Multiple left distributive systems

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Abstract. We describe the free objects in the variety of algebras involving several mutually distributive binary operations. Also, we show how an associative operation can be constructed on such systems in good cases, thus obtaining a two way correspondence between LD-monoids (sets with a left self-distributive and a compatible associative operation) and multi-LD-systems (sets with a family of mutually distributive operations).

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A number of algebraic systems involving a self-distributive operation actually involve several such operations, which in addition are mutually distributive. Such systems will be called here *multi-LD-systems*: so, a multi-LD-system has the form $(S, \{*_i\,;\, i \in \Omega\})$ where each $*_i$ is a binary operation and the identity

$$x *_i (y *_j z) = (x *_i y) *_j (x *_i z)$$

holds for every i, j in Ω, including the case $i = j$. An *LD-system*, *i.e.*, a set equipped with a single left self-distributive operation, corresponds to the case when Ω is a singleton. To give some examples, consider any group G equipped with the conjugacy operations $*_k$ defined by $x *_k y = x^k y x^{-k}$ for k an integer, or any vector space V equipped with the operations $*_\lambda$ defined by $x *_\lambda y = (1-\lambda)x + \lambda y$ for λ a scalar. Observe also that any lattice is a 2-multi-LD-system (we shall speak of a p-multi-LD-system for a multi-LD-system that involves p binary operations), and that the operations $*$ and $\bar{*}$ defined on Artin's braid group B_∞ by

$$x * y = xs(y)\sigma_1 s(x^{-1}), \qquad x \bar{*} y = xs(y)\sigma_1^{-1} s(x^{-1})$$

give to B_∞ the structure of a 2-multi-LD-system ([4]).

The aim of this note is to establish two results about multi-LD-systems, both of which connect a multi-LD-system, *i.e.*, a set with a whole family of operations, and an LD-system, *i.e.*, a set with only one operation (and, in the second case, an additional associative operation). In both cases, the underlying idea is that multi-LD-systems are not really more general than mere LD-systems.

The first result is the description of the free multi-LD-systems in terms of free LD-systems: essentially, a free p-multi-LD-system on n generators is the same as a free LD-system on pn generators. The second result deals with the structure of an LD-monoid, which is an LD-system equipped with an additional associative operation. It had been noted that, starting with an arbitrary LD-monoid, one can always define a family of new mutually distributive operations, thus obtaining a multi-LD-system that we shall call the companion of the initial LD-monoid. Here we describe a converse construction: we show how to construct on convenient multi-LD-systems two new operations so that we obtain an LD-monoid of which the initial multi-LD-system is the companion. The construction applies in particular to the case of free systems, where it extends partial results of [1], [12] and [3].

In the non-associative context of this paper, we shall adopt the convention that missing brackets are to be added on the right: $x * y * z$ will always stand for $x * (y * z)$.

1. Free multi-LD-systems

As was noted by Larue in [10] (*cf.* also [11]), there exists a uniform way how to gather the various operations of a multi-LD-system into a single operation at the expense of enlarging the domain:

Definition 1.1. The *hull* of the multi-LD-system $(S, \{*_i\,;\, i \in \Omega\})$ is the system $(S \times \Omega, *)$, where $*$ is given by

$$(x, i) * (y, j) = (x *_i y, j).$$

The hull of a multi-LD-system is always an LD-system, and the initial operations can clearly be retrieved from the operation of its hull. We shall see here that the hull of a free multi-LD-system is a free LD-system.

Definition 1.2. Assume that $(F, *)$ is a free LD-system based on the set $X \times \Omega$. For j in Ω, I^j is the left ideal of F generated by those elements of the form (x, j) with x in X, and, for i in Ω, f_i^j is the function defined on I^j that maps every element of the form $a_1 * \ldots * a_p * (x, j)$ to the corresponding element $a_1 * \ldots * a_p * (x, i)$. Finally $*_i^j$ is the binary operation defined on I^j by

$$x *_i^j y = f_i^j(x) * y.$$

Proposition 1.3. (i) *Assume that $(F, \{*_i\,;\, i \in \Omega\})$ is a free multi-LD-system based on X; then its hull is a free LD-system based on $X \times \Omega$.*

