

Separation of $(n + 1)$ -families of sets in general position in \mathbf{R}^n

MIRCEA BALAJ

Abstract. In this paper the main result in [1], concerning $(n + 1)$ -families of sets in general position in \mathbf{R}^n , is generalized. Finally we prove the following theorem: If $\{A_1, A_2, \dots, A_{n+1}\}$ is a family of compact convexly connected sets in general position in \mathbf{R}^n , then for each proper subset I of $\{1, 2, \dots, n + 1\}$ the set of hyperplanes separating $\cup\{A_i : i \in I\}$ and $\cup\{A_j : j \in \bar{I}\}$ is homeomorphic to S_n^+ .

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Classification: Primary 52A37; Secondary 47H10

1. Introduction

In this paper we continue the investigation of a previous article [1], regarding the separability of the members of an $(n + 1)$ -family of sets in general position in \mathbf{R}^n . In the beginning we recall some definitions and notations.

A family \mathcal{A} of sets in \mathbf{R}^n is said to be in general position if any m -flat, $0 \leq m \leq n - 1$, intersects at most $m + 1$ members of \mathcal{A} . Let $m = \min\{n + 1, \text{card } \mathcal{A}\}$. It is easy to see that the family \mathcal{A} is in general position if and only if for every choice of sets $A_1, A_2, \dots, A_m \in \mathcal{A}$ and every choice of points $x_1 \in A_1, x_2 \in A_2, \dots, x_m \in A_m$, the set $\{x_1, x_2, \dots, x_m\}$ is affinely independent.

A set $A \subset \mathbf{R}^n$ is called (cf.[5] and [8, p. 174]) convexly connected if there is no hyperplane H such that $H \cap A = \emptyset$ and A contains points in both open halfspaces determined by H .

If A is a compact set and H a hyperplane in \mathbf{R}^n , then the distance between A and H is defined to be $d(A, H) = \min\{\|x - y\| : x \in A, y \in H\}$. If $H = \{x \in \mathbf{R}^n : \langle x, b \rangle = \lambda\}$ is a hyperplane, the corresponding closed halfspace $\{x \in \mathbf{R}^n : \langle x, b \rangle \leq \lambda\}$, $\{x \in \mathbf{R}^n : \langle x, b \rangle \geq \lambda\}$ are denoted respectively by H^{\leq} , H^{\geq} . A set A is said to be separated from a set B by the hyperplane H provided that A lies in one of the closed halfspaces H^{\leq} , H^{\geq} and B lies in the other. The set A is strictly separated from B by H provided that the separating hyperplane H is disjoint from both A and B . If \mathcal{A} is a family of sets containing at least two members, we say that a hyperplane H separates the members of \mathcal{A} if there exists a nontrivial partition $(\mathcal{B}, \mathcal{C})$ of \mathcal{A} such that $\cup \mathcal{B} \subset H^{\leq}$, $\cup \mathcal{C} \subset H^{\geq}$.

The unit sphere in \mathbf{R}^{n+1} and the set $\{x = (x_1, x_2, \dots, x_{n+1}) : \|x\| = 1, x_i \geq 0, 1 \leq i \leq n + 1\}$ are denoted by S_n, S_n^+ respectively. For every subset I of $\{1, 2, \dots, n + 1\}, \bar{I}$ denotes the complement of I in $\{1, 2, \dots, n + 1\}$.

In [1] among other results we have obtained the following

Theorem 1. *Let $\{A_1, A_2, \dots, A_{n+1}\}$ be a family of compact convexly connected sets in general position in \mathbf{R}^n . Then*

(i) *for each proper subset I of $\{1, 2, \dots, n + 1\}$, there exists exactly one hyperplane H such that*

$$(1) \quad H \text{ separates strictly the sets } \cup \{A_i : i \in I\}, \cup \{A_j : j \in \bar{I}\}$$

and

$$(2) \quad d(A_1, H) = d(A_2, H) = \dots = d(A_{n+1}, H);$$

(ii) *there exist exactly $2^n - 1$ hyperplanes satisfying (2).*

In this paper we obtain a generalization of the previous result. Also, we prove that for every nontrivial partition $(\mathcal{B}, \mathcal{C})$ of an $(n + 1)$ -family of compact convexly connected sets in general position in \mathbf{R}^n , the set of hyperplanes separating $\cup \mathcal{B}$ and $\cup \mathcal{C}$ is homeomorphic to S_n^+ .

