# A two-weight inequality for the Bessel potential operator

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*Abstract.* Necessary conditions and sufficient conditions are derived in order that Bessel potential operator $J_{s,\lambda}$ is bounded from the weighted Lebesgue spaces $L^p_v = L^p(\mathbb{R}^n, v(x)dx)$ into $L^q_u$ when $1 < p \leq q < \infty$.

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## §1. Introduction

The Bessel potential operator $J_{s,\lambda}$ is defined via the Fourier transform by

$$(\widehat{J_{s,\lambda}f})(\xi) = \Big(4\pi^2|\xi|^2 + \lambda^{\frac{1}{s}}\Big)^{-\frac{s}{2}}\widehat{f}(\xi)$$

where $\lambda > 0$, $0 < s < n$, $n \in \mathbb{N}^*$ and $\widehat{g}(\xi) = \int_{y\in\mathbb{R}^n} e^{-2i\pi y\cdot\xi}g(y)\,dy$. Our purpose is to characterize the weight functions $u(\cdot)$ and $v(\cdot)$ for which there is $C > 0$ such that

$$(1.1) \qquad \Big(\int_{x\in\mathbb{R}^n}(J_{s,\lambda}f)^q(x)u(x)\,dx\Big)^{\frac{1}{q}} \leq C\Big(\int_{x\in\mathbb{R}^n} f^p(x)v(x)\,dx\Big)^{\frac{1}{p}} \quad \text{for all} \quad f(\cdot) \geq 0,$$

and with $1 < p \leq q < \infty$. A weight means a nonnegative locally integrable function. This inequality implies $J_{s,\lambda}$ is bounded from the weighted Lebesgue space $L^p_v = L^p(\mathbb{R}^n, vdx)$ into $L^q_u$. For the convenience (1.1) will also be denoted by $J_{s,\lambda} : L^p_v \to L^q_u$.

Inequality (1.1) plays a fundamental role in Analysis since it is closely connected with spectral properties of Schrödinger operators [Ch-Wh], [Ke-Sa] and it leads to applications in partial differential equations ([Ad-Pi], [Ma-Ve]), theory of Sobolev spaces ([Ma]), complex analysis, etc. For instance estimate like

$$\int_{y\in\mathbb{R}^n} g^p(y)u(y)\,dy \leq c\int_{y\in\mathbb{R}^n}\Big((-\Delta + \lambda^{\frac{1}{s}})^{\frac{s}{2}}g\Big)^p(y)v(y)\,dy \quad \text{for all smooth functions} \quad g(\cdot),$$

which also appears in partial differential equation related to the operators $(-\Delta+\lambda^{\frac{1}{s}})^{\frac{s}{2}}$, can be derived from $J_{s,\lambda}: L^p_u \to L^p_u$, since $((-\Delta+\lambda^{\frac{1}{s}})^{\frac{s}{2}} J_{s,\lambda})g = (J_{s,\lambda}(-\Delta+\lambda^{\frac{1}{s}})^{\frac{s}{2}} g) = g$.

Compared to the Riesz potential operators $I_s$, $0 < s < n$, defined by

$$(I_s f)(x) = \int_{y\in\mathbb{R}^n} |x-y|^{s-n} f(y)\, dy,$$

few works (see for instance [Ad1], [Sc]) are devoted to the study of $J_{s,\lambda}: L^p_v \to L^q_u$; and people had to be content oneself on $J_{s,\lambda} \le cI_s$ so that a condition for $I_s : L^p_v \to L^q_u$ is also right for $J_{s,\lambda}: L^p_v \to L^q_u$. The strongest up to date results are those of Kerman-Sawyer [Ke-Sa], and Maz'ya-Verbitsky [Ma-Ve]. Indeed a characterization of weights $u(\cdot)$ for which $J_{s,\lambda}: L^p_1 \to L^q_u$ (i.e. $v(\cdot) = 1$) is given in [Ke-Sa], and investigations of weights $w(\cdot)$ which ensure $J_{s,1}: L^p_1 \to L^q_{(J_{s,1}w)^{p'}}$ are presented in [Ma-Ve]. Although a necessary and sufficient condition for $I_s : L^p_v \to L^q_u$ is known ([Sa-Wh]), the analog condition characterizing $J_{s,\lambda}: L^p_v \to L^q_u$ is not clear in the literature. Consequently our intention is to fill this gap.

Although a result due to Sawyer and Wheeden [Sa-Wh] related to $T : L^p_v \to L^q_u$, where $T$ is a potential operator given by a positive kernel $K(x,y)$, could be applied directly to get $J_{s,\lambda}: L^p_v \to L^q_u$, the fast decrease at infinity of the kernel $K_{s,\lambda}$ of $J_{s,\lambda}$ (see §3) leads to conditions more refined than the standard ones used for $T$. Therefore the boundedness $J_{s,\lambda}: L^p_v \to L^q_u$ deserves its own study which is performed in this paper.

The main results are presented in the next section. And §3 is devoted to basic lemmas used for the results whose proofs are given in §4.

## §2. The main results

In this paper we always assume:

$$0 < s < n, \quad \lambda > 0, \quad 1 < p \le q < \infty, \quad p' = \frac{p}{p-1}, \quad q' = \frac{q}{q-1},$$

$$u(\cdot),\ v(\cdot) \text{ are weight functions with } \sigma(\cdot) = v^{-\frac{1}{p-1}}(\cdot) \in L^1_{loc}(\mathbb{R}^n, dx).$$

Our first main result is

**Theorem 1.** *The boundedness $J_{s,\lambda}: L^p_v \to L^q_u$ holds if and only if there are $C, c > 0$ such that*

$$\text{(2.1)}\quad \Big(\int_Q (I_s g)^q(x) u(x)\, dx\Big)^{\frac{1}{q}} \le C\Big(\int_{(3Q)} g^p(x) v(x)\, dx\Big)^{\frac{1}{p}}$$

*for each $g(\cdot) \ge 0$ whose support is $3Q$*

*and*
(2.2)
$$\Big(\int_{y\notin(3Q)} |x_Q-y|^{(s-n)p'}\exp\{-c\lambda^{\frac{1}{2s}}|x_Q-y|\}\sigma(y)\,dy\Big)^{\frac{1}{p'}}\times\Big(\int_{y\in Q} u(y)\,dy\Big)^{\frac{1}{q}}\le C$$
*for all cubes $Q$ centered at $x_Q$ and with* $|Q|^{\frac{1}{n}}=\lambda^{-\frac{1}{2s}}$.

Remind that a cube $Q$ (centered at $x_Q=(x_i)\in\mathbb{R}^n$) is a product of $n$ intervals of the form $[x_i-l,x_i+l]$ where $l>0$. And for $R>0$, $RQ$ is the cube given by the product of $[x_i-Rl,x_i+Rl]$. The Lebesgue measure $\int_{y\in Q}dy$ of $Q$ is denoted by $|Q|$.

Next we give some remarks whose proofs are given in §4.

