A generalization of the exterior product of differential forms combining Hom-valued forms

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Abstract. This article deals with vector valued differential forms on C^∞-manifolds. As a generalization of the exterior product, we introduce an operator that combines $\mathrm{Hom}(\bigotimes^s(W), Z)$-valued forms with $\mathrm{Hom}(\bigotimes^s(V), W)$-valued forms. We discuss the main properties of this operator such as (multi)linearity, associativity and its behavior under pullbacks, push-outs, exterior differentiation of forms, etc. Finally we present applications for Lie groups and fiber bundles.

Keywords: differential forms, exterior product, multilinear algebra

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1. Introduction

The $C^\infty(M)$-module of differential forms on a differentiable manifold M, which we will denote by $\mathcal{A}(M) = \bigoplus_{p=0}^\infty \mathcal{A}_p(M)$, is an associative exterior algebra with respect to the exterior or wedge product $\wedge$. The wedge product can also be extended to vector valued forms, if V, W, Z denote (finite or infinite dimensional) vector spaces and a bilinear mapping $m: V \times W \to Z$ is given, we may define a bilinear exterior product $\wedge_m: (\mathcal{A}(M) \otimes V) \times (\mathcal{A}(M) \otimes W) \to (\mathcal{A}(M) \otimes Z)$ by

$$(\alpha \otimes v) \wedge_m (\beta \otimes w) := (\alpha \wedge \beta) \otimes m(v, w) \quad \text{for all} \quad \alpha, \beta \in \mathcal{A}(M), \quad v \in V, w \in W.$$

If $V = \mathbb{R}$ or $W = \mathbb{R}$ and m is scalar multiplication, we simply use $\wedge$ instead of $\wedge_m$. Also if V is an algebra with multiplication $m: V \times V \to V$, one uses $\wedge_V$ rather than $\wedge_m$, e.g., for a Lie algebra $\mathfrak{g}$ the notation $\wedge_\mathfrak{g}$ implies $m(X, Z) := [X, Y]$. $\wedge_V$ turns the $C^\infty(M)$-module $\mathcal{A}(M) \otimes V$ into a (non-associative) algebra. Let d denote the exterior differentiation of forms, and for a vector field $\mathcal{X} \in \mathcal{D}^1(M)$, let $\imath_\mathcal{X}$ denote the interior product with respect to $\mathcal{X}$ and $L_\mathcal{X}$ denote the Lie differentiation with respect to $\mathcal{X}$, which is given by $L_\mathcal{X} = \imath_\mathcal{X} \circ d + d \circ \imath_\mathcal{X}$. Then for $\alpha_p \in \mathcal{A}_p(M) \otimes V$ and $\omega \in \mathcal{A}(M) \otimes W$,

$$\begin{aligned}
d(\alpha_p \wedge_m \omega) &= d\alpha_p \wedge_m \omega + (-1)^p \alpha_p \wedge_m d\omega, \\
\imath_\mathcal{X}(\alpha_p \wedge_m \omega) &= \imath_\mathcal{X}\alpha_p \wedge_m \omega + (-1)^p \alpha_p \wedge_m \imath_\mathcal{X}\omega, \\
L_\mathcal{X}(\alpha_p \wedge_m \omega) &= L_\mathcal{X}\alpha_p \wedge_m \omega + \alpha_p \wedge_m L_\mathcal{X}\omega.
\end{aligned}$$

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Thus with respect to $\wedge_V$, d and $\imath_{\mathcal{X}}$ are skew-derivations of degree 1, resp., -1 of $\mathcal{A}(M)\otimes V$ and $L_{\mathcal{X}}$ is a derivation of degree 0 of $\mathcal{A}(M)\otimes V$.

Further properties of $\wedge_m$, resp., $\wedge_V$ depend on m. If $m\colon V\times V\to V$ is associative, then $\wedge_V$ is so, too; if $\alpha_r\in\mathcal{A}_r(M)\otimes V$ and $\beta_s\in\mathcal{A}_s(M)\otimes V$, then $\alpha_r\wedge_V\beta_s=(-1)^{rs}\beta_s\wedge_V\alpha_r$ if m is commutative, resp., $\alpha_r\wedge_V\beta_s=(-1)^{rs+1}\beta_s\wedge_V\alpha_r$ if m is anti-commutative.

For some applications one needs generalizations of these wedge products, e.g., to combine a $\mathrm{Hom}(\bigotimes^s W,Z)$-valued r-form χ_r^s with s W-valued p-forms ϕ_p. We will examine the main properties of this Z-valued $(r+sp)$-form $\chi_r^s\bullet\phi_p$. In fact, for the sake of generality, we will consider the case where ϕ also is a Hom-valued form, say $\phi_p^q\in\mathcal{A}_p(M)\otimes\mathrm{Hom}(\bigotimes^q V,W)$. In that case the computations require some multilinear algebra and the derived expressions become quite voluminous. Nevertheless, in many applications, where one or more of the integers p,q,r,s are zero, we obtain more familiar results.

For notational convenience, we will recall the basic definitions from differential geometry on C^∞-manifolds and multilinear algebra according to Helgason [5] and Kobayashi, Numizu [6]. Then we introduce the operator $\bullet$, look for associativity (Section 3) and examine its behavior under pullbacks and push-outs (Section 4). In Section 5 we need to define further operators $\blacktriangleleft$ and $\blacktriangleright$ in order to compute $d(\chi_r^s\bullet\phi_p)$, $\imath_{\mathcal{X}}(\chi_r^s\bullet\phi_p)$ and $L_{\mathcal{X}}(\chi_r^s\bullet\phi_p)$. Finally the last two sections are devoted to applications for Lie groups and fiber bundles.

2. Basic definitions

For any (real) vector space V let $C^\infty(M,V)$ denote the $C^\infty(M)$-module of all weakly differentiable maps from M to V, i.e., all maps $f\colon M\to V$ with $\omega\circ f\in C^\infty(M)$ for every linear functional $\omega\colon V\to\mathbb{R}$. The $C^\infty(M)$-module of all vector fields on M will be denoted by $\mathcal{D}^1(M)$. Every vector field $\mathcal{X}\in\mathcal{D}^1(M)$ differentiably associates with every $x\in M$ an element $\mathcal{X}_x$ in the tangent space $T_x(M)$. Next $\mathcal{D}_p(M,V)$ and $\mathcal{A}_p(M,V)$ denote the $C^\infty(M)$-modules of all $C^\infty(M)$-p-linear, resp., all alternating $C^\infty(M)$-p-linear maps $\alpha_p\colon\mathcal{D}^1(M)\times\cdots\times\mathcal{D}^1(M)\to C^\infty(M,V)$. They associate with every $x\in M$ an element $\phi_x=(\phi_p)_x$ in $\mathrm{Hom}(\bigotimes^p T_x(M),V)$, resp., in $\mathrm{Alt}_p(T_x(M),V)$, where $\mathrm{Alt}_p(W,V)$ means the vector space of all alternating p-linear maps from W^p to V. The alternations $A_p\colon\mathcal{D}_p(M,V)\to\mathcal{D}_p(M,V)$ are the canonical projections of $\mathcal{D}_p(M,V)$ onto $\mathcal{A}_p(M,V)$. We put $\mathcal{D}_*(M,V):=\bigoplus_{p=0}^\infty\mathcal{D}_p(M,V)$ and $\mathcal{A}(M,V):=\bigoplus_{p=0}^\infty\mathcal{A}_p(M,V)$.

The canonical embedding $\imath\colon C^\infty(M)\otimes V\to C^\infty(M,V)$, defined by $[\imath(f\otimes v)](x):=f(x)v\in V$ for all $f\in C^\infty(M)$, $x\in M$ and $v\in V$, is injective and induces canonical embeddings of $\mathcal{D}_*(M)\otimes V$ into $\mathcal{D}_*(M,V)$, resp., of $\mathcal{A}(M)\otimes V$ into $\mathcal{A}(M,V)$.

If $V\cong\mathbb{R}^n$ with its natural differential structure, then $C^\infty(M,V)$, resp., $\mathcal{A}(M,V)$ exactly contain the differentiable maps from M to V, resp., differential forms on M with values in V and the embeddings are bijective. This enables us to identify $\mathcal{A}(M) \otimes V$ with $\mathcal{A}(M, V)$, etc. Of course, we also identify $\mathcal{A}(M, \mathbb{R})$ and $\mathcal{A}(M)$, etc. Omitting $\imath$ we write for $\mathcal{X}^{(i)} \in \mathcal{D}^1(M)$, $f \in C^\infty(M)$, $\omega \in \mathcal{A}_p(M)$, $x \in M$ and $v \in V$:

$$\mathcal{X}(f \otimes v) := \mathcal{X} f \otimes v, \qquad (f \otimes v)(x) := f(x)\, v \in V,$$

$$d(\omega \otimes v) := d\omega \otimes v, \quad (\omega \otimes v)(\mathcal{X}^1, \dots, \mathcal{X}^p) := \omega(\mathcal{X}^1, \dots, \mathcal{X}^p) \otimes v,$$

$$(\omega \otimes v)_x(\mathcal{X}^1_x, \dots, \mathcal{X}^p_x) := (\omega \otimes v)(\mathcal{X}^1, \dots, \mathcal{X}^p)(x) = \omega_x(\mathcal{X}^1_x, \dots, \mathcal{X}^p_x) \otimes v \in V.$$

Analogously to $\mathrm{Alt}_p(W, V)$, the vector space of all symmetric p-linear maps from W^p to V will be denoted by $\mathrm{Sym}_p(W, V)$. For convenience we define $\mathrm{Sym}^\pm(W, V) := \bigoplus_{p=0}^\infty \mathrm{Sym}^\pm_p(W, V)$ by $\mathrm{Sym}^+_p(W, V) := \mathrm{Sym}_p(W, V)$ and $\mathrm{Sym}^-_p(W, V) := \mathrm{Alt}_p(W, V)$.