2. Basic results

We start with the following result which generalizes Lemma 3 in [1].

Lemma 2. *Let $[x_1, x_2, \dots, x_{n+1}]$ be an n -simplex in \mathbf{R}^n . Then for each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n+1}) \in S_n^+$ and for each proper subset I of $\{1, 2, \dots, n + 1\}$ there exists exactly one hyperplane H such that*

$$(3) \quad H \text{ separates the sets } \{x_i : i \in I\} \text{ and } \{x_j : j \in \bar{I}\},$$

$$(4) \quad d(x_i, H) = k\alpha_i \text{ for some } k \text{ and all } i, 1 \leq i \leq n + 1.$$

PROOF: The distance from an arbitrary point x_0 to a hyperplane

$$(5) \quad H = \{x \in \mathbf{R}^n : \langle x, b \rangle = \lambda\}$$

is given by the Ascoli's formula (see [6, p.21])

$$(6) \quad d(x_0, H) = \frac{|\langle x_0, b \rangle - \lambda|}{\|b\|}.$$

Since the pair $(b, \lambda) \in (\mathbf{R}^n \setminus \{0\}) \times \mathbf{R}$ for which the hyperplane H admits the representation (5) is unique up to a non-zero multiplicative constant, the conditions (3) and (4) are equivalent with

$$(7) \quad \langle x_i, b \rangle - \lambda = \beta_i, \quad 1 \leq i \leq n + 1$$

where

$$\beta_i = \begin{cases} \alpha_i, & \text{if } i \in I, \\ -\alpha_i, & \text{if } i \in \bar{I}. \end{cases}$$

Denoting by $(x_{i1}, x_{i2}, \dots, x_{in})$ the coordinates of x_i , $1 \leq i \leq n + 1$, and by (b_1, b_2, \dots, b_n) the coordinates of b , we are lead to an $(n + 1) \times (n + 1)$ linear system

$$(8) \quad x_{i1}b_1 + x_{i2}b_2 + \dots + x_{in}b_n - \lambda = \beta_i, \quad 1 \leq i \leq n + 1.$$

From the affine independence of the points x_1, x_2, \dots, x_{n+1} , it follows that the determinant D of order $n + 1$ having the general row $(x_{i1}, x_{i2}, \dots, x_{in}, 1)$ is different from zero. This proves that the system (8) possesses the unique solution

$$(9) \quad \begin{cases} b_j = \frac{D_j}{D}, & 1 \leq j \leq n \\ \lambda = -\frac{D_{n+1}}{D}, \end{cases}$$

where D_j is the determinant of order $n + 1$ having the general row $(x_{i1}, x_{i2}, \dots, x_{i,j-1}, \beta_i, x_{i,j+1}, \dots, x_{in}, 1)$ and D_{n+1} is the determinant of the same order, with the general row $(x_{i1}, \dots, x_{in}, \beta_i)$. Since at least two β_i are distinct, from (7) it can be easily deduced that $b \neq 0$. All these show that there exists a unique hyperplane H which satisfies the conditions (3) and (4). Note that $d(x_i, H) = \frac{\alpha_i}{\|b\|}$ and that the points x_i , $i \in I$, lie in the closed halfspace H^{\geq} , while the points x_j , $j \in \bar{I}$, lie in H^{\leq} . □

Let a point α_0 lie on the surface S_n^+ with all coordinates equal. The proof of the following lemma repeats the previous proof (taking $I = \{1, 2, \dots, n + 1\}$).