**Remarks.**

(1) A necessary condition for $J_{s,\lambda}:L^p_v\to L^q_u$, which is consequently assumed is $0\le\frac{s}{n}+\frac{1}{q}-\frac{1}{p}$. So for $1<p<\frac{n}{s}$ this boundedness has a sense for $p\le q\le p^*$ with $\frac{1}{p^*}=\frac{1}{p}-\frac{s}{n}$.

(2) Theorem 1 remains true if in conditions (2.1) and (2.2) the cubes $Q$ are chosen such that $|Q|^{\frac{1}{n}}\approx\lambda^{-\frac{1}{2s}}$. This equivalence means $c_1\lambda^{-\frac{1}{2s}}\le|Q|^{\frac{1}{n}}\le c_2\lambda^{-\frac{1}{2s}}$ for some fixed constants $c-1,c_2>0$.

(3) Condition (2.1) in Theorem 1 can be replaced by

$$\Big(\int_{Q_2}(I_sh)^q(x)u(x)\,dx\Big)^{\frac{1}{q}}\le C\Big(\int_{Q_1}h^p(x)v(x)\,dx\Big)^{\frac{1}{p}}\tag{2.3}$$

for each function $h(\cdot)\ge0$ whose support is $Q_1$; and where $Q_1$ and $Q_2$ are cubes with $|Q_1|^{\frac{1}{n}}=|Q_2|^{\frac{1}{n}}=$ [resp. $\approx$] $\lambda^{-\frac{1}{2s}}$ and $\overline{Q}_1\cap\overline{Q}_2\ne\emptyset$.

(4) Also the condition (2.2) can be replaced by

$$\exp(-cm)m^{(s-n)}(\lambda^{-\frac{1}{2s}})^{(s-n)}\Big(\int_{y\in Q_2}\sigma(y)\,dy\Big)^{\frac{1}{p'}}\Big(\int_{x\in Q_1}u(x)\,dx\Big)^{\frac{1}{q}}\le C\tag{2.4}$$

for all integers $m\ge4$ and cubes $Q_1$, $Q_2$ with $|Q_1|^{\frac{1}{n}}=|Q_2|^{\frac{1}{n}}=\lambda^{-\frac{1}{2s}}$ and $\operatorname{dist}(Q_1,Q_2)=\inf\{|x-y|;y\in Q_1,x\in Q_2\}\approx(m\lambda^{-\frac{1}{2s}})>0$. So here we are in the case $\overline{Q}_1\cap\overline{Q}_2=\emptyset$.

(5) The weight function $w(\cdot)$ satisfies the doubling condition if $\int_{(2Q)}w(y)\,dy\le C\int_Q w(y)\,dy$ for some $C>0$ and all cubes $Q$. If one of $u(\cdot)$ and $\sigma(\cdot)=v^{-\frac{1}{p-1}}(\cdot)$ is a doubling weight then an easy condition which ensures (2.4) is

$$(\lambda^{-\frac{1}{2s}})^{(s-n)}\Big(\int_{y\in Q}\sigma(y)\,dy\Big)^{\frac{1}{p'}}\Big(\int_{x\in Q}u(x)\,dx\Big)^{\frac{1}{q}}\le C\tag{2.5}$$
for all cubes $Q$ with $|Q|^{\frac{1}{n}}=\lambda^{-\frac{1}{2s}}$.

Now we can state the following

**Theorem 2.** *Assume that one of $u(\cdot)$ and $\sigma(\cdot) = v^{-\frac{1}{p-1}}(\cdot)$ is a doubling weight function. Then $J_{s,\lambda} : L^p_v \to L^q_u$ if and only if the condition* (2.3) *is satisfied.*

With this theorem and the well known results on the (global) boundedness $I_s : L^p_v \to L^q_u$, we obtain the following more useful statement.

**Proposition 3.** *Let $u(\cdot)$ and $\sigma(\cdot)$ as in Theorem 2. Then $J_{s,\lambda} : L^p_v \to L^q_u$ whenever, for some $t > 1$ with $1 < t < \frac{\frac{1}{q}-\frac{1}{p}+1}{\frac{s}{n}+\frac{1}{q}-\frac{1}{p}}$,*

$$(2.6)\quad |Q|^{\frac{s}{n}+\frac{1}{q}-\frac{1}{p}}\Big(\frac{1}{|Q|}\int_Q u^t(y)\,dy\Big)^{\frac{1}{tq}}\Big(\frac{1}{|Q|}\int_Q \sigma^t(y)\,dy\Big)^{\frac{1}{tp'}} \le A$$
$$\text{for all cubes } Q \text{ with } |Q|^{\frac{1}{n}} \le \lambda^{-\frac{1}{2s}}.$$

*Moreover for $u(\cdot)$ and $\sigma(\cdot)$ satisfying $A_\infty$ condition, then $J_{s,\lambda} : L^p_v \to L^q_u$ if and only if*

$$(2.7)\quad |Q|^{\frac{s}{n}+\frac{1}{q}-\frac{1}{p}}\Big(\frac{1}{|Q|}\int_Q u(y)\,dy\Big)^{\frac{1}{q}}\Big(\frac{1}{|Q|}\int_Q \sigma(y)\,dy\Big)^{\frac{1}{p'}} \le A$$
$$\text{for all cubes } Q \text{ with } |Q|^{\frac{1}{n}} \le \lambda^{-\frac{1}{2s}}.$$

Conditions (2.6) and (2.7) have nontrivial senses since $\frac{s}{n}+\frac{1}{q}-\frac{1}{p} \ge 0$ is assumed (see Remark 1). Recall that $w(\cdot)$ satisfies the $A_\infty$ condition if, for some $r > 1$: $|Q|^{\frac{s}{n}-1}(\frac{1}{|Q|}\int_Q w(y)\,dy)^{\frac{1}{r}}(\frac{1}{|Q|}\int_Q w^{1-r'}(y)\,dy)^{\frac{1}{r'}} \le c$ for all cubes $Q$.

As an example, for each weight function $w(\cdot)$ then $J_{s,1} : L^p_1 \to L^p_{(J_{s,1}w)^{p'}}$ whenever for a $t > 1$: $|Q|^{\frac{s}{n}}(\frac{1}{|Q|}\int_Q (J_{s,1}w)^{tp'}(y)\,dy)^{\frac{1}{tp}} \le C$ for all cubes $Q$ with $|Q|^{\frac{1}{n}} \le 1$. Such a result was proved by a different method in [Ma-Ve]. We will present below another application of Proposition 3.

Although this result yields sufficient condition for $J_{s,\lambda} : L^p_v \to L^q_u$, we are able to state a necessary and sufficient condition for this embedding. However, compared with (2.6), the corresponding characterizing condition is not easy to check in general.