(ii) *Conversely, assume that $(F, *)$ is a free LD-system based on a set $X \times \Omega$. Then, for every j in Ω, $(I^j, \{*_i^j\,;\, i \in \Omega\})$ is a free multi-LD-system based on $X \times \{j\}$.*

In order to prove the result, we represent free systems as quotients of absolutely free systems under convenient congruences. For an arbitrary (nonempty) set X, we denote by T_X the absolutely free binary system generated by X, *i.e.*, the set of all well-formed terms constructed using variables from X and a single binary operator $*$. We denote by $=_{LD}$ the congruence on T_X generated by the pairs $(t_1 * (t_2 * t_3), (t_1 * t_2) * (t_1 * t_3))$, so that $T_X/{=_{LD}}$ is a free LD-system generated by X. Similarly, we denote by T_X^Ω the set of those terms that are constructed using variables in X and binary operators in Ω, and we use again $=_{LD}$ for the congruence on T_X^Ω generated by the pairs $(t_1 *_i (t_2 *_j t_3), (t_1 *_i t_2) *_j (t_1 *_i t_3))$ when i, j range over Ω. Again $T_X^\Omega/{=_{LD}}$ is a free multi-LD-system generated by X.

The point is that the correspondence involved in the construction of the hull can be translated at the level of terms. For t in T_X^Ω and k in Ω, we define inductively a term $f(t,k)$ in $T_{X\times\Omega}$ by the clauses

$$f(t,k) = \begin{cases} (t,k) & \text{if } t \text{ is a variable, } i.e., \text{ for } \ t \in X, \\ f(t_1,i) * f(t_2,k) & \text{if } t \text{ is } \ t_1 *_i t_2. \end{cases}$$

Conversely, for t a term in $T_{X\times\Omega}$, we define $h(t)$ to be the second component of the last variable occurring in t (by hypothesis, the variables occurring in t are pairs in $X \times \Omega$), and we define the term $g(t)$ in T_X^Ω to be the first component of t if t is a variable, and to be $g(t_1) *_{h(t_1)} g(t_2)$ if t is $t_1 * t_2$. In other words, the pair (g,h) maps $T_{X\times\Omega}$ into $T_X^\Omega \times \Omega$, and it obeys the inductive clause

$$(g(t),h(t)) = \begin{cases} t & \text{if } t \text{ is a variable, } i.e., \text{ for } \ t \in X \times \Omega, \\ (g(t_1) *_{h(t_1)} g(t_2), h(t_2)) & \text{if } t \text{ is } \ t_1 * t_2. \end{cases}$$

Lemma 1.4. (i) *The mappings f and (g,h) are inverse one to each other, and, therefore, they are bijective.*

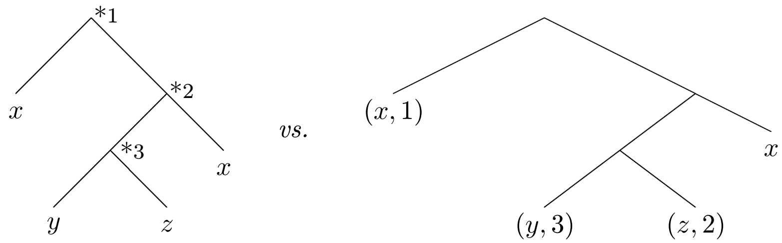
(ii) *The equivalence $f(t,k) =_{LD} f(t',k')$ holds in $T_{X\times\Omega}$ if and only if the indices k and k' are equal and the equivalence $t =_{LD} t'$ holds in T_X^Ω.*

Proof: For (i), one verifies inductively that $f\circ(g,h)$ and $(g,h)\circ f$ are the identity mappings of their domains. For (ii), we observe first that the last variable of any term of the form $f(t,k)$ has the form (x,k) for some x in X. An obvious induction shows that the last variable is invariant under LD-equivalence, *i.e.*, that any two terms that are LD-equivalent must have the same last variable. So the only point to show is that $f(t,k) =_{LD} f(t',k)$ holds in $T_{X\times\Omega}$ if and only if $t =_{LD} t'$ holds in T_X^Ω. Because the congruence $=_{LD}$ is generated as an equivalence relation by those pairs (t,t') such that t' is a 1-expansion of t, *i.e.*, such that t' is obtained from t by replacing exactly one subterm s of the form $s_1 *_i (s_2 *_j s_3)$ by the corresponding term $s' = (s_1 *_i s_2) *_j (s_1 *_i s_3)$, it is enough to make the proof for such pairs. Now we use induction on the size of the term t. Assume that t is $t_1 *_\ell t_2$. If the subterm s involved in the expansion is a subterm of t_1, then, by induction hypothesis, the left subterm t'_1 of t' is a 1-expansion of t_1, $f(t'_1,m) =_{LD} f(t_1,m)$ holds for every m (actually $f(t'_1,m)$ is a 1-expansion of $f(t_1, m)$), and this implies $f(t', k) =_{LD} f(t, k)$. The argument is similar if s is a subterm of the right subterm t_2 of t. The only remaining case is when s is t itself, *i.e.*, when t_1 is s_1, t_2 is $s_2 *_j s_3$ and ℓ is i. Then we have