If $f: M \to N$ is differentiable, we denote the differential of f at $x \in M$ by df_x. We have $[df_x(\mathcal{X}_x)]g = \mathcal{X}_x(g \circ f)$ for all $\mathcal{X}_x \in T_x(M)$, $g \in C^\infty(N)$.

For $\alpha \in \mathcal{D}_r(N, V), r \in \mathbb{N}$ and $X_i \in T_x(M)$, the pullback $f^\star\alpha \in \mathcal{D}_r(M, V)$ is defined by $(f^\star\alpha)_x(X_1, \dots, X_r) = \alpha_{f(x)}(df_x(X_1), \dots, df_x(X_r))$. For $\alpha \in C^\infty(N, V)$ we have $f^\star\alpha := \alpha \circ f$, linear extension defines the pullback on $\mathcal{D}_*(N, V)$. Obviously $f^\star(\mathcal{A}(N, V)) \subseteq \mathcal{A}(M, V)$ and — if we insert $\mathcal{D}_*(M) \otimes V$ into $\mathcal{D}_*(M, V)$ — $f^\star(\mathcal{D}_*(N) \otimes V) \subseteq \mathcal{D}_*(M) \otimes V$ and $f^\star(\mathcal{A}(N) \otimes V) \subseteq \mathcal{A}(M) \otimes V$.

If f is a diffeomorphism, then for $\mathcal{X} \in \mathcal{D}^1(M)$ the push-out $f_\star\mathcal{X} \in \mathcal{D}^1(N)$ is defined by $(f_\star\mathcal{X})_{f(x)} = df_x(\mathcal{X}_x)$ for all $x \in M$.

Let $\mathcal{T}(V)$ denote the tensor algebra of V. Then every linear map $\Lambda: V \to W$ defines a pullback $\Lambda^\star: \mathrm{Hom}(\mathcal{T}(W), Z) \to \mathrm{Hom}(\mathcal{T}(V), Z)$: for $K \in \mathrm{Hom}(\bigotimes^p W, Z)$ and $X_i \in V$ we have $\Lambda^\star K(X_1, \dots, X_p) := K(\Lambda(X_1), \dots, \Lambda(X_p))$, so $\Lambda^\star(\mathrm{Sym}^\pm(W, Z)) \subseteq \mathrm{Sym}^\pm(V, Z)$. $\Lambda_\circ: \mathrm{Hom}(\mathcal{T}(Z), V) \to \mathrm{Hom}(\mathcal{T}(Z), W)$ is defined by $\Lambda_\circ K = \Lambda \circ K$, thus also $\Lambda_\circ(\mathrm{Sym}^\pm(Z, V)) \subseteq \mathrm{Sym}^\pm(Z, W)$.

Finally Λ defines the push-out $\Lambda_\star: \mathcal{D}_*(M, V) \to \mathcal{D}_*(M, W)$ by $\Lambda_\star\omega = \Lambda \circ \omega$. Again $\Lambda_\star(\mathcal{A}(M, V)) \subseteq \mathcal{A}(M, W)$ and $\Lambda_\star(\mathcal{D}_*(M) \otimes V) \subseteq \mathcal{D}_*(M) \otimes W$, where we have $\Lambda_\star(\alpha \otimes v) = \alpha \otimes \Lambda(v)$ for all $\alpha \in \mathcal{D}_*(M)$, $v \in V$.

Pullbacks and push-outs obey $(f \circ g)_\star = f_\star \circ g_\star$, $(f \circ g)^\star = g^\star \circ f^\star$, which one may prove using the chain rule $d(f \circ g)_x = df_{g(x)} \circ dg_x$. We have:

Lemma 2.1. *If $f: M \to N$ is differentiable, $\Lambda: V \to W$ and $G: X \to Y$ linear, $\alpha, \beta \in \mathcal{A}(N) \otimes V$, $\gamma \in \mathcal{A}(N) \otimes W$, $\omega \in \mathcal{A}(N)$ and $K \in \mathrm{Hom}(\mathcal{T}(W), X)$, then*

1. *$f^\star$ and $\Lambda_\star$ commute: $f^\star(\Lambda_\star\alpha) = \Lambda_\star(f^\star\alpha)$, analogously $\Lambda^\star(G_\circ K) = G_\circ(\Lambda^\star K)$;*
2. *$f^\star$ and $\Lambda_\star$ commute with d: $d(f^\star\alpha) = f^\star(d\alpha)$, $d(\Lambda_\star\alpha) = \Lambda_\star(d\alpha)$;*
3. *$f^\star(\omega \wedge \alpha) = (f^\star\omega) \wedge (f^\star\alpha)$, $\Lambda_\star(\omega \wedge \alpha) = \omega \wedge (\Lambda_\star\alpha)$;*
4. *$f^\star(\alpha \wedge_m \gamma) = (f^\star\alpha) \wedge_m (f^\star\gamma)$, for any bilinear $m: V \times W \to Z$;*
5. *$\Lambda_\star(\alpha \wedge_V \beta) = (\Lambda_\star\alpha) \wedge_W (\Lambda_\star\beta)$, if in addition $\Lambda \circ \phi_V = \phi_W \circ (\Lambda \times \Lambda)$, thus $\Lambda_\star$ is an algebra homomorphism, if Λ is one.*

Definition 2.2. *For any $\chi_r^s \in \mathcal{A}_r(M, \mathrm{Hom}(\bigotimes^s W, Z))$, where $s \in \mathbb{N}$, $r \in \mathbb{N}_0$, and $F_j \in \mathrm{Hom}(\bigotimes^q V, W)$, $j = 1, \dots, s$, we define $\chi_r^{F_1,\dots,F_s} \in \mathcal{A}_r(M, \mathrm{Hom}(\bigotimes^{sq} V, Z))$ by*

$$\chi_r^{F_1,\dots,F_s} = [(F_1 \otimes \cdots \otimes F_s)^\star]_\star \chi_r^s.$$

Thus if $\chi_r^s \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\bigotimes^s W, Z)$, then $\chi_r^{F_1,\dots,F_s} \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\bigotimes^{sq} V, Z)$.

Since $(F_1 \otimes \cdots \otimes F_s) \in \mathrm{Hom}(\bigotimes^{sq} V, \bigotimes^q W)$, $\chi_r^{F_1,\dots,F_s}(\mathcal{X}^1, \dots, \mathcal{X}^r)$ is well defined. It is multilinear in F_j: for all $\lambda, \mu \in \mathbb{K}$ and all $j \le s$

$$\chi_r^{F_1,\dots,\lambda F_j + \mu F_j',\dots,F_s} = \lambda \chi_r^{F_1,\dots,F_j,\dots,F_s} + \mu \chi_r^{F_1,\dots,F_j',\dots,F_s}.$$

Note that if $q = 0$ and $E_j \in W$, then $(E_1 \otimes \cdots \otimes E_s)$ is just the canonical evaluation morphism and $\chi_r^{E_1,\dots,E_s} \in \mathcal{A}_r(M, V)$ is the push-out of χ_r^s under this morphism: $\chi_r^{E_1,\dots,E_s} := (E_1 \otimes \cdots \otimes E_s)_\star \chi_r^s$, i.e., for all $x \in M$ and $\mathcal{X}^i \in \mathcal{D}^1(M)$, $i = 1, \dots, r$,

$$(\chi_r^{E_1,\dots,E_s})_x(\mathcal{X}_x^1, \dots, \mathcal{X}_x^r) := (E_1 \otimes \cdots \otimes E_s) \circ (\chi_r^s)_x(\mathcal{X}_x^1, \dots, \mathcal{X}_x^r).$$

Now we are prepared for the definition of the operator $\bullet$.

Definition 2.3. *For $\chi_r^s \in \mathcal{A}_r(M, \mathrm{Hom}(\bigotimes^s W, Z))$ and $\phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W)$, $p, q, r, s-1 \in \mathbb{N}_0$, let $d_{r+sp}^{sq} \in \mathcal{D}_{r+sp}(M, \mathrm{Hom}(\bigotimes^{sq} V, Z))$ with*

$$d_{r+sp}^{sq}(\mathcal{X}^1, \dots, \mathcal{X}^{r+sp})(x) :=$$
$$[\chi_x(\mathcal{X}_x^1, \dots, \mathcal{X}_x^r)] \circ [\phi_x(\mathcal{X}_x^{r+1}, \dots, \mathcal{X}_x^{r+p}) \otimes \cdots \otimes \phi_x(\mathcal{X}_x^{r+(s-1)p+1}, \dots, \mathcal{X}_x^{r+sp})]$$

for all $x \in M$ and define $\chi_r^s \bullet \phi_p^q := A_{r+sp}(d_{r+sp}) \in \mathcal{A}_{r+sp}(M, \mathrm{Hom}(\bigotimes^{sq} V, Z))$. $\chi_r^0 \bullet \phi_p^q := \chi_r^0$ and linear extension defines $\chi \bullet \phi_p^q \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(V), Z))$ for all $\chi \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(W), Z))$.

