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On the Noetherian type of topological spaces

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Abstract. The Noetherian type of topological spaces is introduced. Connections between the Noetherian type and other cardinal functions of topological spaces are obtained.

Keywords: Noetherian type, rank weight

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Let X be a topological space and $\mathcal{B}$ be an open family in X. For a set G let us denote by $\mathcal{B}_G$ the family $\{B \in \mathcal{B} : G \subset B\}$.

We define the *Noetherian type* of $\mathcal{B}$ as the cardinal

$$Nt(\mathcal{B}) = \min\{\alpha : \alpha \text{ is an infinite cardinal} \\ \text{and } |\mathcal{B}_G| < \alpha \text{ for every nonempty open set } G \subset X\}.$$

We define the *lower Noetherian type* of $\mathcal{B}$ as the cardinal

$$lNt(\mathcal{B}) = \sup\{|\mathcal{B}_G| : G \text{ is a nonempty open subset of } X\}.\omega.$$

The cardinal

$$\min\{Nt(\mathcal{B}) : \mathcal{B} \text{ is a base of the space } X\}$$

is called the *Noetherian type* of X and is denoted by $Nt(X)$.

The cardinal

$$\min\{lNt(\mathcal{B}) : \mathcal{B} \text{ is a base of the space } X\}$$

is called the *lower Noetherian type* of X and is denoted by $lNt(X)$.

Considering in the last definitions a π-base in place of a base we obtain definitions of the *Noetherian π-type* and the *lower Noetherian π-type* of X, which are denoted by $N\pi t(X)$ and $lN\pi t(X)$ respectively.

Now let $\mathcal{B}$ be a family of sets. The cardinal

$$rank(\mathcal{B}) = \sup\{|\mathcal{B}'| : \mathcal{B}' \subset \mathcal{B}, \bigcap \mathcal{B}' \neq \emptyset \text{ and } \mathcal{B}' \text{ is an antichain} \\ \text{(by the set theoretic inclusion)}\}$$

is called the *rank of the family B*. We define the *rank weight of a topological space X* as the cardinal

$$w_r(X) = \min\{rank(\mathcal{B}) : \mathcal{B} \text{ is a base of } X\}.\omega.$$

Analogously the *rank π-weight* of X is defined as the cardinal

$$\pi w_r(X) = \min\{rank(\mathcal{B}) : \mathcal{B} \text{ is a } \pi\text{-base of } X\}.\omega.$$

Lemma 1. *Let X be a topological space. Then*
$$w(X) = lNt(X).\pi w(X) \text{ and } lNt(X).\chi(X) = lNt(X).\pi\chi(X).$$

PROOF: Clearly $lNt(X).\pi w(X) \leq w(X)$. Let $\mathcal{B}$ be a base of X such that $|\mathcal{B}| = w(X)$ and $lNt(\mathcal{B}) = lNt(X)$. Let $\mathcal{H}$ be a π-base of X such that $|\mathcal{H}| = \pi w(X)$. For every set $B \in \mathcal{B}$ choose a set $H \in \mathcal{H}$ such that $H \subset B$. We obtain a mapping $f : \mathcal{B} \to \mathcal{H}$. Since $|f^{-1}.(H)| \leq lNt(X)$ and $\mathcal{B} = \bigcup\{f^{-1}.(H) : H \in rng(f)\}$ it follows that $|\mathcal{B}| \leq lNt(X).\pi w(X)$. Consequently, $w(X) = lNt(X).\pi w(X)$. Clearly for every point $x \in X$ we have $\chi(x, X) \leq \pi\chi(x, X).lNt(X)$. Consequently, $\chi(X) \leq \pi\chi(X).lNt(X)$ and $\chi(X).lNt(X) \leq \pi\chi(X).lNt(X)$. The inverse inequality is trivial. □

Lemma 2. *Let X be a topological space. If $Nt(X) < \chi(X)$ then $w_r(X) = \chi(X)$.*