Lemma 3. *Let $\Delta = [x_1, x_2, \dots, x_{n+1}]$ be an n -simplex in \mathbf{R}^n . Then for every $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n+1}) \in S_n^+ \setminus \{\alpha_0\}$ there exists exactly one hyperplane H such that*

- (i) *the simplex Δ is contained in one of the closed half-spaces determined by H , and*
- (ii) *$d(x_i, H) = k\alpha_i$ for some k and all i , $1 \leq i \leq n + 1$.*

The following generalization of Theorem 1 is our main result.

Theorem 4. *Let $\{A_1, A_2, \dots, A_{n+1}\}$ be a family of compact convexly connected sets in general position in \mathbf{R}^n . Then*

- (i) *for each $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n+1}) \in S_n^+$ and for each proper subset I of $\{1, 2, \dots, n + 1\}$ there exists exactly one hyperplane H such that*

$$(10) \quad H \text{ separates the sets } \cup \{A_i : i \in I\} \text{ and } \cup \{A_j : j \in \bar{I}\}$$

and

$$(11) \quad d(A_i, H) = k\alpha_i \text{ for some } k \text{ and all } i, \quad 1 \leq i \leq n + 1;$$

(ii) if for each $\alpha \in S_n^+$, $N(\alpha)$ denotes the number of the hyperplanes H satisfying (11), then

$$N(\alpha) = \begin{cases} 2^n, & \text{if } \alpha \in S_n^+ \setminus \{\alpha_0\} \\ 2^n - 1, & \text{if } \alpha = \alpha_0. \end{cases}$$

PROOF: (i) A similar argument to that used in proving Corollary 7 in [1] permits us to suppose the compact sets A_i being convex. Let $A = A_1 \times A_2 \times \dots \times A_{n+1}$. The elements of A are denoted by \bar{x}, \bar{y}, \dots . Let I be an arbitrary fixed proper subset of $\{1, 2, \dots, n + 1\}$. By Lemma 2, for each $\bar{x} = (x_1, x_2, \dots, x_{n+1}) \in A$, ($x_i \in A_i$, $1 \leq i \leq n + 1$) there exists a unique hyperplane, denoted by $H(\bar{x})$, such that

$$(12) \quad d(x_i, H(\bar{x})) = k\alpha_i \text{ for some } k \text{ (dependent on } \bar{x}) \text{ and all } i, \quad 1 \leq i \leq n + 1,$$

$\{x_i : i \in I\} \subset H^{\geq}(\bar{x})$ and $\{x_j : j \in \bar{I}\} \subset H^{\leq}(\bar{x})$. The equation of $H(\bar{x})$ is $\langle x, b \rangle = \lambda$, where $b = (b_1, b_2, \dots, b_n) \in \mathbf{R}^n \setminus \{0\}$ and $\lambda \in \mathbf{R}$ are given by the formulas (9).

We define the map $f : A \rightarrow 2^A$ by $f(\bar{x}) = P_1(\bar{x}) \times P_2(\bar{x}) \times \dots \times P_{n+1}(\bar{x})$, $\bar{x} \in A$ and $P_i(\bar{x})$ defined by

$$(13) \quad \begin{cases} \text{(a)} & P_i(\bar{x}) = \{x \in A_i : \langle x, b \rangle = \min\{\langle y, b \rangle : y \in A_i\}\}, & i \in I, \\ \text{(b)} & P_i(\bar{x}) = \{x \in A_i : \langle x, b \rangle = \max\{\langle y, b \rangle : y \in A_i\}\}, & i \in \bar{I}. \end{cases}$$

Since the sets A_i are compact, the sets $P_i(\bar{x})$ are nonempty. If $y_i \in P_i(\bar{x})$, $1 \leq i \leq n + 1$, then $P_i(\bar{x})$ coincides with the intersection of the set A_i with the hyperplane through y_i parallel to $H(\bar{x})$. Thus each $P_i(\bar{x})$ is a compact convex set, and $f(\bar{x})$ is a compact convex set for each $\bar{x} \in A$. Using Lemma 4 in [1] it can be easily verified that f is upper semicontinuous.

