Landesman Lazer type results for first order periodic problems

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Abstract. Existence of nonnegative solutions are established for the periodic problem $y' = f(t, y)$ a.e. on $[0, T]$, $y(0) = y(T)$. Here the nonlinearity f satisfies a Landesman Lazer type condition.

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1. Introduction

This paper discusses the nonlinear first order differential equation

$$\begin{cases} y' = f(t,y) \ \text{ a.e. on } \ [0,T], \\ y(0) = y(T), \end{cases} \tag{1.1}$$

where $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ is a L^1-Carathéodory function. By placing a Landesman Lazer type inequality on our nonlinearity f we will establish the existence of nonnegative solutions to (1.1). Of course analogue results could be obtained for nonpositive solutions. The periodic problem (1.1) has been studied by many authors; see [2]–[4, [6], [7] and their references. In [6] the method of upper and lower solutions to (1.1) was discussed. Landesman Lazer type results were obtained in [4], [7]; for example in [7] it is shown that if

$$\begin{cases} \liminf_{x\to\infty} f(t,x) \ge 0 \ \text{ a.e. with strict inequality on a subset} \\ \text{of } \ [0,T] \ \text{ of positive measure} \end{cases} \tag{$H1$}$$

and

$$\begin{cases} \text{there exist } \ \alpha \in L^2[0,T],\ \beta \in L^1[0,T],\ \alpha \ge 0 \ \text{ a.e. with } \ \alpha > 0 \\ \text{on a subset of } \ [0,T] \ \text{ of positive measure such that} \\ \beta(t) \le f(t,y) \le \alpha(t) y \ \text{ for a.e. } \ t \in [0,T] \ \text{ and all } \ y \ge 0 \end{cases} \tag{$H2$}$$

are satisfied then (1.1) has a nonnegative solution. In this paper by placing less restrictive conditions on the nonlinearity f we are able to obtain a new existence result for (1.1). The proof is based on a technique initiated by Mawhin and

Ward [5] in the early 1980's for resonant second order periodic problems. We first prove a result (Theorem 2.1) which can be established from previous results in the literature [4, Chapter 6, p. 71]. However here we provide a new and different proof (our proof also avoids the technicalities associated with guiding functions); the main reason for giving a new proof is that a major part of the reasoning used in the proof of Theorem 2.1 can be used to prove Theorem 2.2 (our new existence result). Our proof of Theorem 2.1 avoids the technicalities associated with guiding functions [4].

To conclude this introduction we state a well known existence principle [2]. First however recall a function $f : [0,T]\times \mathbf{R} \to \mathbf{R}$ is said to be a L^1-Carathéodory function if

(a) the map $u \to f(t,u)$ is continuous for almost all $t \in [0,T]$,

(b) the map $t \to f(t,u)$ is measurable for all $u \in \mathbf{R}$,

(c) for a given $r > 0$ there exists $h_r \in L^1[0,T]$ such that $|u| \le r$ implies $|f(t,u)| \le h_r(t)$ for almost all $t \in [0,T]$.

Theorem 1.1. *Let $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ be a L^1-Carathéodory function and let $q \in L^1[0,T]$ be such that $e^{\int_0^T q(s)\,ds} \neq 1$. Assume that there exists a constant M_0, independent of λ, with $|y|_0 = \sup_{[0,T]} |y(t)| \le M_0$, for any solution y to*

$$(1.1)_\lambda \qquad \begin{cases} y' - q(t)y = \lambda[f(t,y) - q(t)y] \quad \text{a.e. on} \quad [0,T], \\ y(0) = y(T) \end{cases}$$

for $0 < \lambda < 1$. Then (1.1) has a solution $y \in C[0,T]$.

Remark. By a solution to $(1.1)_\lambda$ we mean a function $y \in C[0,T] \cap AC[0,T]$ which satisfies the differential equation in $(1.1)_\lambda$ a.e. on $[0,T]$ and $y(0) = y(T)$.

2. Existence theory

Various existence results are established for the periodic problem (1.1) in this section. We restrict our discussion to the existence of nonnegative solutions.