**Proposition 4.** *Let $u(\cdot)$, $\sigma(\cdot)$ as in the hypotheses of Theorem 2. Then $J_{s,\lambda} : L^p_v \to L^q_u$ if and only if for some $C > 0$:*

$$(2.8)\qquad \Big(\int_{(3Q)} (I_s\sigma\mathbb{I}_Q)^q(y)u(y)\,dy\Big)^{\frac{1}{q}} \le C\Big(\int_Q \sigma(y)\,dy\Big)^{\frac{1}{p}}$$

*and*

$$(2.8^*)\qquad \Big(\int_{(3Q)} (I_s u\mathbb{I}_Q)^{p'}(y)\sigma(y)\,dy\Big)^{\frac{1}{p'}} \le C\Big(\int_Q u(y)\,dy\Big)^{\frac{1}{q'}}$$

*for all dyadic cubes $Q$ with $|Q|^{\frac{1}{n}} \leq \lambda^{-\frac{1}{2s}}$.*

A dyadic cube $Q$ is a product of $n$ intervals of the form $[2^k(x_i - l), 2^k(x_i + l)]$ where $l > 0$, and $\mathbb{I}_E(\cdot)$ is the characteristic function of the measurable set $E$.

Let $w(\cdot)$ be a weight function, and $0 < s < \frac{n}{pr}$ with $1 < r < \infty$. By a result due to Adams [Ad2], there is $C > 0$ such that

$$(2.9) \qquad \int_{x\in\mathbb{R}^n} (I_s f)^p(x) w(x)\,dx \leq C \int_{x\in\mathbb{R}^n} f^p(x)\big(M_{spr} w^r\big)^{\frac{1}{r}}(x)\,dx \quad \text{for all } f(\cdot) \geq 0.$$

Here $M_\beta$, $0 \leq \beta < n$, is the usual fractional maximal operator defined as $(M_\beta g)(x) = \sup\{|Q|^{\frac{\beta}{n}-1} \int_Q |g(y)|\,dy; Q \ni x\}$. Since $J_{s,\lambda}$ is pointwise majorized by $I_s$ then inequality (2.9) remains true with $I_s$ replaced by $J_{s,\lambda}$, and it becomes natural to ask whether (2.9) holds with $J_{s,\lambda}$ and the weight in the second member defined by a smaller operator than $M_\beta$. Therefore we will be interested to get an inequality like

$$(2.10) \qquad \int_{x\in\mathbb{R}^n} (J_{s,\lambda} f)^p(x) w(x)\,dx \leq C \int_{x\in\mathbb{R}^n} f^p(x)\big(M_{spr,\lambda} w^r\big)^{\frac{1}{r}}(x)\,dx \quad \text{for all } f(\cdot) \geq 0,$$

where $(M_{\beta,\lambda} g)(x) = \sup\{|Q|^{\frac{\beta}{n}-1} \int_Q |g(y)|\,dy; Q \ni x \text{ and } |Q|^{\frac{1}{n}} \leq \lambda^{-\frac{1}{2s}}\}$.

Unfortunately (2.10) is false in general. Indeed take $n = 1$, $\lambda = 1$, $w(\cdot) = \mathbb{I}_{[0,1]}(\cdot)$ and $f(\cdot) = \mathbb{I}_{[3,4]}(\cdot)$. Clearly $(M_{\beta,1} w^r)(x) = 0$ for all $|x| \geq 3$ and $\int_{x\in\mathbb{R}^n} f^p(x)(M_{\beta,1} w^r)^{\frac{1}{r}}(x)\,dx = 0$. On the other hand $(J_{s,\lambda} f)(\cdot) \approx (I_s f)(\cdot) \approx 1$, on $[0,1]$ and $\int_{x\in\mathbb{R}^n} (J_{s,\lambda} f)^p(x) w(x)\,dx \approx 1$.

Consequently to get (2.10), some restriction on the weight function $w(\cdot)$ is needed. Really, by Proposition 3, we have

**Corollary 5.** *Let $r > 1$, $0 < s < \frac{n}{pr}$ and $\lambda > 0$. Suppose that one of $w(\cdot)$ and $\sigma(\cdot)$ is a doubling weight function, where $\sigma(\cdot) = (M_{spr,\lambda} w^r)^{\frac{1}{r}(1-p')}(\cdot)$. Then there is $C > 0$ for which (2.10) is true. This constant $C$ depends on $n, p, s$ and the constant on the doubling condition.*

## §3. Preliminaries lemmas

As we have alluded in §1, by arguments in [Ar-Sm] the kernel $K_{s,\lambda}(\cdot)$ of $J_{s,\lambda}$ satisfies

$$(3.1) \quad K_{s,\lambda}(R) \approx R^{s-n} \text{ if } R \leq \lambda^{-\frac{1}{2s}}, \text{ else } K_{s,\lambda}(R) \approx R^{\frac{1}{2}(s-n+1)} \exp(-R\lambda^{\frac{1}{2s}}).$$

These equivalences lead to a better knowledge of the behaviour of $J_{s,\lambda}$.

**Lemma 1.** *Let $0 \le s < n$, $\lambda > 0$. Then*

$$(3.2) \qquad C_1(T_{s,c^{2s}\lambda}f)(\cdot) \le (J_{s,\lambda}f)(\cdot) \le C_2(T_{s,c^{-2s}\lambda}f)(\cdot).$$

*Here $C_1$, $C_2$, $c$ depend only on $n$ and $s$. And the operator $T_{s,\mu}$ ($\mu > 0$) is defined as $(T_{s,\mu}f)(x) = \int_{y\in\mathbb{R}^n} |x-y|^{(s-n)} \exp\{-\mu^{\frac{1}{2s}}|x-y|\} f(y)\,dy$.*

Obviously $(J_{s,\lambda}f)(\cdot) \le C(I_s f)(\cdot)$.

**Lemma 2.** *Let $L > 0$. One can find a family $(Q_l)_{l\in\mathcal{I}}$ of cubes with $|Q_l|^{\frac{1}{n}} = L$ and disjoint interiors such that*

$$(3.3) \qquad \mathbb{R}^n = \bigcup_{l\in\mathcal{I}} Q_l,$$

*and there is an integer $N > 1$ (depending only on $n$) for which the following holds:*

(3.4)
$$(3Q_l) = \bigcup_{l'\in\mathcal{I}_l} Q_{l'} \text{ where } l' \in \mathcal{I}_l \text{ if } \overline{Q}_l \cap \overline{Q}_{l'} \neq \emptyset, \text{ and } \operatorname{card}\{l; l' \in \mathcal{I}_l\} \le N;$$

(3.5)
$$(3Q_l)^c = \{y; y \notin (3Q_l)\} = \bigcup_{m=4}^{\infty} [(m+1)Q_l \setminus (m-1)Q_l] = \bigcup_{m=4}^{\infty} \bigcup_{j\in\mathcal{J}_{m,l}} Q_j,$$