$$f(t, k) = f(s_1, i) * (f(s_2, j) * f(s_3, k)),$$
$$f(t', k) = (f(s_1, i) * f(s_2, j)) * (f(s_1, i) * f(s_3, k)),$$

which shows that $f(t', k)$ is a 1-expansion of $f(t, k)$, and, in particular, that $f(t, k) =_{LD} f(t', k)$ holds. The argument is similar for the converse implication. □

Remark 1.5. Labeled trees can be associated with terms so that the tree associated with the product $t_1 *_i t_2$ has a root labeled $*_i$ (or i) and two subtrees, the left one being the tree associated with t_1 and the right one being the one associated with t_2. In such trees, the leaves correspond to the variables while the inner nodes correspond to the operators. The geometrical meaning of the previous lemma is that, as far as LD-equivalence is concerned, the information contained in the inner nodes can be translated to the leaves without loss. More precisely, the translation shifts the information given at a certain inner node to the rightmost leaf under the left child of that node, as displayed below for the terms $x *_1 ((y *_3 z) *_2 x)$ and its counterpart $(x, 1) * (((y, 3) * (z, 2)) * x)$:



It is now easy to deduce Proposition 1.3. For Point (i), let us consider the function of $T_X^\Omega \times \Omega$ onto the free LD-system generated by $X \times \Omega$ that maps (t, k) to the LD-class of $f(t, k)$. By Lemma 1.4, this function is compatible with the congruence $=_{LD} \times =$ on $T_X^\Omega \times \Omega$, and, therefore, it induces a well defined mapping $\overline{f}$ of $F \times \Omega$ onto $T_{X\times\Omega}/=_{LD}$. Moreover, $\overline{f}$ is a homomorphism with respect to the operation induced on $F \times \Omega$ by the operation defined on $T_X^\Omega \times \Omega$ by

$$(t_1, i_1) * (t_2, i_2) = (t_1 *_{i_1} t_2, i_2),$$

i.e., with respect to the operation of the hull of F. Conversely, a similar argument shows that the function of $T_{X\times\Omega}$ onto $F \times \Omega$ that maps t to the pair (LD-class of $g(t), h(t)$) factorizes through $=_{LD}$, and the induced mapping is an inverse of $\overline{f}$. Hence the latter is an isomorphism.

For Point (ii), observe first that the invariance of the last variable under LD-equivalence guarantees that the functions f_i are well-defined. Then, it is clear from the definition of the operations $*_i^j$ that $(I^j, \{*_i^j\,;\, i \in \Omega\})$ is a multi-LD-system generated by $X \times \{j\}$. That this multi-LD-system is freely based on $X \times \{j\}$ follows again from Lemma 2 for, by construction, the evaluation in I^j of a term t of T_X^Ω is equal to the LD-class of $f(t, j)$. So, two such terms t, t' have the same evaluation if and only if $f(t, j)$ and $f(t', j)$ are LD-equivalent, which takes place if and only if t and t' themselves are LD-equivalent. $\square$

2. The companion multi-LD-system of an LD-monoid

Again, a number of usual examples of LD-systems involve a second, associative operation that is compatible with the left distributive operation, the basic example being a group equipped both with its product and the left distributive operation given by conjugacy. For other examples, and general results, see [1], [2], [12], [7], [9] and [5].