Theorem 2.1. *Assume $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ is a L^1-Carathéodory function. In addition suppose the following conditions are satisfied:*

$$(2.1) \qquad f(t,0) \le 0 \quad \text{for a.e.} \quad t \in [0,T],$$

$$(2.2) \qquad f(t,y) = g(t,y)y + h(t,y) + r(t) \quad \text{for a.e.} \quad t \in [0,T] \quad \text{and all} \quad y \ge 0,$$

$$(2.3) \qquad \begin{gathered} |h(t,y)| \le \phi_1(t)y^\alpha + \phi_2(t) \quad \text{for a.e.} \quad t \in [0,T] \quad \text{and} \quad y \ge 0\,; \\ \text{here} \quad 0 \le \alpha < 1, \end{gathered}$$

$$\phi_1,\ \phi_2,\ r \in L^2[0,T], \tag{2.4}$$

$$\begin{cases} \textit{there exist } \beta,\, \tau \in L^2[0,T] \ \textit{ with } \ \beta(t) \le g(t,y)y \le \tau(t)y \ \textit{ for} \\ \textit{a.e. } \ t \in [0,T] \ \textit{and all } y \ge 0\,; \ \ \textit{here } \tau \ge 0 \ \textit{ a.e. on } [0,T] \ \textit{ and } \tau > 0 \\ \textit{on a subset of } [0,T] \ \textit{of positive measure,} \end{cases} \tag{2.5}$$

(2.6) *there exists* $\rho \in L^1[0,T]$ *with* $h(t,y) \ge \rho(t)$
for a.e. $t \in [0,T]$ *and* $y \ge 0$

and

$$\int_0^T [-r(t)]\,dt < \int_0^T \liminf_{x\to\infty}[g(t,x)x]\,dt + \int_0^T \liminf_{x\to\infty}[h(t,x)]\,dt. \tag{2.7}$$

Then (1.1) *has a nonnegative solution in* $C[0,T] \cap AC[0,T]$.

Proof: Consider the family of problems

$$(2.8)_\lambda \qquad \begin{cases} y' - \tau\, y = \lambda[f^\star(t,y) - \tau\, y] \ \text{ a.e. on } \ [0,T], \\ y(0) = y(T), \end{cases}$$

where $0 < \lambda < 1$, τ is as in (2.5) and

$$f^\star(t,y) = \begin{cases} f(t,0) + y, \ y < 0 \\ f(t,y), \ y \ge 0. \end{cases}$$

We will show that any solution y of $(2.8)_\lambda$ satisfies

$$y(t) \ge 0 \ \text{ for } \ t \in [0,T]. \tag{2.9}$$

Let y be a solution of $(2.8)_\lambda$. Suppose y has a negative global minimum at $t_0 \in [0,T]$. Because of the periodicity we may assume $t_0 \in [0,T)$. Now there exists $t_1 > t_0$ with $y(t) < 0$ on $[t_0,t_1]$ and $y(t) \ge y(t_0)$ for $t \in [t_0,t_1]$. Then

$$\begin{aligned} 0 \le y(t_1) - y(t_0) &= \int_{t_0}^{t_1} [\lambda f^\star(t,y) + (1-\lambda)\tau\, y]\,dt \\ &= \int_{t_0}^{t_1} [\lambda f(t,0) + (1-\lambda)\tau\, y + \lambda\, y]\,dt < 0, \end{aligned}$$

a contradiction. Thus (2.9) is true.

Remark. The above argument also shows that any solution to $(2.8)_1$ is nonnegative.

Next we claim that there exists a constant M_0 with

$$|y|_0 = \sup_{[0,T]} y(t) \leq M_0 \tag{2.10}$$

for any solution y to $(2.8)_\lambda$. If this is not true then there exists a sequence (λ_n) in $(0,1)$ and a sequence (y_n) (here $y_n \in C[0,T] \cap AC[0,T]$ and $y_n(0) = y_n(T)$) with

$$y_n' - \tau\, y_n = \lambda_n [f(t,y_n) - \tau\, y_n] \quad \text{a.e. on} \quad [0,T] \tag{2.11}$$

and

$$|y_n|_0 \to \infty. \tag{2.12}$$

From (2.11) we have

$$y_n' - (1-\lambda_n)\tau\, y_n = \lambda_n [g(t,y_n)y_n + h(t,y_n) + r(t)] \quad \text{a.e. on} \quad [0,T].$$

