# When is IN Lindelöf?

Horst Herrlich, George E. Strecker

*Abstract.*
**Theorem.** *In ZF (i.e., Zermelo-Fraenkel set theory without the axiom of choice) the following conditions are equivalent:*

(1) $\mathbb{N}$ *is a Lindelöf space,*
(2) $\mathbb{Q}$ *is a Lindelöf space,*
(3) $\mathbb{R}$ *is a Lindelöf space,*
(4) *every topological space with a countable base is a Lindelöf space,*
(5) *every subspace of* $\mathbb{R}$ *is separable,*
(6) *in* $\mathbb{R}$*, a point* $x$ *is in the closure of a set* $A$ *iff there exists a sequence in* $A$ *that converges to* $x$*,*
(7) *a function* $f : \mathbb{R} \to \mathbb{R}$ *is continuous at a point* $x$ *iff* $f$ *is sequentially continuous at* $x$*,*
(8) *in* $\mathbb{R}$*, every unbounded set contains a countable, unbounded set,*
(9) *the axiom of countable choice holds for subsets of* $\mathbb{R}$*.*

*Keywords:* axiom of choice, axiom of countable choice, Lindelöf space, separable space, (sequential) continuity, (Dedekind-) finiteness

*Classification:* Primary 03E25, 04A25, 54D20; Secondary 26A03, 26A15, 54A35

## Introduction

Jech (1968) has shown that in ZF (i.e., Zermelo-Fraenkel set theory without the axiom of choice) the space $\mathbb{R}$ of real numbers may fail to be Lindelöf, even though $\mathbb{R}$ has a countable base. Here we will show that — perhaps even more surprisingly — the countable discrete space $\mathbb{N}$ of positive integers can fail to be Lindelöf as well. Naturally, the axiom of countable choice implies that $\mathbb{R}$ and (hence) $\mathbb{N}$ are Lindelöf. Is there a simple set-theoretic condition that is sufficient *and* necessary for $\mathbb{N}$ to be Lindelöf? The purpose of our note is to answer this question.

## Proof of the Theorem

It suffices to establish the implications (8) $\Rightarrow$ (9) $\Rightarrow$ (4) and (1) $\Rightarrow$ (8), since the validity of the implications (4) $\Rightarrow$ (2) $\Rightarrow$ (1), (4) $\Rightarrow$ (3) $\Rightarrow$ (1), (7) $\Rightarrow$ (6) $\Rightarrow$ (8) $\Rightarrow$ (7), and (9) $\Rightarrow$ (5) $\Rightarrow$ (8) is apparent.

(8) $\Rightarrow$ (9) Let $(X_n)$ be a sequence of non-empty subsets of $\mathbb{R}$. For each $n \in \mathbb{N}$ consider an injection $\tau_n\colon \mathbb{R}^n \to ]n, n+1[$.

[Such $\tau_n$ can be constructed effectively, e.g., as follows:
Let $\mathbf{A}$ be the subset of $\{0,1\}^{\mathbb{N}}$ consisting of all non-constant sequences $(x_n)$ with infinitely many zeros.
Let $\alpha\colon\ \mathbb{R} \to\ ]0,1[$ be the bijection defined by $\alpha(x) = 2^{-1} + \pi^{-1}\cdot \arctan x$.
Let $\beta\colon\ \mathbf{A} \to\ ]0,1[$ be the bijection defined by $\beta(x_n) = \sum\limits_{1}^{\infty} 2^{-n}\cdot x_n$.
Consider $\gamma = \beta^{-1}\circ\alpha\colon\ \mathbb{R}\to\mathbf{A}$ and $\gamma^n\colon\ \mathbb{R}^n\to\mathbf{A}^n$.
Let $\sigma_n\colon\ \mathbf{A}^n\to\mathbf{A}$ be the $n$-th squeezing function defined by

$$\sigma_n\Big((x_1^1,x_2^1,\dots),(x_1^2,x_2^2,\dots),\dots,(x_1^n,x_2^n,\dots)\Big) = \\ = (x_1^1,x_1^2,\dots,x_1^n,x_2^1,x_2^2,\dots x_2^n,x_3^1,\dots).$$

Let $\delta_n\colon\ ]0,1[\ \to\ ]n,n+1[$ be the bijection defined by $\delta_n(x) = n+x$.
Then $\tau_n = \delta_n\circ\beta\circ\sigma_n\circ\gamma^n\colon\ \mathbb{R}^n\to\ ]n,n+1[$ is an injection.]