*where $j \in \mathcal{J}_{m,l}$ iff $\operatorname{dist}(Q_j, Q_l) \approx (mL)$, and $\operatorname{card}\{j; j \in \mathcal{J}_{m,l}\} \le N \times m^n$ or $\operatorname{card}\{l; j \in \mathcal{J}_{m,l}\} \le N \times m^n$;*

$$(3.6) \qquad |x-y| \approx |x_{Q_l} - y| \approx (mL) \text{ for all } x \in Q_l,\ y \in Q_j \text{ and } j \in \mathcal{J}_{m,l};$$

$$(3.7) \qquad \sum_{l\in\mathcal{I}} \mathbb{I}_{(3Q_l)}(\cdot) \le N;$$

$$(3.8) \qquad \sum_{l\in\mathcal{I}} \mathbb{I}_{[(m+1)Q_l\setminus(m-1)Q_l]}(\cdot) \le Nm^n \text{ for each integer } m \ge 4.$$

Proof of Lemma 1: Using the property of the exponential like $\lim_{R\to\infty} R^\alpha \exp\{-\beta R\} = 0$, and estimates (3.1) for $K_{s,\lambda}$ then we can find $C_1, C_2, c > 0$ depending only on $s$ and $n$ such that $C_1 R^{s-n} \exp\{-c\lambda^{\frac{1}{2s}} R\} \le K_{s,\lambda}(R) \le C_2 R^{s-n} \exp\{-c^{-1}\lambda^{\frac{1}{2s}} R\}$ for all $R > 0$. With the definition of the operator $T_{s,\mu}$, these inequalities imply (3.2). $\square$

Proof of Lemma 2: This geometrical lemma will be a consequence of the homogeneity property of the euclidean space $\mathbb{R}^n$. Thus the points (3.3) to (3.6) are standard and can be easily seen.

Inequality (3.7) is a consequence of (3.4) since

$$\sum_{l\in\mathcal{I}} \mathbb{I}_{(3Q_l)}(\cdot) = \sum_{l\in\mathcal{I}} \sum_{l'\in\mathcal{I}_l} \mathbb{I}_{Q_{l'}}(\cdot) = \sum_{l'\in\mathcal{I}} \mathbb{I}_{Q_{l'}}(\cdot) \sum_{l;l'\in\mathcal{I}_l} 1 \le N \sum_{l'\in\mathcal{I}} \mathbb{I}_{Q_{l'}}(\cdot) \le N.$$

Inequality (3.8) comes from the cardinality property (3.5) since

$$\sum_{l\in\mathcal{I}} \mathbb{I}_{[(m+1)Q_l\setminus(m-1)Q_l]}(\cdot) = \sum_{l\in\mathcal{I}} \sum_{j\in\mathcal{J}_{m,l}} \mathbb{I}_{Q_j}(\cdot) = \sum_{j\in\mathcal{I}} \mathbb{I}_{Q_j}(\cdot) \sum_{l;j\in\mathcal{J}_{m,l}} 1$$
$$\le N m^n \sum_{j\in\mathcal{I}} \mathbb{I}_{Q_j}(\cdot) \le N m^n.$$

□

## §4. Proofs of results

Proof of Theorem 1: We begin by the sufficient part. By Lemma 1, the proof of $J_{s,\lambda} : L^p_v \to L^q_u$ is reduced to that of $T_{s,c^{-2s}\lambda} : L^p_v \to L^q_u$. Without a loss of generality it can be assumed that $c = 1$. Take a family of cubes $(Q_l)_{l\in\mathcal{I}}$ with common size $L = \lambda^{-\frac{1}{2s}}$ $(= |Q|^{\frac{1}{n}})$ as in Lemma 2. So for $f(\cdot) \ge 0$ we have

$$\int_{x\in\mathbb{R}^n} (T_{s,\lambda}f)^q(x)u(x)\,dx = \sum_{l\in\mathcal{I}} \int_{Q_l} (T_{s,\lambda}f)^q(x)u(x)\,dx \le C\{\mathcal{S}_1 + \mathcal{S}_2\}$$

where

$$\mathcal{S}_1 = \sum_{l\in\mathcal{I}} \int_{Q_l} (T_{s,\lambda}f\mathbb{I}_{(3Q_l)})^q(x)u(x)\,dx,$$
$$\mathcal{S}_2 = \sum_{l\in\mathcal{I}} \int_{Q_l} (T_{s,\lambda}f\mathbb{I}_{(3Q_l)^c})^q(x)u(x)\,dx,$$

and $C > 0$ is a constant which depends on $n$ and $q$. The estimates for $\mathcal{S}_1$ are done as follows

$$\begin{aligned}
\mathcal{S}_1 &\le \sum_{l\in\mathcal{I}} \int_{Q_l} (I_s f\mathbb{I}_{(3Q_l)})^q(x)u(x)\,dx \quad \text{by the definition of } T_{s,\lambda}\\
&\le C\sum_{l\in\mathcal{I}} \Big(\int_{(3Q_l)} f(x)^p v(x)\,dx\Big)^{\frac{q}{p}} \quad \text{by the condition (2.1)}\\
&\le C\Big(\int_{x\in\mathbb{R}^n} \Big[\sum_{l\in\mathcal{I}} \mathbb{I}_{(3Q_l)}(x)\Big] f(x)^p v(x)\,dx\Big)^{\frac{q}{p}} \quad \text{since } \frac{q}{p} \ge 1\\
&\le C N^{\frac{q}{p}} \Big(\int_{x\in\mathbb{R}^n} f(x)^p v(x)\,dx\Big)^{\frac{q}{p}} \quad \text{by (3.7).}
\end{aligned}$$

Let

$$\mathcal{H}(c,Q) = \Big(\int_{y\in(3Q)^c} |x_Q-y|^{(s-n)p'}\exp\{-cL^{-1}|x_Q-y|\}\sigma(y)\,dy\Big)^{\frac{1}{p'}}\Big(\int_{x\in Q} u(x)\,dx\Big)^{\frac{1}{q}}.$$