Integrate from 0 to T to obtain

$$\int_0^T [g(t,y_n)y_n + h(t,y_n) + r(t)]\,dt = -\,\frac{(1-\lambda_n)}{\lambda_n} \int_0^T \tau(t)\, y_n\,dt \leq 0$$

and so

$$\int_0^T [-r(t)]\,dt \geq \int_0^T [g(t,y_n)y_n + h(t,y_n)]\,dt.$$

This together with the fact that $\liminf(s_n + t_n) \geq \liminf(s_n) + \liminf(t_n)$ for sequences s_n and t_n yields

$$\int_0^T [-r(t)]\,dt \geq \liminf_{n\to\infty} \int_0^T g(t,y_n)y_n\,dt + \liminf_{n\to\infty} \int_0^T h(t,y_n)\,dt. \tag{2.13}$$

Notice $g(t,y_n)y_n \geq \beta(t)$ a.e. and $h(t,y_n) \geq \rho(t)$ a.e. so we may apply Fatou's lemma to obtain

$$\int_0^T [-r(t)]\,dt \geq \int_0^T \liminf_{n\to\infty} [g(t,y_n)y_n]\,dt + \int_0^T \liminf_{n\to\infty} [h(t,y_n)]\,dt. \tag{2.14}$$

Let

$$v_n = \frac{y_n}{|y_n|_0}\,.$$

Notice $|v_n|_0 = 1$ and $v_n(0) = v_n(T)$. From $(2.8)_{\lambda_n}$ we have

$$v_n' = [(1-\lambda_n)\tau\, v_n + \lambda_n g(t,y_n)v_n] + \frac{\lambda_n [h(t,y_n) + r(t)]}{|y_n|_0} \quad \text{a.e. on} \quad [0,T]$$

and so

$$\begin{aligned}\|v_n'\|_{L^2}^2 \le 2\int_0^T [(1-\lambda_n)\tau\, v_n + \lambda_n g(t,y_n)v_n]^2\, dt \\ + \frac{2}{|y_n|_0^2}\int_0^T |h(t,y_n)+r(t)|^2\, dt.\end{aligned} \tag{2.15}$$

Notice

$$\begin{aligned}\int_0^T |h(t,y_n)+r(t)|^2\, dt &\le 2\int_0^T |h(t,y_n)|^2\, dt + 2\int_0^T |r(t)|^2\, dt \\ &\le 4\int_0^T \phi_1^2 |y_n|^{2\alpha}\, dt + 4\int_0^T \phi_2^2\, dt + 2\int_0^T r^2\, dt \\ &\le 4|y_n|_0^{2\alpha}\int_0^T \phi_1^2\, dt + 4\int_0^T \phi_2^2\, dt + 2\int_0^T r^2\, dt\end{aligned}$$

and this together with (2.12) and (2.15) implies that there exists an integer n_0 with

$$\|v_n'\|_{L^2}^2 \le 2\int_0^T [(1-\lambda_n)\tau\, v_n + \lambda_n g(t,y_n)v_n]^2\, dt + 1 \quad \text{for} \quad n \ge n_0. \tag{2.16}$$

Next notice since

$$(1-\lambda_n)\tau(t)\, v_n(t) + \lambda_n g(t,y_n(t))v_n(t) \ge \frac{\lambda_n \beta(t)}{|y_n|_0} \quad \text{a.e. on} \quad [0,T]$$

and

$$\begin{aligned}&(1-\lambda_n)\tau(t)\, v_n(t) + \lambda_n g(t,y_n(t))v_n(t) \\ &\quad = \frac{1}{|y_n|_0}\left(\tau(t)\, y_n(t) + \lambda_n[g(t,y_n(t))y_n(t) - \tau(t)\, y_n]\right) \\ &\quad \le \tau(t)\, v_n(t) \quad \text{a.e. on} \quad [0,T]\end{aligned}$$

that

$$\begin{aligned}&|(1-\lambda_n)\tau(t)\, v_n(t) + \lambda_n g(t,y_n(t))v_n(t)| \\ &\qquad \le \max\left\{\tau(t)\, v_n(t),\ \frac{|\beta(t)|}{|y_n|_0}\right\} \quad \text{a.e. on} \quad [0,T].\end{aligned} \tag{2.17}$$

Thus there exists an integer n_1 with for $n \ge n_1$,

$$|(1-\lambda_n)\tau(t)\, v_n(t) + \lambda_n g(t,y_n(t))v_n(t)| \le \max\left\{\tau(t)\, v_n(t),\ |\beta(t)|\right\} \quad \text{a.e. on} \quad [0,T].$$