In other words, the operator $\bullet$ means the following: for any $x \in M$ and $\mathcal{X}^i \in \mathcal{D}^1(M)$, $\chi_x(\mathcal{X}_x^1, \dots, \mathcal{X}_x^r)$ defines an element in $\mathrm{Hom}(\bigotimes^s W, Z)$. Instead of using s vectors in W as input for this map, we may also use s maps in $\mathrm{Hom}(\bigotimes^q V, W)$ as input to obtain an element in $\mathrm{Hom}(\bigotimes^{sq} V, Z)$. But again for any $x \in M$ and $\mathcal{Y}^i \in \mathcal{D}^1(M)$, $\phi_x(\mathcal{Y}_x^1, \dots, \mathcal{Y}_x^p)$ defines such a map in $\mathrm{Hom}(\bigotimes^q V, W)$. Altogether the combination of χ and s factors ϕ defines an element $d_{r+sp}^{sq} \in \mathcal{D}_{r+sp}(M, \mathrm{Hom}(\bigotimes^{sq} V, Z))$. Using the alternation A_{r+sp}, we finally obtain a form in $\mathcal{A}_{r+sp}(M, \mathrm{Hom}(\bigotimes^{sq} V, Z))$.

As was said before, $\bullet$ is a generalization of the wedge product. The following lemma, whose proof is straightforward, makes this more transparent.

Lemma 2.4. *For $p, q, r, s-1 \in \mathbb{N}_0$ and $\phi_p^q = \sum_{i=1}^{m} \phi^i \otimes F_i \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W)$,*

$$\chi_r^s \bullet \phi_p^q = \sum_{i_1,\dots,i_s=1}^{m} \chi_r^{F_{i_1},\dots,F_{i_s}} \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_s}.$$

Thus if $\chi_r^s \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\bigotimes^s W, Z)$, then also $\chi_r^s \bullet \phi_p^q \in \mathcal{A}_{r+sp}(M) \otimes \mathrm{Hom}(\bigotimes^{sq} V, Z)$.

Lemma 2.4 proves that if p is even and $q = 0$, then only the symmetric part of χ_r^s counts: $\chi_r^s \bullet \phi_p^0 = (\mathrm{Sym}_\star \chi_r^s) \bullet \phi_p^0$. On the other hand, if p is odd, Lemma 2.4 yields:

Lemma 2.5. *For $p, q, r, s-1 \in \mathbb{N}_0$, p odd, and $\phi_p^q = \sum_{i=1}^m \phi^i \otimes F_i$, we have*

$$\begin{aligned} \chi_r^s \bullet \phi_p^q &= \sum_{1\le i_1<\cdots<i_s\le m} \sum_{\rho\in S_s} \chi_r^{F_{i_{\rho(1)}},\dots,F_{i_{\rho(s)}}} \wedge \phi^{i_{\rho(1)}} \wedge \cdots \wedge \phi^{i_{\rho(s)}} \\ &= \sum_{1\le i_1<\cdots<i_s\le m} \Big(\sum_{\rho\in S_s} (-1)^\rho \, \chi_r^{F_{i_{\rho(1)}},\dots,F_{i_{\rho(s)}}} \Big) \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_s}. \end{aligned} \tag{1}$$

Thus $\chi_r^s \bullet \phi_p^q = 0$ if $s > m$; if V and W are finite dimensional and $s > \dim W (\dim V)^q$, then $\chi_r^s \bullet \phi_p^q = 0$ for all $\phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W)$.

Proof: $\phi^i \wedge \phi^i = 0$, because p odd, and $\dim \mathrm{Hom}(\bigotimes^q V, W) = \dim W (\dim V)^q$. □

Recall Sym^ς for $\varsigma = \pm$. If $\chi \in \mathcal{A}(M, \mathrm{Sym}^\varsigma(W, Z))$ (e.g., if $\chi = \chi_r^s$ with $s = 0, 1$), it is quite natural to ask for a resulting form $\chi \bullet \phi_p^q \in \mathcal{A}(M, \mathrm{Sym}^\varsigma(V, Z))$. We can achieve this by the push-out $(\mathrm{Sym}^\varsigma)_\star(\chi \bullet \phi_p^q)$. Define

$$\ell := \varsigma^{q+1}(-1)^p = \pm 1, \tag{2}$$

then the following lemma holds.

Lemma 2.6. *For $p, q, r, s-1 \in \mathbb{N}_0$, $\phi_p^q = \sum_{i=1}^m \phi^i \otimes F_i \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W)$ and $\chi_r^s \in \mathcal{A}_r(M, \mathrm{Sym}_s^\varsigma(W, Z))$, we have*

$$\begin{aligned} (\mathrm{Sym}_{sq}^\varsigma)_\star(\chi_r^s \bullet \phi_p^q) &= \sum_{i_1,\dots,i_s=1}^{m} (\mathrm{Sym}_{sq}^\varsigma)_\star(\chi_r^{F_{i_1},\dots,F_{i_s}}) \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_s}, \\ \text{if } (-1)^p = \varsigma^{q+1} = -1: \quad &= s! \sum_{1\le i_1<\cdots<i_s\le m} (\mathrm{Sym}_{sq}^\varsigma)_\star(\chi_r^{F_{i_1},\dots,F_{i_s}}) \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_s}, \\ \text{if } s>1 \text{ and } \ell = -1: \quad &= 0. \end{aligned}$$

Proof: The first equation is trivial from Lemmas 2.4 and 2.1.3. Now for $s > 1$, $\phi^{i_1} \wedge \cdots \wedge \phi^{i_j} \wedge \cdots \wedge \phi^{i_k} \wedge \cdots \wedge \phi^{i_s} = (-1)^p \phi^{i_1} \wedge \cdots \wedge \phi^{i_k} \wedge \cdots \wedge \phi^{i_j} \wedge \cdots \wedge \phi^{i_s}$ and

$$(3) \quad (\mathrm{Sym}^{\varsigma}_{sq})_\star(\chi_r^{F_{i_1},\dots,F_{i_j},\dots,F_{i_k},\dots,F_{i_s}}) = \varsigma^{q+1}(\mathrm{Sym}^{\varsigma}_{sq})_\star(\chi_r^{F_{i_1},\dots,F_{i_k},\dots,F_{i_j},\dots,F_{i_s}})$$

yield

$$(4) \quad (\mathrm{Sym}^{\varsigma}_{sq})_\star(\chi_r^{F_{i_1},\dots,F_{i_j},\dots,F_{i_k},\dots,F_{i_s}}) \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_j} \wedge \cdots \wedge \phi^{i_k} \wedge \cdots \wedge \phi^{i_s}$$
$$= \ell \ (\mathrm{Sym}^{\varsigma}_{sq})_\star(\chi_r^{F_{i_1},\dots,F_{i_k},\dots,F_{i_j},\dots,F_{i_s}}) \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_k} \wedge \cdots \wedge \phi^{i_j} \wedge \cdots \wedge \phi^{i_s}.$$

Thus evaluating $\sum_{\rho \in S_s}$ in (1) proves the rest. $\square$

3. Associativity

In general, $\bullet$ is not associative. Yet the terms $\kappa_t^u \bullet (\chi_r^s \bullet \phi_p^q)$ and $(\kappa_t^u \bullet \chi_r^s) \bullet \phi_p^q$ only differ (at most) by a sign, as the following proposition states.

Proposition 3.1. *Let $\kappa_t^u \in \mathcal{A}_t(M, \mathrm{Hom}(\bigotimes^u X, Y))$, $\chi_r^s \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\bigotimes^s W, X)$ and $\phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W)$ for $p, q, r, s, t, u \in \mathbb{N}_0$. Then*

$$(5) \quad \kappa_t^u \bullet (\chi_r^s \bullet \phi_p^q)$$
$$= (-1)^{prs\frac{u(u-1)}{2}} \ (\kappa_t^u \bullet \chi_r^s) \bullet \phi_p^q \ \in \mathcal{A}_{t+ur+usp}(M, \mathrm{Hom}(\textstyle\bigotimes^{usq} V, Y)).$$

Proof: Let $\chi_r^s = \sum_{j=1}^n \chi^j \otimes G_j$ and $\phi_p^q = \sum_{i=1}^m \phi^i \otimes F_i$. By Lemma 2.4 we find

$$\kappa_t^u \bullet (\chi_r^s \bullet \phi_p^q) = \sum_{j_1,\dots,j_u=1}^{n} \ \sum_{i_{11},\dots,i_{su}=1}^{m} \kappa_t^{G_{j_1}\circ(F_{i_{11}}\otimes\cdots\otimes F_{i_{s1}}),\dots,G_{j_u}\circ(F_{i_{1u}}\otimes\cdots\otimes F_{i_{su}})} \wedge$$
$$\wedge\, \chi^{j_1} \wedge \phi^{i_{11}} \wedge \cdots \wedge \phi^{i_{s1}} \wedge \cdots \wedge \chi^{j_u} \wedge \phi^{i_{1u}} \wedge \cdots \wedge \phi^{i_{su}},$$

while

$$(\kappa_t^u \bullet \chi_r^s) \bullet \phi_p^q = \sum_{j_1,\dots,j_u=1}^{n} \ \sum_{i_{11},\dots,i_{su}=1}^{m} (\kappa_t^{G_{j_1},\dots,G_{j_u}})_t^{F_{i_{11}},\dots,F_{i_{s1}},\dots,F_{i_{1u}},\dots,F_{i_{su}}} \wedge$$
$$\wedge\, \chi^{j_1} \wedge \cdots \wedge \chi^{j_u} \wedge \phi^{i_{11}} \wedge \cdots \wedge \phi^{i_{s1}} \wedge \cdots \wedge \phi^{i_{1u}} \wedge \cdots \wedge \phi^{i_{su}}.$$