By the Fan-Glicksberg-Kakutani fixed point theorem (see [2] and [4]), there is a point $\bar{z} = (z_1, z_2, \dots, z_{n+1}) \in A$ such that $\bar{z} \in f(\bar{z})$. Let $\langle x, b^0 \rangle - \lambda^0 = 0$ be the equation of the hyperplane $H(\bar{z})$, with $b^0 = (b_1^0, b_2^0, \dots, b_n^0)$ and λ^0 given by the formulas (9). For each $i \in I$, $z_i \in H^{\geq}(\bar{z})$ and by definition of f , $z_i \in P_i(\bar{z})$. Thus, we infer from (13a) that $A_i \subset H^{\geq}(\bar{z})$ for all $i \in I$.

Then, for each $i \in I$, we have

$$d(A_i, H(\bar{z})) = \min \left\{ \frac{|\langle x, b^0 \rangle - \lambda^0|}{\|b^0\|} : x \in A_i \right\} = \frac{\langle z_i, b^0 \rangle - \lambda^0}{\|b^0\|} = d(z_i, H(\bar{z})).$$

In a similar manner, we obtain that $A_j \subset H^{\leq}(\bar{z})$ and $d(A_j, H(\bar{z})) = d(z_j, H(\bar{z}))$, for all $j \in \bar{I}$. Therefore $H(\bar{z})$ separates the sets $\cup\{A_i : i \in I\}$, $\cup\{A_j : j \in \bar{I}\}$ and by (12) the sets $\{d(A_1, H(\bar{z})), d(A_2, H(\bar{z})), \dots, d(A_{n+1}, H(\bar{z}))\}$, $\{\alpha_1, \alpha_2, \dots, \alpha_{n+1}\}$ are proportional.

In the second part of the proof we verify the uniqueness of the hyperplane H which satisfies (10) and (11), for an arbitrary fixed set of indices I .

By the way of contradiction, suppose that there exist two distinct hyperplanes $H' = \{x \in \mathbf{R}^n : \langle x, b' \rangle - \lambda' = 0\}$, $H'' = \{x \in \mathbf{R}^n : \langle x, b'' \rangle - \lambda'' = 0\}$ satisfying (10) and (11). For each $i \in \{1, 2, \dots, n + 1\}$ let x'_i and x''_i be points in A_i such that $d(A_i, H') = d(x'_i, H')$, $d(A_i, H'') = d(x''_i, H'')$. Then, for a convenient choice of the pairs (b', λ') , (b'', λ'') we have

$$(14) \quad \begin{cases} (a) & \min\{\langle x, b' \rangle - \lambda' : x \in A_i\} = \langle x'_i, b' \rangle - \lambda' = \alpha_i & \text{if } i \in I, \\ (b) & \max\{\langle x, b' \rangle - \lambda' : x \in A_j\} = \langle x'_j, b' \rangle - \lambda' = -\alpha_j & \text{if } j \in \bar{I} \end{cases}$$

and

$$(15) \quad \begin{cases} (a) & \min\{\langle x, b'' \rangle - \lambda'' : x \in A_i\} = \langle x''_i, b'' \rangle - \lambda'' = \alpha_i & \text{if } i \in I, \\ (b) & \max\{\langle x, b'' \rangle - \lambda'' : x \in A_j\} = \langle x''_j, b'' \rangle - \lambda'' = -\alpha_j & \text{if } j \in \bar{I}. \end{cases}$$

Then, for each $i \in I$, by (14a) and (15a) it follows that $\langle x'_i, b' - b'' \rangle + \lambda'' - \lambda' \leq 0$ and $\langle x''_i, b' - b'' \rangle + \lambda'' - \lambda' \geq 0$. Obviously H' and H'' cannot be parallel, hence $b' \neq b''$. The convexity of A_i implies that the hyperplane $H = \{x \in \mathbf{R}^n : \langle x, b' - b'' \rangle + \lambda'' - \lambda' = 0\}$ intersects all sets A_i , $i \in I$. Using a similar argument we obtain that H intersects all sets A_j , $j \in \bar{I}$. Therefore H intersects each member of the family $\{A_1, A_2, \dots, A_{n+1}\}$ which is in general position. The contradiction obtained completes the proof.