This together with (2.16) implies for $n \geq \max\{n_0, n_1\} \equiv n_2$ that

$$\|v_n'\|_{L^2}^2 \leq 2 \int_0^T [\max\{\tau(t)\, v_n(t),\, |\beta(t)|\}]^2\, dt + 1.$$

Since $|v_n|_0 = 1$ there exists a constant M_1 with

$$\|v_n'\|_{L^2} \leq M_1 \quad \text{for} \quad n \geq n_2. \tag{2.18}$$

Summarizing we have for $n \geq n_2$ that

$$|v_n|_0 = 1 \quad \text{and} \quad \|v_n'\|_{L^2} \leq M_1. \tag{2.19}$$

The Arzela-Ascoli theorem (notice the uniform bound on $\|v_n'\|_{L^2}$ implies the equicontinuity of $\{v_n\}$ since if $x, t \in [0,T]$ we have $|v_n(t) - v_n(x)| \leq \|v_n'\|_{L^2}\, |t - x|^{\frac{1}{2}}$) with a standard result in functional analysis (if E is a reflexive Banach space then any norm bounded sequence in E has a weakly convergent subsequence) implies that there is a subsequence S of $\{n_2, n_2+1, \dots\}$ with

$$v_n \to v \quad \text{in} \quad C[0,T] \quad \text{and} \quad v_n' \rightharpoonup v' \quad \text{in} \quad L^2[0,T] \quad \text{and} \quad \lambda_n \to \lambda \quad \text{as} \quad n \to \infty \quad \text{in} \quad S\,; \tag{2.20}$$

here $\rightharpoonup$ denotes weak convergence.

Remark. Notice $v \geq 0$ on $[0,T]$ since $v_n \geq 0$ on $[0,T]$ for all n.

Let us return to the differential equation

$$\begin{cases} v_n' = [(1-\lambda_n)\tau\, v_n + \lambda_n g(t, y_n) v_n] + \frac{\lambda_n [h(t,y_n) + r(t)]}{|y_n|_0} & \text{a.e. on} \quad [0,T] \\ v_n(0) = v_n(T) \end{cases}$$

for $n \in S$. For $n \in S$ and $\psi \in L^2[0,T]$ we have

$$\int_0^T v_n' \psi\, dt = \int_0^T [(1-\lambda_n)\tau\, v_n + \lambda_n g(t,y_n) v_n]\psi\, dt + \lambda_n \int_0^T \frac{[h(t,y_n) + r(t)]}{|y_n|_0}\, \psi\, dt. \tag{2.21}$$

Notice since

$$\frac{|[h(t,y_n(t)) + r(t)]\psi(t)|}{|y_n|_0} \leq \frac{\phi_1(t)|\psi(t)|\, y_n^\alpha(t) + [\phi_2(t) + |r(t)|]|\psi(t)|}{|y_n|_0} \quad \text{a.e.}$$

and $|y_n|_0 \to \infty$ as $n \to \infty$ we have

$$\lim_{n\to\infty} \lambda_n \int_0^T \frac{[h(t,y_n(t)) + r(t)]}{|y_n|_0}\, \psi(t)\, dt = 0\,; \quad \text{here} \quad n \to \infty \quad \text{in} \quad S. \tag{2.22}$$

Also (2.20) yields

$$\lim_{n\to\infty}\int_0^T v_n'\psi\,dt=\int_0^T v'\psi\,dt\,;\quad\text{here}\quad n\to\infty\quad\text{in}\quad S. \tag{2.23}$$

Now assumption (2.5) implies (as before)

$$\frac{\beta(t)}{|y_n|_0}\le[\lambda_n g(t,y_n)+(1-\lambda_n)\tau]\,v_n\equiv\mu_n(t)\le\tau(t)v_n(t)\quad\text{a.e. on}\quad[0,T]. \tag{2.24}$$

Thus, since $v_n\to v$ in $C[0,T]$ as $n\to\infty$ in S and $|y_n|_0\to\infty$, there exists an integer n_3 with

$$|\mu_n(t)|\le\max\{\tau(t)[v(t)+1],\,|\beta(t)|\}\quad\text{for}\quad n\ge n_3\quad\text{and}\quad n\in S. \tag{2.25}$$