PROOF: Clearly for every base $\mathcal{B}$ of the space X and every point $x \in X$ we have $ord(x, \mathcal{B}) \leq \chi(x, X).lNt(\mathcal{B})$. Consequently, provided that $Nt(X) < \chi(X)$ we have that $w_r(X) \leq \chi(X)$. Now let $\mathcal{B}$ be an arbitrary base of X and let $\mathcal{B}'$ be a base of X such that $Nt(\mathcal{B}') = Nt(X)$. Take an arbitrary cardinal τ such that $Nt(X) \leq \tau < \chi(X)$. Choose a point $x \in X$ such that $\chi(x, X) \geq \tau^+$. Let $\mathcal{B}'_x = \{B' \in \mathcal{B}' : x \in B'\}$ and for every set $B' \in \mathcal{B}'_x$ choose a set $B'' \subset B'$ such that $B'' \in \mathcal{B}_x = \{B \in \mathcal{B} : x \in B\}$. We obtain a family $\mathcal{B}'' \subset \mathcal{B}$ such that $Nt(\mathcal{B}'') \leq Nt(X)$ and $x \in \bigcap \mathcal{B}''$. Besides it is evident that $|\mathcal{B}''| \geq \tau^+$. Choose a maximal antichain $\mathcal{B}''_0$ out of $\mathcal{B}''$. Consider the family of all sets $B \in \mathcal{B}''$ such that B is contained as a proper subset in some set belonging to $\mathcal{B}''_0$. Choose a maximal antichain $\mathcal{B}''_1$ out of this family. Continuing this process we obtain a $Nt(X)$-sequence $(\mathcal{B}''_\xi : \xi < Nt(X))$ such that every its element $\mathcal{B}''_\xi$ is a maximal antichain in the family $\{B \in \mathcal{B}'' : B$ is contained as a proper subset in some set belonging to $\mathcal{B}''_\gamma$ for every $\gamma < \xi\}$. The family $\mathcal{B}''_{Nt(X)} = \bigcup\{\mathcal{B}''_\xi : \xi < Nt(X)\}$ is dense in $\mathcal{B}''$. Therefore it is a local base of x in X. Then $|\mathcal{B}''_{Nt(X)}| \geq \tau^+$ and there exists $\xi < Nt(X)$ such that $|\mathcal{B}''_\xi| \geq \tau^+$. Hence, $rank(\mathcal{B}) \geq \tau^+$. Since it is true for every cardinal τ, such that $Nt(X) \leq \tau < \chi(X)$, we have that $rank(\mathcal{B}) \geq \chi(X)$. Because $\mathcal{B}$ is an arbitrary base of X it follows that $w_r(X) \geq \chi(X)$. □

Definitions. A topological space X is called a *Noetherian space* provided that $Nt(X) = \omega$. A topological space X is called a *weakly Noetherian space* provided that $lNt(X) = \omega$.

Corollary 1. *If X is a Noetherian space then $w_r(X) = \chi(X)$.*

Example 1. Let $X = \omega_1 \bigcup\{\omega_1\}$. Introduce a topology on X as the following. Let every point of ω_1 be isolated. A base of neighborhoods of the point ω_1 is defined as the family $\{(\xi, \omega_1] : \xi < \omega_1\}$ where $(\xi, \omega_1] = \{\eta \leq \omega_1 : \xi < \eta\}$. Evidently X is a regular Lindelöf space for which $lNt(X) = \omega$, $Nt(X) = \omega_1$, $\chi(X) = \omega_1$, and $w_r(X) = \omega$.

Lemma 3. *Let X be a compact Hausdorff space. If $Nt(X)$ is a regular cardinal and $w(X) = Nt(X)$ then $w_r(X) = w(X)$.*

PROOF: Let us assume that $w_r(X) < w(X)$. The space X contains an everywhere dense subset A such that $\pi\chi(x, X) \leq w_r(X)$ for every point $x \in A$ ([2]). Choose a base $\mathcal{B}$ of the space X such that $Nt(\mathcal{B}) = Nt(X)$. Then $ord(x, \mathcal{B}) < Nt(X)$ for every point $x \in A$. By induction in just the same way as in [6], construct a sequence $(S_n : n \in \omega)$ of subsets of the set A such that the following conditions are fulfilled:

(1) if $n, m \in \omega$ and $n < m$ then $S_n \subset S_m$;
(2) $|S_n| < w(X)$ for every $n \in \omega$;
(3) if $n \in \omega$, $\mathcal{B}'$ is a finite subfamily of the family $\{B \in \mathcal{B} : B \bigcap S_n \neq \emptyset\}$ and $A \setminus \bigcup \mathcal{B}' \neq \emptyset$ then $S_{n+1} \bigcap (A \setminus \bigcup \mathcal{B}') \neq \emptyset$.

The set $S = \bigcup\{S_n : n \in \omega\}$ by (3) is an everywhere dense subset of X. In addition $|S| < w(X)$ and $ord(x, \mathcal{B}) < w(X)$ for every point $x \in S$. Then $w(X) < w(X)$, a contradiction. It follows that $w_r(X) = w(X)$. □

Theorem 1. *Let X be a compact Hausdorff space. Then*

$$w(X) = lNt(X).\pi w(X) = lNt(X).\pi\chi(X) = lNt(X).t(X) = lNt(X).\chi(X) = lNt(X).w_r(X) = lNt(X).s(X) = lNt(X).hd(X) = lNt(X).hL(X).$$