Each $Y_n = \tau_n\Big[\prod\limits_1^n X_i\Big]$ is a non-empty subset of $]n,n+1[$. Hence $Y = \bigcup\limits_1^\infty Y_n$ is an unbounded subset of $\mathbb{R}$. By (8), $Y$ contains an unbounded sequence $(y_m)$. For each $m\in\mathbb{N}$ there exists a unique $\nu(m)$ in $\mathbb{N}$ with $y_m\in Y_{\nu(m)}$, thus a unique element $z_m$ of $\prod\limits_1^{\nu(m)} X_i$ with $\tau_{\nu(m)}(z_m) = y_m$. Denote $z_m$ by $\Big(x_1^m,x_2^m,\dots,x_{\nu(m)}^m\Big)$.

Next, let $n$ be an element of $\mathbb{N}$. Since $(y_m)$ is unbounded there exists some $m$ in $\mathbb{N}$ with $n\le\nu(m)$. Define $\mu(n) = \mathrm{Min}\{m\in\mathbb{N}\mid n\le\nu(m)\}$. Then $n\le\nu\Big(\mu(n)\Big)$. Thus $x_n^{\mu(n)}$ belongs to $X_n$, and consequently $\Big(x_n^{\mu(n)}\Big)\in\prod\limits_1^\infty X_n$.

(9) $\Rightarrow$ (4) Let $\mathbf{X}$ be a topological space with a countable base $(B_n)$, and let $\mathfrak{U}$ be an open cover of $\mathbf{X}$. Then the map $\alpha\colon\ \mathfrak{U}\to\mathfrak{P}\mathbb{N}$ from $\mathfrak{U}$ into the powerset of $\mathbb{N}$, defined by $\alpha(U) = \{n\in\mathbb{N}\mid B_n\subseteq U\}$, is injective. For each $n$ in $\mathbb{N}$ define $X_n = \{\alpha(U)\mid B_n\subseteq U\in\mathfrak{U}\}$. Then $M = \{n\in\mathbb{N}\mid X_n\ne\emptyset\}$ is at most countable. Since there exists a bijection between $\mathfrak{P}\mathbb{N}$ and $\mathbb{R}$ condition (9) implies that $\prod\limits_{m\in M} X_m\ne\emptyset$. Let $(x_m)$ be an element of this product. Since $\alpha$ is injective, for each $m\in M$ there exists a unique element $U_m$ in $\mathfrak{U}$ with $\alpha(U_m) = x_m$. In particular, $x_m\in X_m$ implies $B_m\subseteq U_m$. Since $(B_n)$ is a base and $\mathfrak{U}$ is an open cover of $\mathbf{X}$, $\{B_m\mid m\in M\}$ covers $X$. Consequently $\{U_m\mid m\in M\}$ covers $X$.

(1) $\Rightarrow$ (8) Let $A$ be a subset of $\mathbb{R}$ unbounded to the right. Consider a bijection $\alpha\colon\mathbb{N}\to\mathbb{Q}$. Then the map $\beta\colon A\to\mathfrak{P}\mathbb{N}$, given by $\beta(a) = \{n\in\mathbb{N}\mid\alpha(n)<a\}$, is injective. Further $\mathfrak{U} = \{\beta(a)\mid a\in A\}$ is an open cover of $\mathbb{N}$. By (1), $\mathfrak{U}$ contains an at most countable subset $\mathfrak{V}$ that covers $\mathbb{N}$. For each $V\in\mathfrak{V}$ there exists a unique element $a_V\in A$ with $V = \beta(a_V)$. Consequently $\{a_V\mid V\in\mathfrak{V}\}$ is a countable, unbounded subset of $A$. $\square$

**Remarks**

(1) Jaegermann (1965) has constructed a model of ZF in which the condition (7) of our Theorem fails.