Consequently

$$\begin{aligned}
\mathcal{S}_2 &= \sum_{l\in\mathcal{I}}\int_{x\in Q_l}\Big[\int_{y\in(3Q_l)^c}|x-y|^{s-n}\exp\{-L^{-1}|x-y|\}f(y)\,dy\Big]^q u(x)\,dx\\
&\qquad \text{by the definition of } T_{s,\lambda}\\
&\le C\sum_{l\in\mathcal{I}}\Big[\int_{y\in(3Q_l)^c}|x_{Q_l}-y|^{s-n}\exp\{-cL^{-1}|x_{Q_l}-y|\}f(y)\,dy\Big]^q\times\\
&\qquad \times\Big(\int_{x\in Q_l}u(x)\,dx\Big) \quad \text{by property (3.6)}\\
&\le C\sum_{l\in\mathcal{I}}[\mathcal{H}(c,Q_l)]^q\Big(\int_{y\in(3Q_l)^c}\exp\{-cL^{-1}|x_{Q_l}-y|\}f(y)^p v(y)\,dy\Big)^{\frac{q}{p}}\\
&\qquad \text{by the Hölder inequality}\\
&\le CH^q\sum_{l\in\mathcal{I}}\Big(\int_{y\in(3Q_l)^c}\exp\{-cL^{-1}|x_{Q_l}-y|\}f(y)^p v(y)\,dy\Big)^{\frac{q}{p}}\\
&\qquad \text{by the condition (2.2)}\\
&\le CH^q\Big(\sum_{l\in\mathcal{I}}\int_{y\in(3Q_l)^c}\exp\{-cL^{-1}|x_{Q_l}-y|\}f(y)^p v(y)\,dy\Big)^{\frac{q}{p}} \quad \text{since } \frac{q}{p}\ge 1\\
&= CH^q\Big(\sum_{l\in\mathcal{I}}\sum_{m=4}^{\infty}\int_{y\in[((m+1)Q_l)\setminus((m-1)Q_l)]}\exp\{-cL^{-1}|x_{Q_l}-y|\}f(y)^p v(y)\,dx\Big)^{\frac{q}{p}}\\
&\le CH^q\Big(\sum_{m=4}^{\infty}\exp\{-c'm\}\int_{y\in\mathbb{R}^n}\Big[\sum_{l\in\mathcal{I}}\mathbb{I}_{[((m+1)Q_l)\setminus((m-1)Q_l)]}y\Big]f(y)^p v(y)\,dx\Big)^{\frac{q}{p}}\\
&\le NCH^q\Big(\Big[\sum_{m=4}^{\infty}\exp\{-c'm\}\Big]\int_{y\in\mathbb{R}^n}f(y)^p v(y)\,dx\Big)^{\frac{q}{p}} \quad \text{by (3.8)}\\
&\le NC'H^q\Big(\int_{y\in\mathbb{R}^n}f(y)^p v(y)\,dx\Big)^{\frac{q}{p}}\\
&\qquad \text{by the fast decreasing of the exponential function.}
\end{aligned}$$

Conversely suppose $J_{s,\lambda} : L^p_v \to L^q_u$. Then $T_{s,c^{2s}\lambda} : L^p_v \to L^q_u$ (by Lemma 1). Let $Q$ be a cube with $|Q|^{\frac{1}{n}} = \lambda^{-\frac{1}{2s}}$, and $h(\cdot)\ge 0$ a function whose support is $3Q$.

The last boundedness implies

$$\left(\int_Q (T_{s,c^{2s}\lambda}h)^q(x)u(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_{(3Q)} h(x)^p v(x)\,dx\right)^{\frac{1}{p}} \tag{4.1}$$

for a constant $C>0$ which does not depend on $h(\cdot)$ and $Q$. Then for all $x\in Q$

$$\begin{aligned}(T_{s,c^{2s}\lambda}h)(x) &= \int_{y\in(3Q)} |x-y|^{s-n}\exp\{-(c^{2s}\lambda)^{\frac{1}{2s}}|x-y|\}h(y)\,dy\\ &\ge \exp\{-c'\}\int_{y\in(3Q)} |x-y|^{s-n}h(y)\,dy \quad \text{since } |x-y|\le c'\lambda^{-\frac{1}{2s}}\\ &= \exp\{-c'\}(I_s h)(x) \quad \text{since the support of } h(\cdot) \text{ is } (3Q).\end{aligned}$$

This last inequality with (4.1) yields the point (2.1) in Theorem 1. To get (2.2) observe that, by duality, $T_{s,c^{2s}\lambda}: L^p_v \to L^q_u$ is equivalent to

$$\begin{aligned}\left(\int_{y\in\mathbb{R}^n}\left[\int_{x\in\mathbb{R}^n} |x-y|^{s-n}\exp\{-(c^{2s}\lambda)^{\frac{1}{2s}}|x-y|\}f(x)u(x)\,dx\right]^{p'}(x)\sigma(y)\,dy\right)^{\frac{1}{p'}}&\\ \le C\left(\int_{x\in\mathbb{R}^n} f(x)^{q'}u(x)\,dx\right)^{\frac{1}{q'}}&.\end{aligned}$$

Now take a cube $Q$ with $|Q|^{\frac{1}{n}}=\lambda^{-\frac{1}{2s}}$, and $f(\cdot)\ge 0$ equal to 1 in its support $Q$. Then

$$\begin{aligned}\left(\int_{y\in(3Q)^c}\left[\int_{x\in Q} |x-y|^{s-n}\exp\{-(c^{2s}\lambda)^{\frac{1}{2s}}|x-y|\}u(x)\,dx\right]^{p'}(x)\sigma(y)\,dy\right)^{\frac{1}{p'}}&\\ \le C\left(\int_{x\in Q} u(x)\,dx\right)^{\frac{1}{q'}}&.\end{aligned}$$

Since $|x-y|\approx|x_Q-y|$, for all $x\in Q$, $y\in(3Q)^c$, and $\int_Q u(x)\,dx<\infty$ then

$$\begin{aligned}\left(\int_{y\in(3Q)^c} |x_Q-y|^{(s-n)p'}\exp\{-c'p'(c^{2s}\lambda)^{\frac{1}{2s}}|x_Q-y|\}\sigma(y)\,dy\right)^{\frac{1}{p'}}&\\ \times\left(\int_{x\in Q} u(x)\,dx\right)^{\frac{1}{q}} \le C&\end{aligned}$$

which is the condition (2.2) $\square$

Proof of Remark 1: Suppose $J_{s,\lambda}: L^p_v \to L^q_u$. Then $T_{s,c^{2s}\lambda}: L^p_v\to L^q_u$ and $I_s: L^p(Q,v\,dx)\to L^q(Q,u\,dx)$ for all cubes $Q$ with $|Q|^{\frac{1}{n}}\le\lambda^{-\frac{1}{2s}}$. So

$|Q|^{\frac{s}{n}+\frac{1}{q}-\frac{1}{p}}(\frac{1}{|Q|}\int_Q \sigma(y)\,dy)^{\frac{1}{p'}}(\frac{1}{|Q|}\int_Q u(y)\,dy)^{\frac{1}{q}} \leq C$ for a constant $C>0$ not depending on $Q$. By the Lebesgue differentiation theorem, this last inequality yields $0 \leq \frac{s}{n}+\frac{1}{q}-\frac{1}{p}$ unless $u(\cdot)=0$ or $\sigma(\cdot)=0$. $\square$