PROOF: Since X is a compact Hausdorff space it follows from [7] that $\pi\chi(X) \leq t(X)$. By Lemma 1 it implies the first, the third, and the fourth equations. Further, if $\mathcal{B}$ is a base of the space X and $lNt(\mathcal{B}) = lNt(X)$ then $ord(x, \mathcal{B}) \leq lNt(X).\pi\chi(X)$ for every point $x \in X$. By Theorem Mishenko then $|\mathcal{B}| \leq lNt(X).\pi\chi(X)$. It implies the second equation. Now let us assume that $lNt(X).w_r(X) < w(X)$. Then $w_r(X) < w(X)$ and $lNt(X) < w(X)$. By the fourth equation it implies that $w(X) = \chi(X)$ and by Lemma 2 we have $Nt(X) \geq w(X)$. But $w(X) > lNt(X)$, hence $Nt(X) = w(X) = lNt(X)^+$. Then by Lemma 3 we have that $w_r(X) = w(X)$. It is a contradiction, hence $w(X) = lNt(X).w_r(X)$. The other equations are consequences of the inequality $t(X) \leq s(X)$ for compact Hausdorff spaces ([1]). □

Corollary 2. *Let X be a Hausdorff compact weakly Noetherian space. Then $w(X) = \pi w(X) = w_r(X) = \chi(X) = \pi\chi(X) = t(X) = s(X) = hd(X) = hL(X)$.*

Corollary 3. *Let X be a Hausdorff locally compact weakly Noetherian space. Then $w_r(X) = \chi(X)$.*

Corollary 4. *Let X be a compact Hausdorff space. If $Nt(X)$ is a weakly inaccessible cardinal then $w_r(X) = w(X)$.*

Example 2. Let X denote the "two arrows" space. It is known that $w_r(X) = \omega$ ([3]). By Theorem 1 it implies that $lNt(X) = w(X) = 2^\omega$ and because $cf(2^\omega) > \omega$ we get that $Nt(X) = (2^\omega)^+$.

Example 3. Let $X = I^\tau$ where I is the unit segment. Because $lNt(X) = Nt(X) = \omega$ ([5]), it follows from Theorem 1 that $w_r(X) = \tau.\omega$.

Example 4. The space of ordinals $X = \omega_1 \bigcup \{\omega_1\}$ is not weakly Noetherian because $\chi(X) = \omega_1$ and $w_r(X) = \omega$ ([2]).

Theorem 2. *For a compact Hausdorff space X the following conditions are equivalent:*

(1) $w(X) = lNt(X)$;
(2) $lNt(Y) \leq lNt(X)$ *for every subspace Y of X;*
(3) $lNt(Y) \leq lNt(X)$ *for every closed subspace Y of X.*

Proof: Evidently it is sufficient to prove the implication (3) $\to$ (1). Let the condition (3) be fulfilled and suppose that $w(X) > lNt(X)$. Then by Theorem 1 there exists a discrete set $Y \subset X$ such that $|Y| > lNt(X)$. By the assumption $lNt(clY) \leq lNt(X)$. Choose a base $\mathcal{B}$ of the space clY such that $lNt(\mathcal{B}) = lNt(clY)$. Because $ord(y, \mathcal{B}) \leq lNt(clY)$ for every point y belonging to Y, we have by [6] that $w(clY) \leq lNt(clY)$. Hence $|Y| \leq lNt(X)$. It is a contradiction, consequently, $w(X) = lNt(X)$. $\square$

Theorem 3. *If Y is an open or canonical closed subspace of a space X then $lNt(Y) \leq lNt(X)$. If X is a Hausdorff space and $\mathcal{F}$ is a family of compact subspaces of X having in X the character $\leq lNt(X)$, then if $\bigcup \mathcal{F} \subset Y \subset cl \bigcup \mathcal{F}$ then $lNt(Y) \leq lNt(X)$.*

Proof: The first assertion is trivial. To prove the second assertion, take $\mathcal{E}_F = \{A \subset F : A \neq \emptyset, \chi(A, X) \leq lNt(X)\}$ for every $F \in \mathcal{F}$ and put $\mathcal{E} = \bigcup \{\mathcal{E}_F : F \in \mathcal{F}\}$. Let $\mathcal{B}$ be a base of X such that $lNt(\mathcal{B}) = lNt(X)$. If $B \in \mathcal{B}$ and B contains a nonempty open in Y set P, then there exists $E \in \mathcal{E}$ such that $E \subset P$.Because $\chi(E, X) \leq lNt(\mathcal{B})$, it follows that $|\{B \in \mathcal{B} : B \supset E\}|$ is not more than $lNt(\mathcal{B})$. This implies that $lNt(\mathcal{B}|Y) \leq lNt(\mathcal{B})$ and hence $lNt(Y) \leq lNt(X)$. $\square$