Now

$$\chi^{j_1} \wedge \phi^{i_{11}} \wedge \cdots \wedge \phi^{i_{s1}} \wedge \cdots \wedge \chi^{j_u} \wedge \phi^{i_{1u}} \wedge \cdots \wedge \phi^{i_{su}}$$
$$= (-1)^{prs(1+2+\cdots+(u-1))} \chi^{j_1} \wedge \cdots \wedge \chi^{j_u} \wedge \phi^{i_{11}} \wedge \cdots \wedge \phi^{i_{s1}} \wedge \cdots \wedge \phi^{i_{1u}} \wedge \cdots \wedge \phi^{i_{su}}$$
$$= (-1)^{prs\frac{u(u-1)}{2}} \ \chi^{j_1} \wedge \cdots \wedge \chi^{j_u} \wedge \phi^{i_{11}} \wedge \cdots \wedge \phi^{i_{s1}} \wedge \cdots \wedge \phi^{i_{1u}} \wedge \cdots \wedge \phi^{i_{su}}.$$

On the other hand $(F_{i_{11}} \otimes \cdots \otimes F_{i_{s1}} \otimes \cdots \otimes F_{i_{1u}} \otimes \cdots \otimes F_{i_{su}})^\star \circ (G_{j_1} \otimes \cdots \otimes G_{i_u})^\star = [G_{j_1} \circ (F_{i_{11}} \otimes \cdots \otimes F_{i_{s1}}), \dots, G_{j_u} \circ (F_{i_{1u}} \otimes \cdots \otimes F_{i_{su}})]^\star$, so both κ-expressions are identical for each set of indices. $\square$

Corollary 3.2. *If $\kappa \in \mathcal{A}(M, \mathrm{Alt}(X,Y))$, then for p, q, r or s even:*

$$\kappa \bullet (\chi_r^s \bullet \phi_p^q) = (\kappa \bullet \chi_r^s) \bullet \phi_p^q. \tag{6}$$

Proof: Whenever for a κ_t^u in (5) $prs\frac{u(u-1)}{2}$ is odd, $r + sp$ and sq are even and $u > 1$, thus the left side of (5) vanishes by Lemma 2.6. □

In most applications, q will be zero and thus $\kappa \bullet (\chi_r^s \bullet \phi_p^0)$ and $(\kappa \bullet \chi_r^s) \bullet \phi_p^0$ are identical. Nevertheless note that this does not hold for expressions that involve three operators $\bullet$: according to Proposition 3.1, $[\kappa \bullet (\chi_r^s \bullet \phi_p^q)] \bullet \xi_t^0$ and $[(\kappa \bullet \chi_r^s) \bullet \phi_p^q] \bullet \xi_t^0$ will differ, in general.

4. Behavior under pullbacks and push-outs

Now we are looking for the behavior of $\bullet$ under the various pullbacks and push-outs we defined in Section 2. Due to the following lemma, we find that this behavior is canonical.

Lemma 4.1. *Let M, N be C^∞-manifolds and V, W, Y, Z vector spaces.*
1. *If $f\colon M \to N$ is differentiable and $\chi \in \mathcal{A}(N, \mathrm{Hom}(\mathcal{T}(W), Z))$, then*

$$(\forall\, \phi_p^q \in \mathcal{A}_p(N) \otimes \mathrm{Hom}(\textstyle\bigotimes^q V, W))\ f^\star(\chi \bullet \phi_p^q) = (f^\star\chi) \bullet (f^\star \phi_p^q) \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(V), Z));$$

2. *If $A\colon W \to Y$ is linear and $\chi \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(Y), Z))$, then*

$$(\forall\, \phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\textstyle\bigotimes^q V, W))\ \chi \bullet [(A_\circ)_\star \phi_p^q] = [(A^\star)_\star \chi] \bullet \phi_p^q \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(V), Z)),$$

$$(\forall\, \theta_p \in \mathcal{A}_p(M) \otimes W) \quad \chi \bullet (A_\star \theta_p) = [(A^\star)_\star \chi] \bullet \theta_p \in \mathcal{A}(M, Z);$$

3. *If $B\colon Y \to Z$ linear and $\chi \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(W), Y))$, then*

$$(\forall\, \phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\textstyle\bigotimes^q V, W))\ (B_\circ)_\star(\chi \bullet \phi_p^q) = [(B_\circ)_\star \chi] \bullet \phi_p^q \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(V), Z)),$$

$$(\forall\, \theta_p \in \mathcal{A}_p(M) \otimes W) \quad B_\star(\chi \bullet \theta_p) = [(B_\circ)_\star \chi] \bullet \theta_p \in \mathcal{A}(M, Z).$$

Analogous results hold for (anti)symmetrized forms in $\mathcal{A}(M, \mathrm{Sym}^\varsigma(W, Z))$, etc. If in 1. we have $\chi \in \mathcal{A}(N) \otimes \mathrm{Hom}(\mathcal{T}(W), Z)$, the result will be in $\mathcal{A}(M) \otimes \mathrm{Hom}(\mathcal{T}(V), Z)$, etc.

Proof: 1. follows from Lemmas 2.1 and 2.4; 2. and 3. are easily proved directly or by Proposition 3.1: let $a_0^1 := 1 \otimes A \in \mathcal{A}_0(M) \otimes \mathrm{Alt}_1(W, Y)$, then $[(A_\circ)_\star \phi_p^q] = a_0^1 \bullet \phi_p^q$ and $[(A^\star)_\star \chi] = \chi \bullet a_0^1$; analogously with $b_0^1 := 1 \otimes B \in \mathcal{A}_0(M) \otimes \mathrm{Alt}_1(Y, Z)$, $[(B_\circ)_\star \chi] = b_0^1 \bullet \chi$, which is well-defined in this special case. □

5. The operators ◀ and ▶

Let us check (multi)linearity now. Obviously $\chi \bullet \phi_p^q$ is $\mathcal{A}(M)$-linear only in χ. Moreover, if $\chi \in \mathcal{A}(M, \mathrm{Hom}(\bigotimes^s W, Z))$, then

$$\chi \bullet (f\ \phi_p^q) = f^s\ (\chi \bullet \phi_p^q) \quad \text{for all} \quad f \in C^\infty(M). \tag{7}$$

In addition, we would like to give an expression for $\chi \bullet (\phi_p^q + \psi_p^q)$. First we observe that every $\chi_r^s \in \mathcal{A}_r(M, \mathrm{Sym}_s^\varsigma(W, Z))$, $\varsigma = \pm$, naturally defines

$$\chi_r^{s';s''} \in \mathcal{A}_r(M, \mathrm{Sym}_{s'}^\varsigma(W, \mathrm{Sym}_{s''}^\varsigma(W, Z))) \quad \text{for all} \;\; s', s'' \in \mathbb{N}_0,\; s' + s'' = s. \tag{8}$$

For any such combination of s' and s'', $\chi_r^s \bullet (\phi_p^q + \psi_p^q)$ will contain terms, where s' factors of ϕ_p^q and s'' terms of ψ_p^q serve as input for χ_r^s. In order to cover this situation, we need the following two definitions.

Definition 5.1. *For $\chi_r^{s';s''} \in \mathcal{A}_r(M, \mathrm{Hom}(\bigotimes^{s'} W', \mathrm{Hom}(\bigotimes^{s''} W'', Z)))$, $s', s'' \in \mathbb{N}$, $r \in \mathbb{N}_0$, and any $G_i \in \mathrm{Hom}(\bigotimes^q V, W')$, $i = 1, \ldots, s'$, $H_j \in \mathrm{Hom}(\bigotimes^q V, W'')$, $j = 1, \ldots, s''$, we define:*

$$\begin{aligned} \chi_r^{G_1, \ldots, G_{s'}; s''} &:= [(G_1 \otimes \cdots \otimes G_{s'})^\star]_\star \chi_r^{s';s''} \\ &\quad \in \mathcal{A}_r(M, \mathrm{Hom}(\textstyle\bigotimes^{s'q} V, \mathrm{Hom}(\bigotimes^{s''} W'', Z))) \\ \chi_r^{s'; H_1, \ldots, H_{s''}} &:= [((H_1 \otimes \cdots \otimes H_{s''})^\star)_\circ]_\star \chi_r^{s';s''} \\ &\quad \in \mathcal{A}_r(M, \mathrm{Hom}(\textstyle\bigotimes^{s'} W', \mathrm{Hom}(\bigotimes^{s''q} V, Z))) \end{aligned}$$

If $\chi_r^{s',s''} \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\bigotimes^{s'} W', \mathrm{Hom}(\bigotimes^{s''} W'', Z))$, then $\chi_r^{G_1, \ldots, G_{s'}; s''} \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\bigotimes^{sq} V, \mathrm{Hom}(\bigotimes^{s''} W'', Z))$, $\chi_r^{s'; H_1, \ldots, H_{s''}} \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\bigotimes^{s'} W', \mathrm{Hom}(\bigotimes^{s''q} V, Z))$.