Definition 2.1. An *LD-monoid* is a monoid $(M, \cdot, 1)$ equipped with a second binary operation ^ such that the mixed identities

$$(x\hat{\ }y) \cdot x = x \cdot y, \quad x\hat{\ }(y\hat{\ }z) = (x \cdot y)\hat{\ }z, \quad x\hat{\ }(y \cdot z) = (x\hat{\ }y) \cdot (x\hat{\ }z), \quad x\hat{\ }1 = 1$$

are satisfied.

In the sequel, we shall refer to the monoid operation of an LD-monoid as its product, and to the second operation as its exponentiation (the above axioms take the rather pleasant form ${}^{x}y \cdot x = x \cdot y$, ${}^{x}({}^{y}z) = {}^{x \cdot y}z$, ${}^{x}(y \cdot z) = {}^{x}y \cdot {}^{x}z$, ${}^{x}1 = 1$ when the second operation is denoted ${}^{x}y$ — however, we shall not use this notation here, as it makes reading of long expressions tedious). One immediately verifies that the exponentiation of an LD-monoid is a left distributive operation, and that $1\hat{\ }x = x$ holds for every x: $x \hat{\ } (y \hat{\ } z) = (x \cdot y) \hat{\ } z = ((x \hat{\ } y) \cdot x) \hat{\ } z = (x \hat{\ } y) \hat{\ } (x \hat{\ } z)$, and $1 \hat{\ } x = (1 \hat{\ } x) \cdot 1 = 1 \cdot x = x$.

Our specific interest comes presently from the following easy observation.

Proposition 2.2. *Assume that M is an LD-monoid, and define, for e in M, a new binary operation $*_e$ by*

$$x *_e y = (x \hat{\ } e) \cdot y.$$

Then $(M, \{*_e\,;\, e \in M\})$ is a multi-LD-system.

The verification is straightforward. It is also easy to see that, in all usual cases, none of the self-distributive operations $*_e$ coincides with the operation ^ (which is self-distributive as well), and this is why we used here completely different notations for these operations.

Definition 2,3. In the above situation, the multi-LD-system $(M, \{*_e\,;\, e \in M\})$ is the *companion* of the LD-monoid M; more generally, if E is a subset of M, the multi-LD-system $(M, \{*_e\,;\, e \in E\})$ is the E-companion of M.

The question we address here is as to whether the initial operations of an LD-monoid can be retrieved from the operations of the companion multi-LD-system. More generally, we start with a multi-LD-system (possibly an LD-system), and we try to construct on that system (or on one of its subsets) two binary operations, the one associative and the other left distributive, so that the new operations form an LD-monoid and the initial multi-LD-system is the companion of this LD-monoid.

It is not hard to see that some conditions are necessary. As usual, we shall denote by L_a the left translation associated with an element a of a given binary system. Similarly, in the case of a multiple system involving a family of operations $\{*_i\,;\, i \in \Omega\}$, we shall use L_a^i for the left translation corresponding to the operation $*_i$, *i.e.*, for the mapping $x \mapsto a *_i x$.

Definition 2.4. (i) The multi-LD-system $(S, \{*_i\,;\, i \in \Omega\})$ is *strongly tame* if any equality of the form $L_{a_1}^{i_1} \circ \ldots \circ L_{a_p}^{i_p} = L_{a'_1}^{i'_1} \circ \ldots \circ L_{a'_{p'}}^{i'_{p'}}$ implies the equalities

$$L_{c *_k a_1}^{i_1} \circ \ldots \circ L_{c *_k a_p}^{i_p} = L_{c *_k a'_1}^{i'_1} \circ \ldots \circ L_{c *_k a'_{p'}}^{i'_{p'}}$$

for every c in S and every k in Ω.
(ii) The element g of S is *generic* in $(S, \{*_i\,;\, i \in \Omega\})$ if any equality of the form $a_1 *_{i_1} \ldots\ a_p *_{i_p} g = a'_1 *_{i'_1} \ldots\ a'_{p'} *_{i'_{p'}} g$ implies the equality $L_{a_1}^{i_1} \circ \ldots \circ L_{a_p}^{i_p} = L_{a'_1}^{i'_1} \circ \ldots \circ L_{a'_{p'}}^{i'_{p'}}$.

The present notion of a strongly tame multi-LD-system is a natural strengthening of the notion of a tame multi-LD-system, defined as the property that $L_a^i = L_{a'}^{i'}$ implies $L_{c *_k a}^i = L_{c *_k a'}^{i'}$ for every c, k — a condition that was used in [3] and [9] for instance.