(2) Jech (1968) has shown that in any model of ZF that violates the following condition

(*) every infinite subset of $\mathbb{R}$ is Dedekind-infinite,[1]

(e.g., in Cohen's basic model) the conditions (3), (5), (6), and (7) of our Theorem must fail. Obviously, condition (5) implies (*). We do not know whether (*) is properly weaker than the conditions of our Theorem. It is, however, easy to see that (*) is equivalent to the following strong form of the Bolzano-Weierstraß-Theorem:

(SBW) in $\mathbb{R}$, every bounded, infinite set contains a convergent, injective sequence.

In contrast to this, the ordinary Bolzano-Weierstraß-Theorem

(BW) in $\mathbb{R}$, for every bounded, infinite set there exists an accumulation point

is easily seen to hold in ZF.

(3) Sierpiński (1916) has shown that the conditions (6) and (7) of our Theorem are equivalent to each other and to the following (somewhat unattractive) set-theoretic-condition:

(P) "Pour toute suite infinie des ensembles de nombres réels $X_1, X_2, X_3, \ldots$ , [non vides] sans points communs, existe au moins une suite infinie de nombres réels $x_1, x_2, x_3, \ldots$ , dont les termes correspondants aux indices différents appartiennent toujours aux différents ensembles $X_n$."

In contrast to the above, Sierpiński (1918) proved that a function $f: \mathbb{R} \to \mathbb{R}$ is continuous iff it is sequentially continuous.

In contrast to this, Herrlich and Steprāns proved that the equivalence of continuity and sequential continuity for functions between metric spaces (equivalently: for functions from metric spaces into $\mathbb{R}$) is equivalent to the axiom of countable choice.

(4) In the wider realm of pseudometric spaces the following hold:

(a) (Herrlich (1996)) Equivalent are:

($\alpha$) Heine-Borel-compactness (i.e., every open cover contains a finite one) implies Alexandroff-Urysohn-compactness (i.e., every infinite set has a complete accumulation point),

[1] A set $A$ is called *Dedekind-infinite* provided there exists some injection from $\mathbb{N}$ into $A$.

($\beta$) the axiom of choice.

(b) (Bentley and Herrlich) Equivalent are:

($\alpha$) sequential compactness implies Heine-Borel-compactness,

($\beta$) Heine-Borel-compactness implies separability,

($\gamma$) the Lindelöf property implies separability,

($\delta$) the countable base condition (= second axiom of countability) implies separability,

($\epsilon$) subspaces of separable spaces are separable,

($\zeta$) the Baire Category Theorem holds for complete spaces with countable base,

($\eta$) the axiom of countable choice.

(c) (Bentley and Herrlich) Equivalent are:

($\alpha$) sequential compactness implies Bolzano-Weierstraß-compactness (i.e., every infinite set has an accumulation point),

($\beta$) every infinite set is Dedekind-infinite.

(d) (Bentley and Herrlich) The Baire category theorem holds for separable complete spaces.

## References

Bentley H.L., Herrlich H., *Countable choice and pseudometric spaces*, in preparation.

Herrlich H. (1996), *Compactness and the Axiom of Choice*, Appl. Categ. Struct. **4** (1996), 1–14.

Herrlich H., Steprāns J., *Maximal Filters, continuity, and choice principles*, to appear in Quaestiones Math.

Jaegermann M. (1965), *The axiom of choice and two definitions of continuity*, Bulletin de l'Acad. Polonaise des Sciences, Ser. Math. **13** (1965), 699–704.

Jech T. (1968), *Eine Bemerkung zum Auswahlaxiom*, Časopis pro pěstování matematiky **9** (1968), 30–31.

Sierpiński W. (1916), *Sur le rôle de l'axiome de M. Zermelo dans l'Analyse moderne*, Compt. Rendus Hebdomadaires des Séances de l'Academie des Sciences, Paris **193** (1916), 688–691.

Sierpiński W. (1918), *L'axiome de M. Zermelo et son rôle dans la théorie des ensembles et l'analyse*, Bulletin de l'Académie des Sciences de Cracovie, Classe des Sciences Math., Sér. A (1918), 97–152.

Fachbereich 3, Universität Bremen, 28359 Bremen, Germany

Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA

(*Received* June 7, 1996)