Definition 5.2. *For any $\chi_r^{s';s''} \in \mathcal{A}_r(M, \mathrm{Hom}(\bigotimes^{s'} W', \mathrm{Hom}(\bigotimes^{s''} W'', Z)))$ and any $\phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W')$, where $p, q, r, s', s'' \in \mathbb{N}_0$, let $Z' := \mathrm{Hom}(\bigotimes^{s''} W'', Z)$ and $\widetilde{\chi}_r^{s'} := \chi_r^{s';s''} \in \mathcal{A}_r(M, \mathrm{Hom}(\bigotimes^{s'} W', Z'))$, and define*

$$\chi_r^{s';s''} \blacktriangleleft \phi_p^q := \widetilde{\chi}_r^{s'} \bullet \phi_p^q \in \mathcal{A}_{r+s'p}(M, \mathrm{Hom}(\textstyle\bigotimes^{s'q} V, \mathrm{Hom}(\bigotimes^{s''} W'', Z))).$$

Be $\psi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W'')$ and $\jmath \colon \bigotimes^{s'} W' \to [\mathrm{Hom}(\bigotimes^{s'} W', Z) \to Z]$ the evaluation morphism. Define $\chi_r^{s';s''} \blacktriangleright \psi_p^q \in \mathcal{A}_{r+s''p}(M, \mathrm{Hom}(\bigotimes^{s'} W', \mathrm{Hom}(\bigotimes^{s''q} V, Z)))$ by

$$\jmath(w^1 \otimes \cdots \otimes w^{s'})_\star (\chi_r^{s';s''} \blacktriangleright \psi_p^q) := [\jmath(w^1 \otimes \cdots \otimes w^{s'})_\star \chi_r^{s';s''}] \bullet \psi_p^q \quad \textit{for all} \;\; w^i \in W'.$$

Thus for $\chi \in \mathcal{A}_r(M, \mathrm{Hom}(\bigotimes^s W, Z))$, the direction of the triangle indicates whether the second form is used as input for the first s' or the last s'' factors in $\chi_x(\mathcal{X}_x^1, \ldots, \mathcal{X}_x^r) \in \mathrm{Hom}(\bigotimes^s W, Z)$.

Analogously to Lemma 2.4, we obtain for the new operators:

Lemma 5.3. *Using the notation of the previous definitions, we have*

$$\chi_r^{s';s''} \blacktriangleleft \phi_p^q = \sum_{j_1,\dots,j_{s'}=1}^{m} \chi_r^{G_{j_1},\dots,G_{j_{s'}};s''} \wedge \phi^{j_1} \wedge \dots \wedge \phi^{j_{s'}} \quad \text{if} \quad \phi_p^q = \sum_{j=1}^{m} \phi^j \otimes G_j,$$

$$\chi_r^{s';s''} \blacktriangleright \psi_p^q = \sum_{k_1,\dots,k_{s''}=1}^{m} \chi_r^{s';H_{k_1},\dots,H_{k_{s''}}} \wedge \psi^{k_1} \wedge \dots \wedge \psi^{k_{s''}} \quad \text{if} \quad \psi_p^q = \sum_{k=1}^{m} \psi^k \otimes H_k.$$

$\chi_r^{s';s''} \blacktriangleleft \phi_p^q \in \mathcal{A}_{r+s'p}(M) \otimes \mathrm{Hom}(\bigotimes^{s'q} V, \mathrm{Hom}(\bigotimes^{s''} W'', Z))$ *and* $\chi_r^{s';s''} \blacktriangleright \psi_p^q \in \mathcal{A}_{r+s''p}(M) \otimes \mathrm{Hom}(\bigotimes^{s'} W', \mathrm{Hom}(\bigotimes^{s''q} V, Z))$ *if*
$\chi_r^{s';s''} \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\bigotimes^{s'} W', \mathrm{Hom}(\bigotimes^{s''} W'', Z))$.

Thus the terms in the sums for $(\chi_r^{s';s''} \blacktriangleleft \phi_{p'}^q) \blacktriangleright \psi_{p''}^q$, resp., $(\chi_r^{s';s''} \blacktriangleright \psi_{p''}^q) \blacktriangleleft \phi_{p'}^q$ contain exterior products of s' p'-forms ϕ^j in front of s'' p''-forms ψ^k, resp., s'' p''-forms ψ^k in front of s' p'-forms ϕ^j. As a consequence, $(\chi_r^{s';s''} \blacktriangleleft \phi_{p'}^q) \blacktriangleright \psi_{p''}^q$ and $(\chi_r^{s';s''} \blacktriangleright \psi_{p''}^q) \blacktriangleleft \phi_{p'}^q$ differ by a factor $(-1)^{p'p''s's''}$:

Lemma 5.4. *Let* $\chi_r^{s';s''}$, $\phi_{p'}^q$ *and* $\psi_{p''}^q$ *be defined as before. Then*

$$\begin{aligned}&(\chi_r^{s';s''} \blacktriangleleft \phi_{p'}^q) \blacktriangleright \psi_{p''}^q \\ &\quad = (-1)^{p'p''s's''} (\chi_r^{s';s''} \blacktriangleright \psi_{p''}^q) \blacktriangleleft \phi_{p'}^q \in \mathcal{A}_{r+s'p'+s''p''}(M, \mathrm{Hom}(\textstyle\bigotimes^{sq} V, Z))\end{aligned}$$

For $\chi_r^s \in \mathcal{A}_r(M, \mathrm{Sym}_s^\varsigma(W, Z))$ *with* $\varsigma = \pm$ *and* $\chi_r^{s';s''}$ *from* (8),

$$(\mathrm{Sym}_{sq}^\varsigma)_\star[(\chi_r^{s';s''} \blacktriangleleft \phi_{p'}^q) \blacktriangleright \psi_{p''}^q] = \varsigma^{(q+1)s's''} (\mathrm{Sym}_{sq}^\varsigma)_\star[(\chi_r^{s'';s'} \blacktriangleright \phi_{p'}^q) \blacktriangleleft \psi_{p''}^q].$$

Proof: With the previous notation, the first two terms are both equal to

$$\sum_{i_1,\dots,i_{s'+s''}=1}^{m} \chi_r^{G_{i_1},\dots,G_{i_{s'}};H_{i_{s'+1}},\dots,H_{i_{s'+s''}}} \wedge \phi^{i_1} \wedge \dots \wedge \phi^{i_{s'}} \wedge \psi^{i_{s'+1}} \wedge \dots \wedge \psi^{i_{s'+s''}};$$

$$\begin{aligned}&(\mathrm{Sym}_{sq}^\varsigma)_\star(\chi_r^{G_{i_1},\dots,G_{i_{s'}},H_{i_{s'+1}},\dots,H_{i_s}}) \\ &= \varsigma^{(q+1)s's''} (\mathrm{Sym}_{sq}^\varsigma)_\star(\chi_r^{H_{i_{s'+1}},\dots,H_{i_s},G_{i_1},\dots,G_{i_{s'}}})\end{aligned}$$

from (3) proves the second equation. □

With these new operators we can evaluate at least the (anti)symmetrized forms $(\mathrm{Sym}_{sq}^\varsigma)_\star[\chi_r^s \bullet (\phi_p^q + \psi_p^q)]$ (recall that this means no restriction for $q = 0$). To this purpose, we also introduce generalizations $\binom{s}{k}_\pm$ of the ordinary binomial coefficients:

$$(9) \quad \binom{s}{k}_+ := \binom{s}{k}, \quad \binom{s}{k}_- := \begin{cases} 0, & \text{if } s \text{ even and } k \text{ odd}, \\ \binom{[s/2]}{[k/2]}, & \text{else (for } r \in \mathbb{R},\ [r] := \max_{z \in \mathbb{Z}}\{z \le r\}). \end{cases}$$

Note that $\binom{s}{k}_\pm = \binom{s}{s-k}_\pm$ as before.

Proposition 5.5. *For $p,q,r,s \in \mathbb{N}_0$, let $\phi_p^q, \psi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W)$ and $\chi_r^s \in \mathcal{A}_r(M, \mathrm{Sym}_s^\varsigma(W,Z))$. Define ℓ as in (2). Then*

$$\begin{aligned}
&(\mathrm{Sym}_{sq}^\varsigma)_\star[\chi_r^s \bullet (\phi_p^q + \psi_p^q)] \\
&= \sum_{k=0}^{s} \tbinom{s}{k}_\ell (\mathrm{Sym}_{sq}^\varsigma)_\star[(\chi_r^{k;s-k} \blacktriangleleft \phi_p^q) \blacktriangleright \psi_p^q] \\
&= \sum_{k=0}^{s} (-1)^{pk(s-1)} \tbinom{s}{k}_\ell (\mathrm{Sym}_{sq}^\varsigma)_\star[(\chi_r^{k;s-k} \blacktriangleright \psi_p^q) \blacktriangleleft \phi_p^q] \\
&= \sum_{k=0}^{s} \tbinom{s}{k}_\ell (\mathrm{Sym}_{sq}^\varsigma)_\star[(\chi_r^{k;s-k} \blacktriangleleft \psi_p^q) \blacktriangleright \phi_p^q] \\
&= \sum_{k=0}^{s} (-1)^{pk(s-1)} \tbinom{s}{k}_\ell (\mathrm{Sym}_{sq}^\varsigma)_\star[(\chi_r^{k;s-k} \blacktriangleright \phi_p^q) \blacktriangleleft \psi_p^q].
\end{aligned}$$

Whenever $(\mathrm{Sym}_{sq}^\varsigma)_\star[\chi_r^s \bullet (\phi_p^q + \psi_p^q)]$ is nonzero according to Lemma 2.6, $\binom{s}{k}_\ell = \binom{s}{k}$.