Proof of Remark 3: If $J_{s,\lambda} : L^p_v \to L^q_u$ then $T_{s,c^{2s}\lambda} : L^p_v \to L^q_u$ and the condition (2.3) is satisfied. Indeed if $Q_1$ and $Q_2$ are cubes with $|Q_1|^{\frac{1}{n}} = |Q_2|^{\frac{1}{n}} = \lambda^{-\frac{1}{2s}}$ and $\overline{Q}_1 \cap \overline{Q}_2 \neq \emptyset$, then $|x-y| \leq c'\lambda^{-\frac{1}{2s}}$ for $x \in Q_1$ and $y \in Q_2$, and the operator $T_{s,c^{2s}\lambda}$ can be replaced by $I_s$. To see that (2.3) implies (2.1), let $Q$ be a cube with $|Q|^{\frac{1}{n}} = \lambda^{-\frac{1}{2s}}$ and $h(\cdot) \geq 0$ supported in $(3Q)$. By (3.4), $(3Q) = \bigcup_l Q_l$ with $|Q_l|^{\frac{1}{n}} = |Q|^{\frac{1}{n}}$, $\overline{Q} = \overline{Q}_l \neq \emptyset$, and so by (2.3) the condition (2.1) appears since

$$\begin{aligned}
\int_Q (I_s h)^q(x)u(x)\,dx &\leq C\sum_l \int_Q (I_s h\mathbb{I}_{Q_l})^q(x)u(x)\,dx \\
&\leq C\sum_l \Big(\int_{Q_l} h(x)^p v(x)\,dx\Big)^{\frac{q}{p}} \\
&\leq C\Big(\int_{x\in\mathbb{R}^n}\Big[\sum_l \mathbb{I}_{Q_l}(x)\Big]h(x)^p v(x)\,dx\Big)^{\frac{q}{p}} \\
&= C\Big(\int_{(3Q)} h(x)^p v(x)\,dx\Big)^{\frac{q}{p}}.
\end{aligned}$$

$\square$

Proof of Remark 4: Suppose (2.2) is true. To get (2.4) let $Q_1$, $Q_2$ be cubes with $|Q_1|^{\frac{1}{n}} = |Q_2|^{\frac{1}{n}} = \lambda^{-\frac{1}{2s}}$ and $\operatorname{dist}(Q_1,Q_2) \approx (m\lambda^{-\frac{1}{2s}})$ where $m \geq 4$. Since $Q_2 \subset (3Q_1)^c$ then, taking $Q = Q_1$ in (2.2) and using $|x_{Q_1}-y| \approx \operatorname{dist}(Q_1,Q_2) \approx (m\lambda^{-\frac{1}{2s}})$ for all $y \in Q_2$, we obtain (2.4). Conversely suppose this last condition is satisfied for some constant $c_0>0$. For a cube $Q$ with $|Q|^{\frac{1}{n}} = \lambda^{-\frac{1}{2s}}$ and $c = c_0c_1^{-1}$, with $c_1$ a fixed constant depending only on $n$, then

$$\begin{aligned}
&\Big(\int_{(3Q)^c} |x_Q-y|^{(s-n)p'}\exp\{-(2c)\lambda^{-\frac{1}{2s}}|x_Q-y|\}\sigma(y)\,dy\Big)\Big(\int_Q u(x)\,dx\Big)^{\frac{p'}{q}} \\
&= \sum_{m=4}^{\infty}\sum_{l\in\mathcal{J}(m,Q)}\Big(\int_{Q_l} |x_Q-y|^{(s-n)p'}\exp\{-(2c)\lambda^{-\frac{1}{2s}}|x_Q-y|\}\sigma(y)\,dy\Big) \\
&\qquad\times\Big(\int_Q u(x)\,dx\Big)^{\frac{p'}{q}} \\
&\leq C\sum_{m=4}^{\infty}\sum_{l\in\mathcal{J}(m,Q)} (m\lambda^{-\frac{1}{2s}})^{s-n}\exp\{-(2cc_1)m\}\Big(\int_{Q_l}\sigma(y)\,dy\Big)\Big(\int_Q u(x)\,dx\Big)^{\frac{p'}{q}}
\end{aligned}$$

$$\text{here } |x_Q - y| \approx \operatorname{dist}(Q, Q_1) \approx (m\lambda^{-\frac{1}{2s}})$$

$$\leq C \sum_{m=4}^{\infty} m^{s-n} \exp\{-c_0 m\} \sum_{l \in \mathcal{J}(m,Q)} \exp\{-c_0 m\}(m\lambda^{-\frac{1}{2s}})^{s-n} \Big( \int_{Q_l} \sigma(y)\, dy \Big)$$

$$\times \Big( \int_Q u(x)\, dx \Big)^{\frac{p'}{q}}$$

$$\leq C'C \sum_{m=4}^{\infty} m^{s-n} \exp\{-c_0 m\} \sum_{l \in \mathcal{J}(m,Q)} 1 \quad \text{by the condition (2.4)}$$

$$\leq NC'C \sum_{m=4}^{\infty} m^{s} \exp\{-c_0 m\} = NC'CC'' \quad \text{by (3.5).}$$

□

Proof of Remark 5: Since the arguments are the same, we can suppose that $\sigma(\cdot)$ satisfies the doubling condition. This hypothesis implies $\int_{(tQ)} \sigma(y)\, dy \leq C_1 t^{n\rho} \int_Q \sigma(y)\, dy$ for all $t > 1$. The constant $\rho$, $C_1 > 0$, depend on the doubling condition. Suppose (2.5) is satisfied. To get (2.4) let $Q_1$, $Q_2$ with $|Q_1|^{\frac{1}{n}} = |Q_2|^{\frac{1}{n}} = \lambda^{-\frac{1}{2s}}$ and $\operatorname{dist}(Q_1, Q_2) \approx (m\lambda^{-\frac{1}{2s}})$, $m \geq 4$. Since $Q_2 \subset (c_1 m Q_1)$ for a fixed constant $c_1$ (depending only on $n$), then $\int_{Q_2} \sigma(y)\, dy \leq \int_{(c_1 m Q_1)} \sigma(y)\, dy \leq C_1 m^{n\rho} \int_{Q_1} \sigma(y)\, dy$. With this last inequality the conclusion appears, since for all $c > 0$

$$\exp(-cm) m^{(s-n)} (\lambda^{-\frac{1}{2s}})^{(s-n)} \Big( \int_{y \in Q_2} \sigma(y)\, dy \Big)^{\frac{1}{p'}} \Big( \int_{x \in Q_1} u(x)\, dx \Big)^{\frac{1}{q}}$$

$$\leq C_2 \exp(-cm) m^{[s - n(1 - \frac{\rho}{p'})]} (\lambda^{-\frac{1}{2s}})^{(s-n)} \Big( \int_{Q_1} \sigma(y)\, dy \Big)^{\frac{1}{p'}} \Big( \int_{Q_1} u(x)\, dx \Big)^{\frac{1}{q}}$$

$$\leq C_0 C_3 C_2$$

where $C_0$ is from the condition (2.5) and $C_3$ a constant which exists by the property of the exponential function ($\lim_{R \to \infty} R^{\beta} \exp\{-\gamma R\} = 0$, $\gamma > 0$) and does not depend on $m$. □

Proof of Theorem 2: By Theorem 1, Remarks 3 and 4 then $J_{s,\lambda} : L^p_v \to L^q_u$ iff both (2.3) and (2.4) hold. So we have just to prove that (2.3) implies (2.4). Taking $Q_1 = Q_2 = Q$ (with $|Q|^{\frac{1}{n}} = \lambda^{\frac{1}{2s}}$) in (2.3) then $I_s : L^p(Q, v dx) \to L^q(Q, u dx)$, with a constant independent of $Q$. So, as in the proof of Remark 1, (2.5) is satisfied. By Remark 5, this last condition implies (2.4). □