Consequently there exists a constant M_2 with

$$\|\mu_n\|_{L^2}\le M_2\quad\text{for}\quad n\ge n_3\quad\text{and}\quad n\in S. \tag{2.26}$$

Let S_1 denote those $n\in S$ with $n\ge n_3$. Notice (2.26) implies that μ_n has a weakly convergent subsequence in $L^2[0,T]$ i.e. there exists a subsequence S_2 of S_1 with

$$\mu_n\rightharpoonup\mu\quad\text{in}\quad L^2[0,T]\quad\text{as}\quad n\to\infty\quad\text{in}\quad S_2\,; \tag{2.27}$$

here μ is the weak limit (as $n\to\infty$ in S_2) in $L^2[0,T]$ of μ_n. Now let $n\to\infty$ in S_2 in (2.21), using (2.22), (2.23) and (2.27), to obtain

$$\int_0^T v'\psi\,dt=\int_0^T \mu\psi\,dt. \tag{2.28}$$

Also

$$v(0)=v(T). \tag{2.29}$$

Next we claim that

$$\mu(t)\ge 0\quad\text{for a.e.}\quad t\in[0,T]. \tag{2.30}$$

Let m be an integer. Fix m and let $\epsilon=\frac{1}{m}$. Then from (2.24) there exists $n_4\in S_2$ with

$$-\epsilon\le\mu_n(t)\le\tau(t)[v(t)+\epsilon]\quad\text{for}\quad n\ge n_4\quad\text{and}\quad n\in S_2. \tag{2.31}$$

Let

$$K=\Big\{u\in L^2[0,T]:\,-\epsilon\le u(t)\le\tau(t)[v(t)+\epsilon]\quad\text{for a.e.}\quad t\in[0,T]\Big\}.$$

Notice K is convex and strongly closed. Hence K is weakly closed [9]. Now since μ is the weak limit (as $n \to \infty$ in S_2) in $L^2[0,T]$ of μ_n and $\mu_n \in K$ for $n \geq n_4$, $n \in S_2$ then $\mu \in K$. Hence

$$-\epsilon \leq \mu(t) \leq \tau(t)[v(t)+\epsilon] \quad \text{for a.e.} \quad t \in [0,T]. \tag{2.32}$$

We can do this for each $\epsilon = \frac{1}{m}$, $m \in \{1, 2 \dots\}$. Thus

$$0 \leq \mu(t) \leq \tau(t)\, v(t) \quad \text{for a.e.} \quad t \in [0,T] \tag{2.33}$$

and so (2.30) is true.

Now (2.30) together with (2.28) implies that v is nondecreasing on $[0,T]$. Consequently, since $v(0) = v(T)$,

$$v \equiv c \geq 0, \;\; c \;\; \text{a constant.} \tag{2.34}$$

Now if $c = 0$ we have a contradiction since $|v|_0 = 1$. Thus

$$v \equiv c > 0. \tag{2.35}$$

Thus there exists $n_5 \in S$ with

$$\frac{y_n(t)}{|y_n|_0} = v_n(t) \geq \frac{c}{2} \quad \text{for each} \quad t \in [0,T] \quad \text{and} \quad n \geq n_5,\, n \in S.$$

Hence

$$y_n(t) \to \infty \quad \text{for each} \quad t \in [0,T] \quad \text{as} \quad n \to \infty \quad \text{through} \quad S. \tag{2.36}$$

Now (2.36) together with (2.14) implies

$$\int_0^T [-r(t)]\, dt \geq \int_0^T \liminf_{x\to\infty} [g(t,x)x]\, dt + \int_0^T \liminf_{x\to\infty} [h(t,x)]\, dt.$$

This contradicts (2.7) and so (2.10) is true. Existence of a solution to (1.1) is now guaranteed from Theorem 1.1. □

Remarks. (i) If β, ϕ_1, ϕ_2, r, $\tau \in L^p[0,T]$, $1 < p < 2$ then the result of Theorem 2.1 is again true; in this case use $\|v_n'\|_{L^p}$ instead of $\|v_n'\|_{L^2}$ in the proof.