Proof: The equations are trivial for $s = 0$ and $s = 1$, so assume $s > 1$. Let $\phi_p^q = \sum_{i=1}^m \phi^i \otimes F_i$ and $\psi_p^q = \sum_{i=1}^m \psi^i \otimes F_i$. Then with $\widetilde{\chi}_r^{i_1,\dots,i_s} := (\mathrm{Sym}_{sq}^\varsigma)_\star(\chi_r^{F_{i_1},\dots,F_{i_s}})$,

$$(\mathrm{Sym}_{sq}^\varsigma)_\star(\chi_r^s \bullet (\phi_p^q + \psi_p^q)) = \sum_{i_1,\dots,i_s=1}^{m} \widetilde{\chi}_r^{i_1,\dots,i_s} \wedge (\phi^{i_1} + \psi^{i_1}) \wedge \cdots \wedge (\phi^{i_s} + \psi^{i_s}),$$

$$\text{and}\quad (\mathrm{Sym}_{sq}^\varsigma)_\star[(\chi_r^{k;s-k} \blacktriangleleft \phi_p^q) \blacktriangleright \psi_p^q] = \sum_{i_1,\dots,i_s=1}^{m} \widetilde{\chi}_r^{i_1,\dots,i_s} \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_k} \wedge \psi^{i_{k+1}} \wedge \cdots \wedge \psi^{i_s}.$$

We proceed by induction on s. Thus

$$\begin{aligned}
&(\mathrm{Sym}_{sq}^\varsigma)_\star(\chi_r^s \bullet (\phi_p^q + \psi_p^q)) \\
&= \sum_{i_s=1}^{m} (\mathrm{Sym}_{sq}^\varsigma)_\star (\mathrm{Sym}_{(s-1)q}^\varsigma)_\star (\chi_r^{s-1;F_{i_s}} \bullet (\phi_p^q + \psi_p^q)) \wedge (\phi^{i_s} + \psi^{i_s}) \\
&= \sum_{k=0}^{s-1} \tbinom{s-1}{k}_\ell \sum_{i_1,\dots,i_s=1}^{m} \widetilde{\chi}_r^{i_1,\dots,i_s} \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_k} \wedge \psi^{i_{k+1}} \wedge \cdots \wedge \psi^{i_{s-1}} \wedge (\phi^{i_s} + \psi^{i_s}) \\
&= \sum_{k=0}^{s} [\tbinom{s-1}{k}_\ell + \ell^{s-k} \tbinom{s-1}{k-1}_\ell] \sum_{i_1,\dots,i_s=1}^{m} \widetilde{\chi}_r^{i_1,\dots,i_s} \wedge \phi^{i_1} \wedge \cdots \wedge \phi^{i_k} \wedge \psi^{i_{k+1}} \wedge \cdots \wedge \psi^{i_s},
\end{aligned}$$

where we have used (4). Recursion $\binom{s}{k}_\ell = \binom{s-1}{k}_\ell + \ell^{s-k}\binom{s-1}{k-1}_\ell$ proves the formulae for $\binom{s}{k}_\ell$. Lemma 5.4 and interchanging ϕ_p^q and ψ_p^q finally yield the rest. □

6. Exterior derivative, interior product and Lie derivative

Finally we will also derive formulae for the exterior derivative of $\chi_r^s \bullet \phi_p^q$, as well as for its interior product and its Lie derivative with respect to a vector field $\mathcal{X} \in \mathcal{D}^1(M)$. Let us start with the (anti)symmetrized forms.

Proposition 6.1. *Let* $\phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W)$ *and* $\chi_r^s \in \mathcal{A}_r(M) \otimes \mathrm{Sym}_s^\varsigma(W, Z)$ *for* $p,q,r,s \in \mathbb{N}_0$. *Define* $\binom{s}{1}_\ell$ *as in* (9). *Then*

$$\begin{aligned} d[(\mathrm{Sym}_{sq}^\varsigma)_\star(\chi_r^s \bullet \phi_p^q)] &= (\mathrm{Sym}_{sq}^\varsigma)_\star[(d\chi)_{r+1}^s \bullet \phi_p^q] \\ &\quad + (-1)^r \binom{s}{1}_\ell (\mathrm{Sym}_{sq}^\varsigma)_\star[(\chi_r^{1;s-1} \blacktriangleleft (d\phi)_{p+1}^q) \blacktriangleright \phi_p^q] \\ &= (\mathrm{Sym}_{sq}^\varsigma)_\star[(d\chi)_{r+1}^s \bullet \phi_p^q] \\ &\quad + (-1)^{r+p(s-1)} \binom{s}{1}_\ell (\mathrm{Sym}_{sq}^\varsigma)_\star[(\chi_r^{s-1;1} \blacktriangleleft \phi_p^q) \blacktriangleright (d\phi)_{p+1}^q]. \end{aligned}$$

Proof: With the notation of the previous proof, Lemmas 2.1 and 2.6 yield

$$\begin{aligned} d[(\mathrm{Sym}_{sq}^\varsigma)_\star(\chi_r^s \bullet \phi_p^q)] &= \sum_{i_1,\dots,i_s=1}^m d\widetilde{\chi}_r^{i_1,\dots,i_s} \wedge \phi^{i_1} \wedge \dots \wedge \phi^{i_s} \\ &+ \sum_{j=1}^s \sum_{i_1,\dots,i_s=1}^m (-1)^{r+p(j-1)} \widetilde{\chi}_r^{i_1,\dots,i_s} \wedge \phi^{i_1} \wedge \dots \wedge \phi^{i_{j-1}} \wedge d\phi^{i_j} \wedge \phi^{i_{j+1}} \wedge \dots \wedge \phi^{i_s} \\ &= (\mathrm{Sym}_{sq}^\varsigma)_\star[(d\chi)_{r+1}^s \bullet \phi_p^q] \\ &+ (-1)^r \sum_{j=1}^s \sum_{i_1,\dots,i_s=1}^m \ell^{j-1}\, \widetilde{\chi}_r^{i_1,\dots,i_s} \wedge d\phi^{i_1} \wedge \phi^{i_2} \wedge \dots \wedge \phi^{i_s} \\ &= (\mathrm{Sym}_{sq}^\varsigma)_\star[(d\chi)_{r+1}^s \bullet \phi_p^q] + (-1)^r \binom{s}{1}_\ell \sum_{i_1,\dots,i_s=1}^m \widetilde{\chi}_r^{i_1,\dots,i_s} \wedge d\phi^{i_1} \wedge \phi^{i_2} \wedge \dots \wedge \phi^{i_s}, \end{aligned}$$

where we used (2) and (3) in the second step. Lemma 5.4 proves the rest. □

Since we only used the fact that d is a skew-derivation of $\mathcal{A}(M)$, Proposition 6.1 also holds for $\imath_{\mathcal{X}}$ instead of d, and for $L_{\mathcal{X}}$, if one drops $(-1)^r$. Tracing the previous proof we get for the general case:

Corollary 6.2. *If* $\chi_r^s \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\bigotimes^s W, Z)$, $\phi_p^q \in \mathcal{A}_p(M) \otimes \mathrm{Hom}(\bigotimes^q V, W)$ *and* $\mathcal{X} \in \mathcal{D}^1(M)$, *then*

$$\begin{aligned} &d(\chi_r^s \bullet \phi_p^q) \\ &\quad = (d\chi)_{r+1}^s \bullet \phi_p^q + (-1)^r \sum_{j=0}^{s-1} (-1)^{jp} [(\chi_r^{j;s-j} \blacktriangleleft \phi_p^q)^{1;s-j-1} \blacktriangleleft (d\phi)_{p+1}^q] \blacktriangleright \phi_p^q, \end{aligned}$$

$$\imath_{\mathcal{X}}(\chi_r^s \bullet \phi_p^q)$$
$$= (\imath_{\mathcal{X}}\chi)_{r-1}^s \bullet \phi_p^q + (-1)^r \sum_{j=0}^{s-1}(-1)^{jp}[(\chi_r^{j;s-j} \blacktriangleleft \phi_p^q)^{1;s-j-1} \blacktriangleleft (\imath_{\mathcal{X}}\phi)_{p-1}^q] \blacktriangleright \phi_p^q,$$
$$L_{\mathcal{X}}(\chi_r^s \bullet \phi_p^q)$$
$$= (L_{\mathcal{X}}\chi)_r^s \bullet \phi_p^q + \sum_{j=0}^{s-1}(-1)^{jp}[(\chi_r^{j;s-j} \blacktriangleleft \phi_p^q)^{1;s-j-1} \blacktriangleleft (L_{\mathcal{X}}\phi)_{p]}^q] \blacktriangleright \phi_p^q.$$

7. Lie groups and Lie group actions

Suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and adjoint action $\mathrm{Ad}: G \to \mathrm{Gl}(\mathfrak{g})$. If λ_g and ρ_g denote multiplication with $g \in G$ form the left, resp., from the right, then the canonical left, resp., right invariant 1-forms $\Theta^L \in \mathcal{A}_1^L(G, \mathfrak{g})$, resp., $\Theta^R \in \mathcal{A}_1^R(G, \mathfrak{g})$ are given by

$$\Theta_g^L(\mathcal{X}_g) := d\lambda_{g^{-1}}(\mathcal{X}_g), \quad \Theta_g^R(\mathcal{X}_g) := d\rho_{g^{-1}}(\mathcal{X}_g) \quad \text{for all} \;\; g \in G,\ \mathcal{X} \in \mathcal{D}^1(G).$$

Both are connected via $\Theta_g^R = \mathrm{Ad}(g) \circ \Theta_g^L$. Using $\bullet$ we can get rid of the argument g and may write this identity as $\Theta^R = \mathrm{Ad} \bullet \Theta^L$.