Theorem 4. *Let $\{Z_\alpha : \alpha \in A\}$ be a family of topological spaces and $\prod \{Z_\alpha : \alpha \in A\}$ is denoted by Z. Then*

$$Nt(Z) \leq \sup\{Nt(Z_\alpha) : \alpha \in A\} \quad \textit{and} \quad lNt(Z) \leq \sup\{lNt(Z_\alpha) : \alpha \in A\}.$$

Proof: To prove this, choose a base $\mathcal{B}_\alpha$ of the space Z_α for every $\alpha \in A$ such that $Nt(\mathcal{B}_\alpha) = Nt(Z_\alpha)$. It is easy to see that a base of Z consisting of sets of the form $B_{\alpha_1} \times \dots \times B_{\alpha_n} \times \prod \{Z_\alpha : \alpha \in A \setminus \{\alpha_1, \dots, \alpha_n\}\}$, where $B_{\alpha_i} \in \mathcal{B}_{\alpha_i}$ for $i = 1, \dots, n$, have the Noetherian type that is not greater than $\sup\{Nt(Z_\alpha) : \alpha \in A\}$. Analogously it can be proved that $lNt(Z) \leq \sup\{lNt(Z_\alpha) : \alpha \in A\}$. $\square$

Remark. The following theorem, which has been proved in [4], is an essential supplement of Theorem 4:

$$\textit{if } |A| \geq \sup\{w(Z_\alpha) : \alpha \in A\} \textit{ then } Nt(Z) = \omega.$$

Theorem 5. *Let X be a topological space such that $\pi w(X) > N\pi t(X)$ and let κ be a cardinal such that κ^+ is a calibre of X. If $\pi w(X) > \kappa$ then $\pi w_r(X) > \kappa$.*

PROOF: Let $\mathcal{H}$ be a π-base of X. Choose a π-base $\mathcal{H}'$ of X such that $Nt(\mathcal{H}') = N\pi t(X)$. Now choose a cardinal τ such that $N\pi t(X).\kappa \leq \tau < \pi w(X)$. Take a mapping $f : \mathcal{H}' \to \mathcal{H}$ such that $f(H') \subset H'$ and put $\mathcal{H}'' = rng(f)$. It is evident that $\mathcal{H}''$ is a π-base of X. Also it is clear that $|\mathcal{H}''| \geq \pi w(X)$ and $Nt(\mathcal{H}'') = N\pi t(X)$. Consequently $|\mathcal{H}''| \geq \tau^+$. By induction construct a $N\pi t$-sequence $(\mathcal{H}''_\xi : \xi < N\pi t(X))$ of subsets of $\mathcal{H}''$ such that the following conditions are fulfilled:

(1) $\mathcal{H}''_0$ is a maximal antichain (by inclusion) in $\mathcal{H}''$;
(2) if $\xi > 0$ then $\mathcal{H}''_\xi$ is a maximal antichain in the family $\{H \in \mathcal{H}'' :$ for every $\eta < \xi$ there exists $H' \in \mathcal{H}''_\eta$ such that $H \subset H'$ and $H \neq H'\}$.

Put $\mathcal{H}''_{N\pi t(X)} = \bigcup\{\mathcal{H}''_\xi : \xi < N\pi t(X)\}$. It is easy to see that $\mathcal{H}''_{N\pi t(X)}$ is a π-base of X. This implies that $|\mathcal{H}''_{N\pi t(X)}| \geq \tau^+$. Because $N\pi t(X) \leq \tau$, there exists $\xi < N\pi t(X)$ such that $|\mathcal{H}''_\xi| \geq \tau^+$. Since $\kappa \leq \tau$, there exists $\widehat{\mathcal{H}}_\xi \subset \mathcal{H}''_\xi$ such that $|\widehat{\mathcal{H}}_\xi| > \kappa$ and $\bigcap \widehat{\mathcal{H}}_\xi \neq \emptyset$. Because $\mathcal{H}$ is an arbitrary π-base we get that $\pi w_r(X) > \kappa$.

Corollary 5. *Let X be a topological space such that $\pi w(X) > N\pi t(X)$. Then the following assertions are fulfilled:*

(a) *if $\pi w(X) > d(X)$ then $\pi w_r(X) = \pi w(X)$;*
(b) *if $\pi w(X) > sh(X)$ then $\pi w_r(X) > sh(X)$.*

The following problem was raised by P. Bir'ukov:

Is there a compact Hausdorff space X such that $|X| > 2^{w_r(X)}$?

In view of the above mentioned assertions the space may be encountered only where $Nt(X) = w(X)^+$ or $Nt(X) = w(X)$ is a singular cardinal.

Corollary 6 (MA)**.** *Let X be a compact Hausdorff space. If $Nt(X) \leq 2^\omega$ then $|X| \leq 2^{w_r(X)}$.*

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