We can refine Proposition 2.2 as

Proposition 2.5. *Assume that M is an LD-monoid. Then the companion of M is a strongly tame multi-LD-system, and the unit of M is generic for this multi-LD-system.*

Proof: Assume that the equality

$$L_{a_1}^{e_1} \circ \ldots \circ L_{a_p}^{e_p} = L_{a'_1}^{e'_1} \circ \ldots \circ L_{a'_{p'}}^{e'_{p'}}$$

holds in the multi-LD-system $(M, \{*_e\,;\, e \in M\})$. By construction this means that

$$(a_1 \,\hat{}\, e_1) \cdot \ldots \cdot (a_p \,\hat{}\, e_p) \cdot x = (a'_1 \,\hat{}\, e'_1) \cdot \ldots \cdot (a'_{p'} \,\hat{}\, e'_{p'}) \cdot x$$

holds for every x, and, in particular, for $x = 1$. So $(a_1 \,\hat{}\, e_1) \cdot \ldots \cdot (a_p \,\hat{}\, e_p)$ and $(a'_1 \,\hat{}\, e'_1) \cdot \ldots \cdot (a'_{p'} \,\hat{}\, e'_{p'})$ are equal. Applying exponentiation by $c \,\hat{}\, e$ and developing, we obtain

$$(c\hat{}e)\hat{}(a_1\hat{}e_1) \cdot \ldots \cdot (c\hat{}e)\hat{}(a_p\hat{}e_p) = (c\hat{}e)\hat{}(a'_1\hat{}e'_1) \cdot \ldots \cdot (c\hat{}e)\hat{}(a'_{p'}\hat{}e'_{p'}),$$

and, therefore,

$$(((c\hat{}e) \cdot a_1)\hat{}e_1) \cdot \ldots \cdot (((c\hat{}e) \cdot a_p)\hat{}e_p) = (((c\hat{}e) \cdot a'_1)\hat{}e'_1) \cdot \ldots \cdot (((c\hat{}e) \cdot a'_{p'})\hat{}e'_{p'}),$$

which is

$$((c *_e a_1)\hat{}e_1) \cdot \ldots \cdot ((c *_e a_p)\hat{}e_p) = ((c *_e a'_1)\hat{}e'_1) \cdot \ldots \cdot ((c *_e a'_{p'})\hat{}e'_{p'}).$$

By multiplying both sides by x on the right, we conclude that $L^{e_1}_{c*_e a_1} \circ \ldots \circ L^{e_p}_{c*_e a_p}$ and $L^{e'_1}_{c*_e a'_1} \circ \ldots \circ L^{e'_{p'}}_{c*_e a'_{p'}}$ are equal, and so M is strongly tame.

The genericity of 1 is obvious, for

$$(a_1 \,\hat{}\, e_1) \cdot \ldots \cdot (a_p \,\hat{}\, e_p) \cdot 1 = (a'_1 \,\hat{}\, e'_1) \cdot \ldots \cdot (a'_{p'} \,\hat{}\, e'_{p'}) \cdot 1$$

immediately implies

$$(a_1 \,\hat{}\, e_1) \cdot \ldots \cdot (a_p \,\hat{}\, e_p) \cdot x = (a'_1 \,\hat{}\, e'_1) \cdot \ldots \cdot (a'_{p'} \,\hat{}\, e'_{p'}) \cdot x$$

for every x. □

The previous conditions turn out to be sufficient for our purpose, and we obtain an exact characterization.

Proposition 2.6. (i) *Assume that $(S, \{*_i\,;\, i \in \Omega\})$ is a strongly tame multi-LD-system, and that g is generic in this system. Define on the left ideal I_g of S generated by g two new binary operations $\cdot$ and $\hat{}$ by the formulas*

$$(a_1 *_{i_1} \ldots\ a_p *_{i_p} g) \cdot (b_1 *_{j_1} \ldots\ b_q *_{j_q} g) = a_1 *_{i_1} \ldots\ a_p *_{i_p} b_1 *_{j_1} \ldots\ b_q *_{j_q} g,$$
$$(a_1 *_{i_1} \ldots\ a_p *_{i_p} g) \,\hat{}\, (b_1 *_{j_1} \ldots\ b_q *_{j_q} g) = c_1 *_{j_1} \ldots\ c_q *_{j_q} g,$$