Proof of Proposition 3: By Theorem 2 and Remark 2, to get $J_{s,\lambda} : L^p_v \to L^q_u$ it is sufficient to get (2.3), which can be written as

$$\text{(4.2)} \quad \Big( \int_{x \in \mathbb{R}^n} (I_s f)^q(x) \widetilde{u}(x)\, dx \Big)^{\frac{1}{q}} \leq C \Big( \int_{x \in \mathbb{R}^n} f(x)^p \widetilde{v}(x)\, dx \Big)^{\frac{1}{p}} \quad \text{for all } f(\cdot) \geq 0.$$

Here $\widetilde{u}(\cdot) = u(\cdot)\mathbb{I}_{Q_2}(\cdot)$, $\widetilde{v}(\cdot) = v(\cdot)\mathbb{I}_{Q_1}(\cdot)$ and $Q_1$, $Q_2$ are cubes with $|Q_1|^{\frac{1}{n}} = |Q_2|^{\frac{1}{n}} = \frac{1}{3}\lambda^{-\frac{1}{2s}}$ and $\overline{Q}_1 \cap \overline{Q}_2 \neq \emptyset$. We emphasize that $C > 0$ is a constant which does not depend on $Q_1$ and $Q_2$. Sawyer and Wheeden [Sa-Wh] proved that (4.2) holds if for some $t > 1$ and $S > 0$

$$|Q|^{\frac{s}{n}+\frac{1}{q}-\frac{1}{p}}\Big(\frac{1}{|Q|}\int_Q \widetilde{u}(y)^t\,dy\Big)^{\frac{1}{tq}}\Big(\frac{1}{|Q|}\int_Q \widetilde{\sigma}(y)^t\,dy\Big)^{\frac{1}{tp'}} \leq S \tag{4.3}$$

for any cube $Q$ of arbitrary size and where $\widetilde{\sigma}(\cdot) = \sigma(\cdot)\mathbb{I}_{Q_1}(\cdot)$.

Precisely they found $C = cS$ where $c > 0$ depends only on $s, n, p, q$. Of course the constant $S > 0$ in (4.3) must depend on $\widetilde{u}(\cdot)$ and $\widetilde{\sigma}(\cdot)$. Thus to get (4.2), by using this Sawyer-Wheeden's result, we have to prove that in our context really $S$ in (4.3) depends only on $u(\cdot)$ and $v(\cdot)$ but not on the cubes $Q_1$ and $Q_2$.

Call $\mathcal{A}(\widetilde{u},\widetilde{\sigma},Q)$ the left member of (4.3), and where $Q$ is an arbitrary cube. First consider the case $|(3Q_1)|^{\frac{1}{n}} \leq |Q|^{\frac{1}{n}}$. Note that $\int_Q \widetilde{u}^t(y)\,dy \leq \int_{Q_2} u^t(y)\,dy \leq \int_{(3Q_1)} u^t(y)\,dy$ and $\int_Q \widetilde{\sigma}^t(y)\,dy \leq \int_{(3Q_1)} \sigma^t(y)\,dy$. Using these estimates and $1 < t < \frac{\frac{1}{q}-\frac{1}{p}+1}{\frac{s}{n}+\frac{1}{q}-\frac{1}{p}}$ then

$$\mathcal{A}(\widetilde{u},\widetilde{\sigma},Q) \leq \mathcal{A}(u,\sigma,(3Q_1)) \leq A.$$

This last inequality is true since $|3Q_1|^{\frac{1}{n}} = 3|Q_1|^{\frac{1}{n}} \leq \lambda^{-\frac{1}{2s}}$, and $A > 0$ which depends on $u(\cdot)$, $v(\cdot)$ comes from (2.6). Next suppose $|Q|^{\frac{1}{n}} \leq |3Q_1|^{\frac{1}{n}}$. Since $\int_Q \widetilde{u}^t(y)\,dy \leq \int_Q u^t(y)\,dy$ and $\int_Q \widetilde{\sigma}^t(y)\,dy \leq \int_Q \sigma^t(y)\,dy$ then, again by (2.6),

$$\mathcal{A}(\widetilde{u},\widetilde{\sigma},Q) \leq \mathcal{A}(u,\sigma,Q) \leq A \quad \text{ here } |Q|^{\frac{1}{n}} \leq |3Q_1|^{\frac{1}{n}} \leq \lambda^{-\frac{1}{2s}}.$$

Therefore $\mathcal{A}(\widetilde{u},\widetilde{\sigma},Q) \leq A$ for any cube of arbitrary size, and with $A > 0$ independent of $Q_1$, $Q_2$. Then (4.2) is satisfied and so $J_{s,\lambda} : L^p_v \to L^q_u$.

If moreover both $u(\cdot)$ and $\sigma(\cdot)$ satisfy the Muckenhoupt $A_\infty$ condition then, as above, both $\widetilde{u}(\cdot)$ and $\widetilde{\sigma}(\cdot)$ satisfy $A_\infty$ with constants depending on $u(\cdot)$ and $\sigma(\cdot)$ but not on $Q_1$ and $Q_2$. It is known from [Sa-Wh] that condition (4.3), with $t = 1$, is a sufficient condition which ensures the embedding (4.2). Condition (4.3) with $t = 1$ and a constant $S > 0$ not depending on $Q_1$ and $Q_2$ can be obtained from (2.7). □

Proof of Proposition 4: Choose the family of dyadic cubes $(Q_l)_{l\in\mathcal{I}}$, in Lemma 2, with common size equal to $2^k$, where $k$ is an integer such that $2^k \leq \lambda^{-\frac{1}{2s}} < 2^{k+1}$. Again we have to get (4.2) (where $Q_1$ and $Q_2$ are dyadic cubes with $|Q_1|^{\frac{1}{n}} = |Q_2|^{\frac{1}{n}} = 2^k$ and $\overline{Q}_1 \cap \overline{Q}_2 \neq \emptyset$). By the Sawyer's theorem [Sa-Wh], then (4.2) holds iff for some $S > 0$

$$\Big(\int_{y\in\mathbb{R}^n} (I_s\widetilde{\sigma}\mathbb{I}_Q)^q(y)\widetilde{u}(y)\,dy\Big)^{\frac{1}{q}} \leq S\Big(\int_Q \widetilde{\sigma}(y)\,dy\Big)^{\frac{1}{p}} \tag{4.4}$$

and

$$(4.4^*)\qquad \Big(\int_{y\in\mathbb{R}^n}(I_s\widetilde{u}\mathbb{I}_Q)^{p'}(y)\widetilde{\sigma}(y)\,dy\Big)^{\frac{1}{p'}}\le S\Big(\int_Q\widetilde{u}(y)\,dy\Big)^{\frac{1}{q'}}$$

for each dyadic cube $Q$ with an arbitrary size. Therefore it remains to prove that the condition (2.8) (respectively (2.8*)) implies (4.4) (respectively (4.4*)) and the corresponding contact $S>0$ depends only on $u(\cdot)$, $\sigma(\cdot)$ but not on $Q_1$ and $Q_2$. Conditions (4.4) and (4.4*) can be written as

$$(4.5)\qquad \Big(\int_{Q\cap Q_2}(I_s\sigma\mathbb{I}_{Q\cap Q_1})^{q}(y)u(y)\,dy\Big)^{\frac{1}{q}}\le S\Big(\int_{Q\cap Q_1}\sigma(y)\,dy\Big)^{\frac{1}{p}}$$

and

$$(4.5^*)\qquad \Big(\int_{Q\cap Q_2}(I_su\mathbb{I}_{Q\cap Q_2})^{p'}(y)\sigma(y)\,dy\Big)^{\frac{1}{p'}}\le S\Big(\int_{Q\cap Q_2}u(y)\,dy\Big)^{\frac{1}{q'}}$$