(ii) We now state a more general version of Theorem 2.1. Suppose (2.1)–(2.5) hold and also that there exists $\theta \in (-\infty, 1)$ with the following conditions satisfied:

(2.6)* there exists $\rho \in L^1[0,T]$ with $f(t,y) \geq \rho(t)\, y^\theta$
for a.e. $t \in [0,T]$ and $y \geq 0$

and

$$(2.7)^\star \qquad 0 < \int_0^T \liminf_{x\to\infty} [f(t,x)\, x^{-\theta}]\, dt.$$

Then (1.1) has a solution.

The proof follows the reasoning in Theorem 2.1. The only change occurs from equation (2.12) to (2.14). In this case multiply (2.11) by $y_n^{-\theta}$ and integrate from 0 to T to obtain

$$\int_0^T [f(t,y_n)\, y_n^{-\theta}]\, dt = -\,\frac{(1-\lambda_n)}{\lambda_n} \int_0^T \tau(t)\, y_n^{1-\theta}\, dt \le 0$$

and so

$$0 \ge \liminf_{n\to\infty} \int_0^T [f(t,y_n)\, y_n^{-\theta}]\, dt.$$

We may apply Fatou's lemma (because of $(2.6)^\star$) to obtain

$$0 \ge \int_0^T \liminf_{n\to\infty} [f(t,y_n)\, y_n^{-\theta}]\, dt.$$

Essentially the same reasoning as in Theorem 2.1 establishes the result.

It is of interest to establish another type of result when (2.6) may not be true. Our next theorem gives such a result.

Theorem 2.2. *Assume* $f : [0,T]\times\mathbf{R}\to\mathbf{R}$ *is a* L^1*-Carathéodory function and suppose* (2.1)–(2.5) *are satisfied. In addition suppose*

$$(2.37) \qquad \begin{cases} \text{there exists a constant } \ M>0 \ \text{ such that} \\ \int_0^T [g(t,y(t))y(t) + h(t,y(t)) + r(t)]\, dt \ge 0 \\ \text{for all } \ y\in C[0,T]\cap AC[0,T] \ \text{ with} \\ y(0)=y(T) \ \text{ and } \ \min_{[0,T]} y(t) \ge M \end{cases}$$

is satisfied. Then (1.1) *has a nonnegative solution.*

Proof: Let y be a solution to $(2.8)_\lambda$. Assume (2.10) does not hold. Then there is a sequence (λ_n) in $(0,1)$ and a sequence (y_n) such that (2.11) and (2.12) hold. As in Theorem 2.1 we have

$$(2.38) \qquad \int_0^T [g(t,y_n)y_n + h(t,y_n) + r(t)]\, dt = -\,\frac{(1-\lambda_n)}{\lambda_n} \int_0^T \tau(t)\, y_n\, dt.$$

Also we know (Theorem 2.1) that there exists a subsequence S of integers with

$$v_n \to v \ \text{ in } \ C[0,T] \ \text{ as } \ n\to\infty \ \text{ in } \ S \ \text{ and } \ v\equiv c>0\,;$$

here $v_n = \frac{y_n}{|y_n|_0}$. Thus there exists $n_6 \in S$ with

$$(2.39)\quad v_n(t) \geq \frac{c}{2} \quad \text{i.e} \quad y_n(t) \geq \frac{c}{2}|y_n|_0 \quad \text{for each} \quad t \in [0,T] \quad \text{and} \quad n \geq n_6,\, n \in S.$$

Let S_3 denote those $n \in S$ with $n \geq n_6$. Since $|y_n|_0 \to \infty$ as $n \to \infty$ there exists a subsequence S_4 of S_3 with

$$(2.40)\qquad y_n(t) \geq M \quad \text{for each} \quad t \in [0,T] \quad \text{and} \quad n \in S_4\,;$$

here M is as in (2.37). Now (2.38) and (2.40) imply that

$$\int_0^T [g(t,y_n)y_n + h(t,y_n) + r(t)]\,dt < 0 \quad \text{for} \quad n \in S_4\,;$$

notice also $\min_{[0,T]} y_n(t) \geq M$. This contradicts (2.37). □

The above results have "dual versions". We will just give the dual version of Theorem 2.1.

