

We put, for the abbreviation,

$$L = \cot\varphi. \tag{15}$$

Then (12) and (13) give

$$(\overline{b})'_w = -2(\overline{b}tL)'_x \tag{16}$$

and (14), (15) imply

$$\left(\frac{t}{\sqrt{L^2+1}}\right)'_x = \left(\frac{L}{\sqrt{L^2+1}}\right)'_w. \tag{17}$$

The last equation can be written in the form

$$t'_x(L^2+1) \ - \ tLL'_x \ = \ L'_w. \tag{18}$$

After the substitution from (11) into (16) we get easily

$$-tL'_x \ + \ Lt'_x + t^{-1}t'_w = A(w), \tag{19}$$

where

$$A(w) = \frac{GG'}{2(G^2-1)}. \tag{20}$$

Now, (19) and (18) can be re-written in the form

$$L'_x = t^{-1}(Lt'_x + u), \tag{21}$$

$$L'_w = t'_x - Lu, \tag{22}$$

where

$$u = t^{-1}t'_w - A(w). \tag{23}$$

From the theory explained in the Section 10.4 [3], it follows that *the second ruled foliation on $M$ exists if and only if the system* (21), (22) *of partial differential equations for $L$ has a solution.*

We check directly that the system (21), (22) has no solution if $B$ and $G$ in (8) are both constant. (This is in accordance with Proposition 10.18 from [3].)

Now, the integrability condition of (21), (22) can be written in the form

$$2(\log t)''_{wx}L = t(\log t)''_{xx} + t^{-1}(Au - u'_w). \tag{24}$$

Here we can assume that the coefficient $(\log t)''_{wx}$ is nonzero, because otherwise we get $B'(w) = G'(w) = 0$, a case which was just excluded. So, we can express $L$ from (24) and substitute in (21) and (22). In each case, one obtains a sum of fractions which can be taken to the (simplest) common denominator, and the corresponding equation will be satisfied if and only if the numerator is equal to zero. The last equation is always of the form

$$a_0 + \sum_{k=1}^{3} a_k t^{2k} + \sum_{k=1}^{3} b_k t^{2(k-1)} \sin(2x+B) = 0, \tag{25}$$

where $a_k$, $b_k$ are certain functions depending only on $B(w)$, $G(w)$ and their derivatives. Hence all coefficients $a_k$, $b_k$ must be put equal to zero.

Substituting from (24) in (21), we see that only the coefficient $a_2$ is nonzero and we obtain just one differential equation of the form

$$\begin{aligned} &-2(G^2-1)^2B'B''+3G(G^2-1)G'(B')^2 \\ -2(G^2-1)G'G''+5G(G')^3+8(G^2-1)^2G' \;&=\; 0. \end{aligned} \tag{26}$$

Now, if $G'=0$, we get $B'B''=0$ and thus $B(w)=pw+q$, where $p\neq 0$ and $q$ are constants. (Moreover, we can assume $q=0$.) Using the obvious identity $pt'_x=2t'_w$, we obtain easily the function $L$ from (24) in a simple explicit form, and we can check that the system (21), (22) is satisfied for this $L$.

Thus, we can assume $G'\neq 0$ in the sequel, and the equation (26) can be re-written in the form (3).

Substituting from (24) in (22), we obtain much more complicated situation, and just here the computer assistance with MAPLE software was used. One gets five nontrivial coefficients in (25) and hence five ordinary differential equations of order 3 involving $G$, $G'$, $G''$, $G'''$, $B'$, $B''$ and $B'''$. Substituting for $G''$ from (3) and for $G'''$ from the first derivative of (3), one can show that these five equations are reduced to only one, which splits in two factors: one factor gives $B'=0$ and the second factor gives the equation (4). But (4) contains $B'=0$ as a particular case.

This completes the proof of the main theorem.

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