The crucial fact we use is the well-known property of dyadic cubes which asserts that: *for a given closed dyadic cubes $Q$, $Q_0$ the only cases which can occur are*: (a) $Q\cap Q_0=\emptyset$; (b) $Q\cap Q_0=\partial Q\cap\partial Q_0$; (c) $Q\subset Q_1$; (d) $Q_1\subset Q$.

First take a dyadic cube $Q$ with $|Q|^{\frac{1}{n}}\le 2^k$. We can assume $\int_{Q\cap Q_1}\sigma(y)\,dy\neq 0$ (respectively $\int_{Q\cap Q_2}u(y)\,dy\neq 0$) else (4.5) (respectively (4.5*)) is trivially satisfied. Suppose $Q\subset Q_1$ (respectively $Q\subset Q_2$). For $Q_1\neq Q_2$ then (4.5) (respectively (4.5*)) is trivially satisfied since necessarily $\int_{Q\cap Q_2}u(y)\,dy=0$ (respectively $\int_{Q\cap Q_1}\sigma(y)\,dy=0$). But for $Q_1=Q_2$ then $Q\cap Q_1=Q$ (respectively $Q\cap Q_2=Q$) and (4.5) (respectively (4.5*)) is reduced to

$$(4.6)\qquad \Big(\int_Q(I_s\sigma\mathbb{I}_Q)^{q}(y)u(y)\,dy\Big)^{\frac{1}{q}}\le S\Big(\int_Q\sigma(y)\,dy\Big)^{\frac{1}{p}}$$

(respectively

$$(4.6^*)\qquad \Big(\int_Q(I_su\mathbb{I}_Q)^{p'}(y)\sigma(y)\,dy\Big)^{\frac{1}{p'}}\le S\Big(\int_Q u(y)\,dy\Big)^{\frac{1}{q'}}\ ).$$

Since $|Q|^{\frac{1}{n}}\le 2^k\le\lambda^{-\frac{1}{2s}}$, then by (2.8) (respectively (2.8*)) the condition (4.6) (respectively (4.6*)) is satisfied with $S=C$, a constant which depends only on $u(\cdot)$ and $\sigma(\cdot)$.

Next consider $2^k<$ and assume $\int_{Q\cap Q_1}\sigma(y)\,dy\neq 0$ (respectively $\int_{Q\cap Q_2}u(y)\,dy\neq 0$) else there is nothing to prove. If $Q_1\subset Q$ (respectively $Q_2\subset Q$) then (4.5) (respectively (4.5*)) is reduced to

$$(4.7)\qquad \Big(\int_{Q\cap Q_2}(I_s\sigma\mathbb{I}_{Q_1})^{q}(y)u(y)\,dy\Big)^{\frac{1}{q}}\le S\Big(\int_{Q_1}\sigma(y)\,dy\Big)^{\frac{1}{p}}$$

(respectively

$$(4.7^*) \qquad \Big(\int_{Q\cap Q_1}(I_s u\mathbb{I}_{Q_2})^{p'}(y)\sigma(y)\,dy\Big)^{\frac{1}{p'}} \le S\Big(\int_{Q_2} u(y)\,dy\Big)^{\frac{1}{q'}}\ ).$$

If moreover $\int_{Q\cap Q_2} u(y)\,dy \neq 0$ (respectively $\int_{Q\cap Q_1}\sigma(y)\,dy \neq 0$) then necessarily $Q_2 \subset Q$ (respectively $Q_1 \subset Q$) and (4.7) (respectively (4.7*)) is the same as

$$(4.8) \qquad \Big(\int_{Q_2}(I_s\sigma\mathbb{I}_{Q_1})^{q}(y)u(y)\,dy\Big)^{\frac{1}{q}} \le S\Big(\int_{Q_1}\sigma(y)\,dy\Big)^{\frac{1}{p}}$$

(respectively

$$(4.8^*) \qquad \Big(\int_{Q_1}(I_s u\mathbb{I}_{Q_2})^{p'}(y)\sigma(y)\,dy\Big)^{\frac{1}{p'}} \le C\Big(\int_{Q_2} u(y)\,dy\Big)^{\frac{1}{q'}}\ ).$$

Since $|Q_1|^{\frac{1}{n}} = |Q_2|^{\frac{1}{n}} = 2^k \le \lambda^{-\frac{1}{2s}}$ and $Q_2 \subset (3Q_1)$ (respectively $Q_1 \subset (3Q_2)$) then the condition (4.8) (respectively (4.8*)) is satisfied with S=C by (2.8) (respectively (2.4*)).

□

Proof of Corollary 5: Let $v(\cdot) = (M_{spr,\lambda}w^r)^{\frac{1}{r}}(\cdot)$. It remains to prove $J_{s,\lambda} : L^p_v \to L^p_w$. Since one of $w(\cdot)$ and $\sigma(\cdot) = v^{-\frac{1}{p-1}}(\cdot)$ is a doubling weight function, then by Proposition 3, (2.6) is a sufficient condition in order to get the above embedding. So we have to estimate $|Q|^{\frac{s}{n}}(\frac{1}{|Q|}\int_Q w^r(y)\,dy)^{\frac{1}{rp}}(\frac{1}{|Q|}\int_Q \sigma^r(y)\,dy)^{\frac{1}{rp'}} = \mathcal{F}_r(Q)$ by a constant which does nor depend on $Q$ with $|Q|^{\frac{1}{n}} \le \lambda^{-\frac{1}{2s}}$. By the definition of $M_{\beta,\lambda}$ then $\sigma^r(x) \le (|Q|^{\frac{spr}{n}}\frac{1}{|Q|}\int_Q w^r(y)\,dy)^{1-p'}$ for each cube $Q$ with $|Q|^{\frac{1}{n}} \le \lambda^{-\frac{1}{2s}}$ and for all $x \in Q$. Consequently $\mathcal{F}_r(Q) \le |Q|^{\frac{s}{n}}(\frac{1}{|Q|}\int_Q w^r(y)\,dy)^{\frac{1}{rp}}$ $(|Q|^{\frac{spr}{n}}\frac{1}{|Q|}\int_Q w^r(y)\,dy)^{-\frac{1}{rp}} = 1.$ □

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