Theorem 2.3. *Assume $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ is a L^1-Carathéodory function and suppose (2.1)–(2.4) hold. In addition assume the following conditions are satisfied:*

$$(2.41)\quad \begin{cases} \textit{there exist } \beta,\, \tau \in L^2[0,T] \textit{ with } -\tau(t)\,y \leq g(t,y)y \leq \beta(t) \textit{ for} \\ \textit{a.e. } t \in [0,T] \textit{ and all } y \geq 0\,; \textit{ here } \tau \geq 0 \textit{ a.e. on } [0,T] \textit{ and } \tau > 0 \\ \textit{on a subset of } [0,T] \textit{ of positive measure,} \end{cases}$$

(2.42)
there exists $\rho \in L^1[0,T]$ with $h(t,y) \leq \rho(t)$ for a.e. $t \in [0,T]$ and $y \geq 0$,

and

$$(2.43)\qquad \int_0^T [-r(t)]\,dt > \int_0^T \limsup_{x\to\infty}[g(t,x)x]\,dt + \int_0^T \limsup_{x\to\infty}[h(t,x)]\,dt.$$

Then (1.1) has a nonnegative solution.

Proof: Consider the family of problems

$$(2.44)_\lambda\qquad \begin{cases} y' + \tau\,y = \lambda[f^\star(t,y) + \tau\,y] \quad \text{a.e. on} \quad [0,T], \\ y(0) = y(T), \end{cases}$$

where $0 < \lambda < 1$ and $f^\star$ is as in Theorem 2.1. Let y be a solution to $(2.44)_\lambda$. We show

$$(2.45)\qquad y(t) \geq 0 \quad \text{for} \quad t \in [0,T].$$

Suppose y has a negative global minimum at $t_0 \in [0,T)$. Then there exists $t_1 > t_0$ with $y(t) < 0$ on $[t_0, t_1]$ and $y(t) \geq y(t_0)$ for $t \in [t_0, t_1]$. Now the differential equation yields

$$\frac{d}{dt}\left(e^{\int_0^t \tau(x)\,dx}\, y(t)\right) = e^{\int_0^t \tau(x)\,dx}\,[\lambda f^\star(t, y(t)) + \lambda\tau(t)\, y(t)] \quad \text{a.e. on} \quad [t_0, t_1].$$

Consequently

$$e^{\int_0^{t_1} \tau(x)\,dx} y(t_1) - e^{\int_0^{t_0} \tau(x)\,dx} y(t_0) \\ = \int_{t_0}^{t_1} e^{\int_0^s \tau(x)\,dx}\,[\lambda f^\star(s,0) + \lambda\tau(s)\, y(s) + \lambda y(s)]\, ds < 0$$

and so

$$y(t_1) < e^{-\int_{t_0}^{t_1} \tau(x)\,dx}\, y(t_0) \leq y(t_0),$$

a contradiction. Thus (2.45) is true.

Next we claim that there exists a constant M_0 with $|y|_0 \leq M_0$ for any solution y to $(2.44)_\lambda$. If not there exists a sequence (λ_n) in $(0,1)$ and a sequence (y_n) with

$$y_n' + \tau\, y_n = \lambda_n[f(t, y_n) + \tau\, y_n] \quad \text{a.e. on} \quad [0,T]$$

and

$$|y_n|_0 \to \infty.$$

Of course

$$\int_0^T [g(t,y_n)y_n + h(t,y_n) + r(t)]\, dt = \frac{(1-\lambda_n)}{\lambda_n} \int_0^T \tau(t)\, y_n\, dt \geq 0$$

and essentially the same reasoning as in Theorem 2.1 (except we use lim sup instead of lim inf) establishes the result. □

Examples are easy to construct so that Theorems 2.1–2.3 may be applied. For completeness we supply one such example.

Example. Suppose our nonlinearity $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ is given by

$$f(t,y) = t^2\, |y|^{\frac{1}{2}} - t^{-\gamma}, \quad 0 \leq \gamma < \frac{1}{2}.$$

Then (1.1) has a nonnegative solution.

To see this we apply Theorem 2.1. For a.e. $t \in [0,T]$ and $y \geq 0$ let $h(t,y) = t^2\, y^{\frac{1}{2}}$, $g(t,y) = 0$ and $r(t) = -t^{-\gamma}$. Notice (2.1), (2.2), (2.3) (with $\phi_1 = t^2$, $\phi_2 = t^{-\gamma}$ and $\alpha = \frac{1}{2}$), (2.4), (2.5), (2.6) and (2.7) are satisfied.

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