(ii) From (i) we deduce that there exist exactly $2^n - 1$ hyperplanes which satisfy (11) and separate the members of the family $\{A_1, A_2, \dots, A_{n+1}\}$.

$N(\alpha_0) = 2^n - 1$ is the assertion (ii) in Theorem 1. If $\alpha \in S_n^+ \setminus \{\alpha_0\}$, arguing as above, Lemma 3 yields a unique hyperplane which leaves all sets A_i on the same side and which satisfies (11). □

Let $\{A_1, A_2, \dots, A_{n+1}\}$ be a family of compact convexly connected sets in general position in R^n . For each proper subset I of $\{1, 2, \dots, n + 1\}$ let $\mathcal{H}(I)$ denote the set of hyperplanes which separate the sets $\cup\{A_i : i \in I\}$ and $\cup\{A_j : j \in \bar{I}\}$. To each hyperplane $H \in \mathcal{H}(I)$ there corresponds a unique point $(b^H, \lambda^H) = (b_1^H, b_2^H, \dots, b_n^H, \lambda^H) \in S_n$ such that $H = \{x \in \mathbf{R}^n : \langle x, b^H \rangle = \lambda^H\}$ and $\cup\{A_i : i \in I\} \subset H^\geq$. This correspondence permits to identify $\mathcal{H}(I)$ with a subset of S_n , namely $\{(b^H, \lambda^H) : H \in \mathcal{H}(I)\}$.

The following known results are needed in the proof of Theorem 7.

Lemma 5 [7, Theorem 1]. *If M is a compact convex set in \mathbf{R}^n , then the function $h : \mathbf{R}^n \rightarrow \mathbf{R}$ defined by $h(b) = \max\{\langle x, b \rangle : x \in M\}$ is continuous.*

Lemma 6 [3, p.207, Lemma 3]. *Let X and Y be topological spaces, X compact and Y separated. If $f : X \rightarrow Y$ is a continuous bijection, then f is a homeomorphism.*

Theorem 7. Let $\{A_1, A_2, \dots, A_{n+1}\}$ be a family of compact convexly connected sets in general position in \mathbf{R}^n . Then for every proper subset I of $\{1, 2, \dots, n+1\}$ the sets $\mathcal{H}(I)$ and S_n^+ are homeomorphic.

PROOF: Let I be a proper subset of $\{1, 2, \dots, n+1\}$ arbitrarily fixed. Define $f : \mathcal{H}(I) \rightarrow S_n^+$ by $f(H) = \frac{1}{\|d_H\|} d_H$, where $d_H = (d(A_1, H), d(A_2, H), \dots, d(A_{n+1}, H))$. By Theorem 4, f is a bijection. By Lemma 5, each component of f is continuous, hence f is continuous too. Then, taking into account the quoted identification, $\mathcal{H}(I) = f^{-1}(S_n^+)$ is a closed subset of the compact set S_n . So $\mathcal{H}(I)$ is compact and the assertion of Theorem 7 follows now from Lemma 6. \square

Remark. Theorems 4 and 7 can be reformulated obtaining analogous informations about the hyperplanes which strictly separate the members of the family $\{A_1, A_2, \dots, A_{n+1}\}$. For instance we have:

Let $\{A_1, A_2, \dots, A_{n+1}\}$ be a family of compact convexly connected sets in general position in \mathbf{R}^n . Then for each proper subset I of $\{1, 2, \dots, n+1\}$ the set of hyperplanes strictly separating $\cup\{A_i : i \in I\}$ and $\cup\{A_j : j \in \bar{I}\}$ is homeomorphic to $\{(\alpha_1, \alpha_2, \dots, \alpha_{n+1}) \in S_n : \alpha_i > 0, 1 \leq i \leq n+1\}$.

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DEPARTMENT OF MATHEMATICS, ORADEA UNIVERSITY, STR. ARMATEI ROMÂNE NR. 5,
3700 ORADEA, ROMANIA

E-mail: balmir@lego.soroscj.ro

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