where c_k is $a_1 *_{i_1} \ldots\ a_p *_{i_p} b_k$. Then $(I_g, \cdot, g, \hat{})$ is an LD-monoid, and the multi-LD-system $(I_g, \{*_i \upharpoonright I_g\,;\, i \in \Omega\})$ is a companion of this LD-monoid, according to the formula

$$x *_i y = (x \,\hat{}\, (g *_i g)) \cdot y.$$

(ii) *Conversely, if M is an LD-monoid and the above construction is applied to the companion multi-LD-system of M, the operations obtained in this way are the initial operations of M.*

Proof: (i) First, we must verify that the operations are well defined. For the product, this follows from the genericity of g: if $a_1 *_{i_1} \ldots\ a_p *_{i_p} g$ and $a'_1 *_{i'_1}$

$\dots\ a'_{p'} *_{i'_{p'}} g$ are two decompositions of an element a of I_g, then the translations $L^{i_1}_{a_1} \circ \dots \circ L^{i_p}_{a_p}$ and $L^{i'_1}_{a'_1} \circ \dots \circ L^{i'_{p'}}_{a'_{p'}}$ are equal, and, therefore, $a_1 *_{i_1} \dots\ a_p *_{i_p} b_1 *_{j_1} \dots\ b_q *_{j_q} g$ and $a'_1 *_{i'_1} \dots\ a'_{p'} *_{i'_{p'}} b_1 *_{j_1} \dots\ b_q *_{j_q} g$ coincide. So, for every b, the product $a \cdot b$ does not depend on the decomposition of a, and it is clear from the definition that it does not depend either on the decomposition of b. For exponentiation, the same argument as above shows that the factors c_k involved in the definition of $a\hat{}b$ do not depend on the choice of the decomposition of a, so $a\hat{}b$ itself does not either. For the second factor, assume that $b_1 *_{j_1} \dots\ b_q *_{j_q} g$ and $b'_1 *_{j'_1} \dots\ b'_{q'} *_{j'_{q'}} g$ are two decompositions of b. By genericity of g, the mappings $L^{j_1}_{b_1} \circ \dots \circ L^{j_q}_{b_q}$ and $L^{j'_1}_{b'_1} \circ \dots \circ L^{j'_{q'}}_{b'_{q'}}$ are equal. Then the hypothesis that S is strongly tame successively implies the equality of $L^{j_1}_{a_p *_{i_p} b_1} \circ \dots \circ L^{j_q}_{a_p *_{i_p} b_q}$ and $L^{j'_1}_{a_p *_{i_p} b'_1} \circ \dots \circ L^{j'_{q'}}_{a_p *_{i_p} b'_{q'}}$, then the equality of

$$L^{j_1}_{a_{p-1} *_{i_{p-1}} a_p *_{i_p} b_1} \circ \dots \circ L^{j_q}_{a_{p-1} *_{i_{p-1}} a_p *_{i_p} b_q}$$

and

$$L^{j'_1}_{a_{p-1} *_{i_{p-1}} a_p *_{i_p} b'_1} \circ \dots \circ L^{j'_{q'}}_{a_{p-1} *_{i_{p-1}} a_p *_{i_p} b'_{q'}},$$

and so on. Finally $L^{j_1}_{c_1} \circ \dots \circ L^{j_q}_{c_q}$ and $L^{j'_1}_{c'_1} \circ \dots \circ L^{j'_{q'}}_{c'_{q'}}$ are equal, where c_k is $a_1 *_{i_1} \dots\ a_p *_{i_p} b_k$ and c'_k is $a_1 *_{i_1} \dots\ a_p *_{i_p} b'_k$. So exponentiation is well defined.