For $S = L, R$, let $\psi^S: \mathrm{Alt}_p(\mathfrak{g}, V) \to \mathcal{A}_p^S(G, V)$ denote the isomorphisms between the vector spaces of alternating p-linear maps on $\mathfrak{g}^p$ and of left, resp., right invariant p-forms on G (where the inverse morphisms are the evaluation at the identity $e \in G$). Then $\Theta^S = \psi^S(\mathrm{id}_{\mathfrak{g}})$ and if $1 \in C^\infty(M)$ denotes the constant map onto $1 \in \mathbb{R}$ (here $M = G$), then ψ^S is given by

$$\psi^S(K) = (1 \otimes K) \bullet \Theta^S \in \mathcal{A}^S(G, V) \quad \text{for all} \quad K \in \mathrm{Alt}(\mathfrak{g}, V). \tag{10}$$

Note that for any linear $\Lambda: V \to W$, Lemma 4.1.3 combined with (10) yields

$$\Lambda_\star \psi^S(K) = \Lambda_\star[(1 \otimes K) \bullet \Theta^S] = [1 \otimes (\Lambda_\circ K)] \bullet \Theta^S = \psi^S(\Lambda_\circ K) \in \mathcal{A}^S(G, W).$$

For any manifold M, $C^\infty(M, G)$ is a group with respect to pointwise multiplication and inversion. For any differentiable $f, g: M \to G$, the expressions $f \cdot g$ and f^{-1} are understood within this group. From Lemma 4.1.1 we obtain for the so-called left and right differential of $f \in C^\infty(M, G)$:

$$f^\star\Theta^R = (\mathrm{Ad} \circ f) \bullet f^\star\Theta^L, \qquad f^\star\Theta^L = (\mathrm{Ad} \circ f^{-1}) \bullet f^\star\Theta^R.$$

Moreover, the generalized product rule $d(f \cdot g)_x = (d\rho_{g(x)})_{f(x)} df_x + (d\lambda_{f(x)})_{g(x)} dg_x$ for all $x \in M$ yields the following relations:

$$(f \cdot g)^\star\Theta^L = (\mathrm{Ad} \circ g^{-1}) \bullet f^\star\Theta^L + g^\star\Theta^L,$$
$$(f \cdot g)^\star\Theta^R = f^\star\Theta^R + (\mathrm{Ad} \circ f) \bullet g^\star\Theta^R,$$
$$(f^{-1})^\star\Theta^L = -(\mathrm{Ad} \circ f) \bullet f^\star\Theta^L \;\; = \;\; -f^\star\Theta^R,$$
$$(f^{-1})^\star\Theta^R = -(\mathrm{Ad} \circ f^{-1}) \bullet f^\star\Theta^R \;\; = \;\; -f^\star\Theta^L.$$

Now (10) and Lemma 4.1.1 yield for $K \in \mathrm{Alt}(\mathfrak{g}, V)$:

$$(f \cdot g)^\star \psi^L(K) = (1 \otimes K) \bullet [(\mathrm{Ad} \circ g^{-1}) \bullet f^\star \Theta^L + g^\star \Theta^L],$$
$$(f \cdot g)^\star \psi^R(K) = (1 \otimes K) \bullet [f^\star \Theta^R + (\mathrm{Ad} \circ f) \bullet g^\star \Theta^R].$$

(In addition, $(f^{-1})^\star \psi^L(K) = (-1)^p f^\star \psi^R(K)$ if $K \in \mathrm{Alt}_p(\mathfrak{g}, V)$.)

Again for $S = L, R$ and any manifold P let $S: G \times P \to P$ denote a left, resp., right LIE group action. (For notational convenience, we write G on the left, even for a right action.) Thus if $S_g(p) := S(g, p)$, then all $S_g: P \to P$ are diffeomorphisms and we will identify S with $S: G \to \mathrm{Diff}(P)$. We put $\mathrm{sgn}(S) = -1$ for $S = L$ and $\mathrm{sgn}(S) = +1$ for $S = R$. Then the following lemma holds.

Lemma 7.1. *Let $S: G \times P \to P$ be a LIE group action and $L': G \to \mathrm{Gl}(W)$ be a left representation. If $\varphi_r \in \mathcal{A}_r(P) \otimes W$ and $\chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(W), V))$ are equivariant in the sense that $S_g^\star \varphi_r = L'(g^{-\mathrm{sgn}(S)})_\star \varphi_r$ and $S_g^\star \chi = (L'(g^{\mathrm{sgn}(S)})^\star)_\star \chi$ for all $g \in G$, then $\chi \bullet \varphi_r$ is invariant.*

PROOF: $S_g^\star(\chi \bullet \varphi_r) = (S_g^\star \chi) \bullet (S_g^\star \varphi_r) = \chi \bullet [(L'(g^{\mathrm{sgn}(S)})^\star)_\star S_g^\star \varphi_r] = \chi \bullet \varphi_r$, where the first equality follows from Lemma 4.1.1 and the second from 4.1.2. □

Every representation $S: G \to \mathrm{Gl}(V)$ induces a representation $s = dS_e: \mathfrak{g} \to \mathrm{gl}(V)$ of the LIE algebra such that $S \circ \exp X = e^{s(X)}$ for all $X \in \mathfrak{g}$. As for S, we will identify $s = l, r$ with the induced bilinear mappings $s: \mathfrak{g} \times V \to V$. We have the following relations for all $g \in G$, $X \in \mathfrak{g}$ and $v \in V$:

$$s(X, S(g, v)) = S(g, s(\mathrm{Ad}(g^{\mathrm{sgn}(S)})X, v)),$$
$$S(g, s(X, v)) = s(\mathrm{Ad}(g^{-\mathrm{sgn}(S)})X, S(g, v))$$

and thus obtain:

Proposition 7.2. *Let $S: G \to \mathrm{Gl}(V)$ be a representation and $s: \mathfrak{g} \times V \to V$ be the induced bilinear map. Then for any differentiable $f: M \to G$ and forms $\omega \in \mathcal{A}(M, \mathfrak{g})$ and $\phi \in \mathcal{A}(M) \otimes V$,*

$$(S \circ f) \bullet (\omega \wedge_s \phi) = [(\mathrm{Ad} \circ f^{-\mathrm{sgn}(S)}) \bullet \omega] \wedge_s [(S \circ f) \bullet \phi], \tag{11}$$
$$d[(S \circ f) \bullet \phi] = (S \circ f) \bullet (f^\star \Theta^S \wedge_s \phi + d\phi). \tag{12}$$

PROOF: Only (12) still needs to be proved. For $S = L$, observe that for all $g \in G$, $L \circ \lambda_g = \lambda'_{L(g)} \circ L$ with $\lambda'_{L(g)}: \mathrm{Gl}(V) \to \mathrm{Gl}(V): A \mapsto L(g) \circ A$. For any vector field $\mathcal{X} \in \mathcal{D}^1(M)$ and $x \in M$, this yields $[d(L \circ f)]\mathcal{X}(x) = dL_{f(x)} \circ d\lambda_{f(x)}(f^\star \Theta^L)_x \mathcal{X}_x = \lambda'_{L(f(x))} \circ l \circ (f^\star \Theta^L)_x \mathcal{X}_x$, and thus $[d(L \circ f)] \bullet \phi = (L \circ f) \bullet (f^\star \Theta^L \wedge_l \phi)$. Now (12) follows from Proposition 6.1. Analogous arguments hold for $S = R$. □

As a corollary, we get for any $f \in C^\infty(M,G)$ and all $\omega, \phi \in \mathcal{A}(M,\mathfrak{g})$,

$$(\mathrm{Ad}\circ f) \bullet (\omega \wedge_{\mathfrak{g}} \phi) = [(\mathrm{Ad}\circ f) \bullet \omega] \wedge_{\mathfrak{g}} [(\mathrm{Ad}\circ f) \bullet \phi], \tag{13}$$

$$d[(\mathrm{Ad}\circ f) \bullet \phi] = (\mathrm{Ad}\circ f) \bullet (f^\star \Theta^L \wedge_{\mathfrak{g}} \phi + d\phi). \tag{14}$$

In all cases, the operator $\bullet$ leads to a quite compact notation, since we need not refer to the points $g \in G$ nor the vector fields that the forms act on.