It is immediate to verify that the product $\cdot$ is associative, and that the identities $x \cdot g = g \cdot x = g\ \hat{}\ x$, $x\ \hat{}\ g = g$ hold in I_g (by extending in the natural way the defining formulas for the cases $p = 0$ or $q = 0$). For the remaining identities, assume $a = a_1 *_{i_1} \dots\ a_p *_{i_p} g$ and $b = b_1 *_{j_1} \dots\ b_q *_{j_q} g$. We have successively

$$\begin{aligned}(a\hat{}b) \cdot a &= ((a_1 *_{i_1} \dots\ a_p *_{i_p} g)\hat{}(b_1 *_{j_1} \dots\ b_q *_{j_q} g)) \cdot (a_1 *_{i_1} \dots\ a_p *_{i_p} g)\\ &= (c_1 *_{j_1} \dots\ c_q *_{j_q} g) \cdot (a_1 *_{i_1} \dots\ a_p *_{i_p} g)\\ &= c_1 *_{j_1} \dots\ c_q *_{j_q} a_1 *_{i_1} \dots\ a_p *_{i_p} g\end{aligned}$$

where c_k is $a_1 *_{i_1} \dots\ a_p *_{i_p} b_k$, while we find by applying p times left distributivity

$$\begin{aligned}a \cdot b &= a_1 *_{i_1} \dots\ a_p *_{i_p} b_1 *_{j_1} \dots\ b_q *_{j_q} g\\ &= a_1 *_{i_1} \dots\ a_{p-1} *_{i_{p-1}} (a_p *_{i_p} b_1) *_{j_1} \dots\ (a_p *_{i_p} b_q) *_{j_q} a_p *_{i_p} g\\ &\quad \dots\\ &= c_1 *_{j_1} \dots\ c_q *_{j_q} a_1 *_{i_1} \dots\ a_p *_{i_p} g = (a\hat{}b) \cdot a,\end{aligned}$$

so the identity $(x\hat{\ }y) \cdot x = x \cdot y$ holds in I_g. The verification of the identities $(x \cdot y)\hat{\ }z = x\hat{\ }(y\hat{\ }z)$ and $x\hat{\ }(y \cdot z) = (x\hat{\ }y) \cdot (x\hat{\ }z)$ is similar. Hence $(I_g, \cdot, g, \hat{\ })$ is an LD-monoid.

Finally, with the same notations, we have

$$\begin{aligned}(a\hat{\ }(g *_i g)) \cdot b &= ((a_1 *_{i_1} \dots\ a_p *_{i_p} g)\hat{\ }(g *_i g)) \cdot (b_1 *_{j_1} \dots\ b_q *_{j_q} g)\\ &= ((a_1 *_{i_1} \dots\ a_p *_{i_p} g) *_i g) \cdot (b_1 *_{j_1} \dots\ b_q *_{j_q} g)\\ &= (a *_i g) \cdot (b_1 *_{j_1} \dots\ b_q *_{j_q} g)\\ &= a *_i b_1 *_{j_1} \dots\ b_q *_{j_q} g = a *_i b,\end{aligned}$$

and, on I_g, the operation $*_i$ is the one derived from the LD-monoid operations as in Definition 2.1 with $e = g *_i g$.

(ii) Assume now that $(M, \cdot, 1, \hat{\ })$ is an LD-monoid and that $(M, \{*_e\, ;\, e \in M\})$ is the companion multi-LD-system. Let $\bar{\cdot}$, $\bar{\hat{\ }}$ denote the associative and the distributive operations associated as in (i) with the latter (strongly tame) multi-LD-system and the generic element 1. For every element a of M, we have

$$1 *_a 1 = (1 \,\hat{\ }\, a) \cdot 1 = a.$$

This proves first that the left ideal of $(M, \{*_e\, ;\, e \in M\})$ generated by 1 is the whole of M. Then we obtain, for arbitrary elements a, b,

$$\begin{aligned} a \,\bar{\cdot}\, b &= (1 *_a 1) \,\bar{\cdot}\, (1 *_b 1) = 1 *_a 1 *_b 1 = (1 \,\hat{\ }\, a) \cdot (1 \,\hat{\ }\, b) = a \cdot b\\ a \,\bar{\hat{\ }}\, b &= (1 *_a 1) \,\bar{\hat{\ }}\, (1 *_b 1) = (1 *_a 1) *_b 1 = a \,\hat{\ }\, b,\end{aligned}$$

which shows that the new product and exponentiation coincide with the initial ones. □

Remarks. (i) In the situation of Proposition 3.(i), the mixed identities

$$x *_i (y \cdot z) = (x *_i y) \cdot z \qquad \text{and} \qquad x \,\hat{\ }\, (y *_i g) = (x \cdot y) *_i g$$

hold in I_g. By taking $y = 1$ in the first, we deduce in particular the formula $x *_i z = (x *_i g) \cdot z$.

(ii) In the situation of Proposition 3.(ii), it is not necessary to consider the maximal companion multi-LD-system $(M, \{*_e\, ;\, e \in M\})$ in order to make sure that the construction of (i) gives the initial operations: it suffices to consider any X-companion where X is a subset of M that generates it as an LD-monoid. Indeed, the left ideal of $(M, \{*_e; e \in X\})$ generated by 1 is the whole of M.

(iii) Let us denote by $\mathcal{L}(S)$ the left multiplication monoid of the multi-LD-system S, *i.e.*, the submonoid of the monoid $\mathrm{End}(S)$ generated by the left translations L^i_a. If S is a strongly tame (multi-)LD-system, the mapping L can be used

to define on the left multiplication monoid $\mathcal{L}(S)$ the structure of an LD-monoid, the associative operation being composition, and the distributive operation being obtained by carrying those of S according to the formula $L^i_a * L^j_b = L^j_{a *_i b}$: the LD-monoid thus obtained is the "adjoint" of S considered in [3] — in the case of LD-systems. If g is a generic element of S, the mapping $\varphi \mapsto \varphi(g)$ gives an isomorphism of the adjoint LD-monoid $\mathcal{L}(S)$ onto the LD-monoid of Proposition 3.(i).
(iv) If $(S, \{*_i\,;\, i \in \Omega\})$ is a multi-LD-system, the mapping $L^i_a \mapsto L_{(a,i)}$ establishes an isomorphism between the left multiplication monoid of S and that of its hull. It is easy to verify that $(S, \{*_i\,;\, i \in \Omega\})$ is strongly tame if and only if its hull is strongly tame, and that an element g of S is generic in $(S, \{*_i\,;\, i \in \Omega\})$ if and only if the elements (g, i) are generic in its hull. Then the LD-monoids given by Proposition 2.6(i) from $(S, \{*_i\,;\, i \in \Omega\})$ and from its hull are isomorphic.

The previous construction applies to various examples, and, in particular, to the case of free LD-systems. Assume that $(F, \{*_i\,;\, i \in \Omega\}))$ is a free multi-LD-system based on the set X. Verifying that F is strongly tame amounts to showing that a term equivalence of the form

$$t_1 *_{i_1} \ldots\ t_p *_{i_p} x =_{LD} t'_1 *_{i'_1} \ldots\ t'_{p'} *_{i'_{p'}} x'$$

with x, x' in X implies the equivalence

$$(t *_i t_1) *_{i_1} \ldots\ (t *_i t_p) *_{i_p} x =_{LD} (t *_i t'_1) *_{i'_1} \ldots\ (t *_i t'_{p'}) *_{i'_{p'}} x'$$

for every term t and every i in Ω. This is easy, for the hypothesis first implies $p' = p$ and $x' = x$, and, then, it suffices to consider the case of 1-expansions as in the proof of Lemma 1.2. The same argument shows that every element of X is generic, and that the conditions of Proposition 3(i) are fulfilled. It is easy to verify that the LD-monoid obtained in this way is a free LD-monoid, and that the present construction corresponds to that considered (in the case of a monogenerated LD-system) in [12] and [14].

To give other examples, let us consider the case of the cyclic LD-monoids A_n of [6], [8], [13]. Denote the associative and the distributive operations of A_n respectively by $\circ_n$ and $*_n$, both defined on $\{1, \ldots, 2^n\}$. The companion operation at e is given by

$$x *_{n,e} y = (x *_n e) \circ_n y.$$

In this case, it is trivial to reconstruct $*_n$ and $\circ_n$ from the companion operations, and even from $*_{n,1}$ solely, for the latter is the preimage of $*_n$ under the bijective map $x \mapsto x *_n 1$; for $e \le 2^n$, the operation $*_{n,e}$ is obtained similarly from the distributive operation $(x, y) \mapsto x_{[e]} *_n y$ — observe that, for any LD-system $(S, *)$ and any term $t(x)$ involving a single variable x, the formula $(x, y) \mapsto t(x) * y$ defines a new left distributive operation.

Finally, we refer to [5] where the case of the distributive operations on Artin's braid group B_∞ and its extensions is investigated.

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