8. Fiber bundles and connections

Let $P(M,G)$ denote a principal bundle with base manifold $M = \bigcup_{\alpha\in A} U_\alpha$, projection $\pi: P \to M$, fiber G, right action $R: P \times G \to P$ and local trivializations $\psi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times G$ with local projections $\pi_\alpha = \mathrm{pr}_G \circ \psi_\alpha$. Recall that any connection Γ on P defines horizontal and vertical projections of vector fields, not only on P, but also on every associated fiber bundle $B(M,F,G) = P \times_G F$ with fiber F, such that the vertical fields are tangential to the fiber. We thus obtain projections h, v of differential forms via

$$\omega h(\dots, \mathcal{X}^i, \dots) := \omega(\dots, h\mathcal{X}^i, \dots), \qquad \omega v(\dots, \mathcal{X}^i, \dots) := \omega(\dots, v\mathcal{X}^i, \dots)$$

for all $\omega \in \mathcal{A}(B,V)$. Obviously h and v commute with $\bullet$, $\blacktriangleleft$ and $\blacktriangleright$: e. g. for $\chi \in \mathcal{A}(B, \mathrm{Hom}(\mathcal{T}(W), Z))$ and $\phi_r^q \in \mathcal{A}_r(B) \otimes \mathrm{Hom}(\bigotimes^q V, W)$,

$$(\chi \bullet \phi_r^q)h = \chi h \bullet \phi_r^q h, \qquad (\chi \bullet \phi_r^q)v = \chi v \bullet \phi_r^q v.$$

Let $d^\Gamma: \mathcal{A}(P) \otimes V \to \mathcal{A}(P)h \otimes V$ denote the exterior covariant differentiation on P, which is defined by $d^\Gamma \omega := (d\omega)h$. Then we immediately obtain from Corollary 6.2:

Proposition 8.1. *For differential forms $\chi_r^s \in \mathcal{A}_r(P) \otimes \mathrm{Hom}(\bigotimes^s W, Z)$ and $\phi_p^q \in \mathcal{A}_p(P) \otimes \mathrm{Hom}(\bigotimes^q V, W)$,*

$$d^\Gamma(\chi_r^s \bullet \phi_p^q)$$
$$= (d^\Gamma \chi)_{r+1}^s \bullet \phi_p^q h + \sum_{j=0}^{s-1} (-1)^{r+jp} [(\chi_r^{j;s-j} h \blacktriangleleft \phi_p^q h)^{1;s-j-1} \blacktriangleleft (d^\Gamma \phi)_{p+1}^q] \blacktriangleright \phi_p^q h.$$

Let $\omega^\Gamma \in \mathcal{A}_1(P,\mathfrak{g})$ and $\Omega^\Gamma = d^\Gamma \omega^\Gamma \in \mathcal{A}_2(P,\mathfrak{g})$ denote the connection 1-form, resp., the curvature 2-form of Γ. Both are equivariant with respect to R and Ad. If $\sigma_{\alpha,e}: U_\alpha \to \pi^{-1}(U_\alpha)$ are the local sections given by $\sigma_{\alpha,e}(x) := \psi_\alpha^{-1}(x,e)$, then the gauge potentials A^α and the gauge fields F^α are given by

$$\begin{aligned} \mathrm{A}^\alpha &:= \sigma_{\alpha,e}^\star(\omega^\Gamma|_{\pi^{-1}(U_\alpha)}) \in \mathcal{A}_1(U_\alpha,\mathfrak{g}), \\ \mathrm{F}^\alpha &:= \sigma_{\alpha,e}^\star(\Omega^\Gamma|_{\pi^{-1}(U_\alpha)}) \in \mathcal{A}_2(U_\alpha,\mathfrak{g}). \end{aligned} \tag{15}$$

On the other hand, the collection of A^α and F^α determines ω^Γ and Ω^Γ:

$$\begin{aligned}\omega^\Gamma|_{\pi^{-1}(U_\alpha)} &= (\mathrm{Ad}\circ\pi_\alpha^{-1}) \bullet (\pi^\star \mathrm{A}^\alpha) + \pi_\alpha^\star \Theta^L,\\ \Omega^\Gamma|_{\pi^{-1}(U_\alpha)} &= (\mathrm{Ad}\circ\pi_\alpha^{-1}) \bullet (\pi^\star \mathrm{F}^\alpha)\end{aligned} \tag{16}$$

(recall that $\pi_\alpha^{-1}\colon U_\alpha \to G$ means the pointwise inverse). On all $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, the transition functions $g_{\alpha\beta}\colon U_{\alpha\beta} \to G$ are given by $g_{\alpha\beta} = (\pi_\alpha|_{U_{\alpha\beta}}) \circ (\sigma_{\beta,e}|_{U_{\alpha\beta}})$ such that $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$. Using (15) and (16) we may compute the behavior of the A^α and F^α under a change of the local trivialization. In our compact notation we obtain:

$$\begin{aligned}\mathrm{A}^\alpha|_{U_{\alpha\beta}} &= (\mathrm{Ad}\circ g_{\alpha\beta}) \bullet \mathrm{A}^\beta|_{U_{\alpha\beta}} + g_{\beta\alpha}^\star \Theta^L \\ &= (\mathrm{Ad}\circ g_{\alpha\beta}) \bullet (\mathrm{A}^\beta|_{U_{\alpha\beta}} - g_{\alpha\beta}^\star \Theta^L),\end{aligned} \tag{17}$$

$$\mathrm{F}^\alpha|_{U_{\alpha\beta}} = (\mathrm{Ad}\circ g_{\alpha\beta}) \bullet \mathrm{F}^\beta|_{U_{\alpha\beta}}. \tag{18}$$

Since the gauge potentials and the gauge fields play an important role in all field theories in theoretical physics, these formulae prove to be very useful for all computations in those field theories like electromagnetism, Yang-Mills theories, etc. Of course these formulae are not new, only the notation is new. Also note that equations (13) to (16) prove that the so-called field equations

$$\mathrm{F}^\alpha = d\,\mathrm{A}^\alpha + \frac{1}{2}\,\mathrm{A}^\alpha \wedge_{\mathfrak{g}} \mathrm{A}^\alpha \qquad \text{and} \qquad d\,\mathrm{F}^\alpha = -\,\mathrm{A}^\alpha \wedge_{\mathfrak{g}} \mathrm{F}^\alpha$$

are equivalent to the structure equation and Bianchi's identity,

$$\Omega^\Gamma = d\omega^\Gamma + \frac{1}{2}\omega^\Gamma \wedge_{\mathfrak{g}} \omega^\Gamma \qquad \text{and} \qquad d^\Gamma\Omega^\Gamma = d\Omega^\Gamma + \omega^\Gamma \wedge_{\mathfrak{g}} \Omega^\Gamma = 0.$$

Finally, we may also use $\bullet$ for the definition of characteristic classes of a principal bundle, cf. Greub, Halperin, Vanstone [1]. Let $C \in \mathrm{Sym}_k(\mathfrak{g}, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$ be invariant under $\mathrm{Ad}^\star$. Then $(1 \otimes C) \in C^\infty(P, \mathrm{Sym}_k(\mathfrak{g}, \mathbb{F}))$ is equivariant and thus $(1 \otimes C) \bullet \Omega^\Gamma \in \mathcal{A}_{2k}(P, \mathbb{F})$ is invariant under R according to Lemma 7.1. Since it is also horizontal, it is a pullback of a form "$C \bullet \mathrm{F}$" $\in \mathcal{A}_{2k}(M, \mathbb{F})$. (The notation $C \bullet \mathrm{F}$ reminds to the fact that on U_α, this form is given by $(1 \otimes C) \bullet \mathrm{F}^\alpha$ with $1 \in C^\infty(U_\alpha)$.) $C \bullet \mathrm{F}$ defines a characteristic cohomology class of P. In fact, because $\pi^\star \circ d = d^\Gamma \circ \pi^\star$, we obtain from Proposition 8.1 and Bianchi's identity that $d(C \bullet \mathrm{F}) = 0$. (Recall that $[C \bullet \mathrm{F}] \in H^{2k}(M, \mathbb{F})$ is invariant of the special choice of connection, cf. [1, p. 264].)

In all these applications, we have $r = 0$ in Definition 2.3. Further applications for Lie transformation groups and fiber bundles that involve $r \neq 0$, are given in [2] and [3], whereas [4] contains applications to field theories in theoretical physics. E.g., the operator $\bullet$ is essential for the local description of vertical forms on a fiber bundle. Using Lemma 7.1, one can prove ([3]) that for any G-equivariant form $\chi \in \mathcal{A}(F, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))$, the combination of its local vertical projections $(\pi_\alpha^\star \chi)v^\alpha$ and the gauge fields defines a *global* form on the bundle $B(M, F, G)$, since on the overlaps,

$$[(\pi_\beta^\star \chi)v^\beta] \bullet (\pi^\star \mathrm{F}^\beta) = [(\pi_\alpha^\star \chi)v^\alpha] \bullet (\pi^\star \mathrm{F}^\alpha).$$

Finally, Corollary 6.2 is needed to compute the exterior derivative of this global form, cf. [2].

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