Full regularity of weak solutions to a class of nonlinear fluids in two dimensions – stationary, periodic problem

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Abstract. We prove the existence of regular solution to a system of nonlinear equations describing the steady motions of a certain class of non-Newtonian fluids in two dimensions. The equations are completed by requirement that all functions are periodic.

Keywords: non-Newtonian fluids, shear dependent viscosity, regularity, Hölder continuity of gradients

Classification: 76F10, 35Q35, 35J65

0. Introduction

Let Ω be a two-dimensional square $(0,L)\times(0,L)$, $L\in(0,\infty)$. We consider the following problem in $\mathbb{R}^2$: to find $v=(v_1,v_2)$ and π which are periodic with the period L at each variable x_i, $i=1,2$, and solve the equations

$$\begin{aligned}&\operatorname{div} v = 0\\ &v_k\frac{\partial v}{\partial x_k}-\operatorname{div}(T(D(v)))+\nabla\pi=f,\end{aligned}\tag{0.1}$$

where $f=(f_1,f_2)$ is a given periodic vector field in $\mathbb{R}^2$ with zero mean value. Throughout the whole paper we use the summation convention; thus

$$v_k\frac{\partial v}{\partial x_k}=\sum_{k=1}^{2}v_k\frac{\partial v}{\partial x_k}$$

etc.

Let $\mathbb{S}$ be the set of symmetric matrices of the type 2×2. Then $D(v)$ belonging to $\mathbb{S}$ denotes the symmetrized ∇v and has components $D_{ij}(v)=\frac{1}{2}(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i})$, $i,j=1,2$. Further we put the following assumptions on nonlinear tensor function $T:\mathbb{S}\to\mathbb{S}$:

(i) there exists $F:\mathbb{R}_0^+\to\mathbb{R}_0^+$ such that $F\in C^2\left(\mathbb{R}_0^+\right)$ and for all $i,j=1,2$

$$T_{ij}(\eta)=\frac{\partial F(|\eta|^2)}{\partial p_{ij}}=2F'(|\eta|^2)\eta_{ij},\quad\forall\,\eta\in\mathbb{S},\quad F(0)=\frac{\partial F(0)}{\partial p_{ij}}=0;\tag{0.2}$$

The research was supported by the Grant Agency of the Czech Republic; J. Málek by the grant No. 201/96/0228; J. Stará by the grant No. 201/96/0311. J. Stará was also partially supported by the Grant Agency of the Charles University — the grant 189/96.

(ii) for a certain $p \in (1,\infty)$ there exist $C_1, C_2 > 0$ such that

$$\frac{\partial^2 F(|\eta|^2)}{\partial p_{ij}\partial p_{kl}}\xi_{ij}\xi_{kl} \geq C_1(1+|\eta|^2)^{\frac{p-2}{2}}|\xi|^2, \quad \forall\,\eta,\xi \in \mathbb{S}; \tag{0.3}$$

and (for all $i,j,k,l = 1,2$)

$$\left|\frac{\partial^2 F(|\eta|^2)}{\partial p_{ij}\partial p_{kl}}\right| \leq C_2\left(1+|\eta|^2\right)^{\frac{p-2}{2}}, \quad \forall\,\eta \in \mathbb{S}. \tag{0.4}$$

The purpose of this paper is to show that no matter whatever $p \in (1,\infty)$ is given, a weak solution to (0.1) exists and in fact it is as regular as data (smoothness of f and F) allow starting from the assumption

$$f \in L^{p'}(\Omega) \text{ if } p \in (1,2) \text{ or } f \in L^r(\Omega),\ r > 2, \text{ if } p \geq 2.$$

We start with the notion of weak solution to (0.1). Let us denote[1]

$$V_p = \left\{\varphi \in W^{1,p}_{\mathrm{loc}}\left(\mathbb{R}^2,\mathbb{R}^2\right); \varphi \text{ periodic},\ \operatorname{div}\varphi = 0, \int_\Omega \varphi\,\mathrm{d}x = 0\right\}.$$

We say that $v \in V_p$ is a weak solution to (0.1) if

$$\int_\Omega v_k\frac{\partial v_i}{\partial x_k}\varphi_i\,\mathrm{d}x + \int_\Omega T_{ij}(D(v))D_{ij}(\varphi)\,\mathrm{d}x = \int_\Omega f_i\varphi_i\,\mathrm{d}x \tag{0.5}$$

for all φ smooth, periodic and divergence-free. Let us remark that without any additional information on v, the first integral is well-defined only for $p \geq \frac{4}{3}$. The method of the proof, however, provides directly better regularity of v. More precisely, we obtain below $v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2) \cap V_p$.

It is worth noticing that the nonlinearity T has some useful properties that follows from (0.2)-(0.4). Namely: there exist $C_i,\ i = 3,4,5$, such that

$$T(\eta)\cdot\eta \geq C_4\left(|\eta|^p - 1\right), \quad \forall\,\eta \in \mathbb{S}, \tag{0.6}$$

$$|T(\eta)| \leq C_5\left(1+|\eta|^2\right)^{\frac{p-1}{2}}, \quad \forall\,\eta \in \mathbb{S}, \tag{0.7}$$

$$\begin{gathered}(T(\eta)-T(\xi))\cdot(\eta-\xi) \geq C_5\,|\eta-\xi|^2, \quad \forall\,\eta,\xi \in \mathbb{S}\\ \text{with } C_5 = C_5(\eta,\xi) \equiv C_1\int_0^1\left(1+|\xi - s(\eta-\xi)|^2\right)^{\frac{p-2}{2}}\mathrm{d}s.\end{gathered} \tag{0.8}$$

(If $A, B \in \mathbb{S}$, then $A\cdot B \equiv A_{ij}B_{ij}$.)

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[1] We use the standard notation for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$, and their norms $\|.\|_p$ and $\|.\|_{k,p}$ respectively.

In particular, the last condition means that the corresponding elliptic operator is strictly monotone. If the Leray-Lions theory of monotone operators is applied directly to (0.1), the existence of weak solutions is obtained for $p \geq \frac{3}{2}$ (in dimension d, the corresponding bound reads $p \geq \frac{3d}{d+2}$, which is the case when testing by $\varphi \in V_p$ is allowed in (0.5))(see [4] for details).

Finer techniques, using strict monotonicity of T and a construction of a special L^∞-test function, provide the existence of weak solution to (0.1) for $p \geq \frac{4}{3}$ (generally for $p \geq \frac{2d}{d+1}$, cf. [3]). The bound corresponds to the situation when $v_k \frac{\partial v}{\partial x_k} \in L^1(\Omega)$ for $v \in V_p$.

In two dimensions, these results can be improved because of an additional[2] cancellation in the convective term. More precisely, using the fact that $\frac{\partial u_1}{\partial x_1} = -\frac{\partial u_2}{\partial x_2}$, we can easily check that

$$\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_s}\frac{\partial u_i}{\partial x_s} = 0, \tag{0.9}$$

(see [12] for the proof of it).

This cancellation brings in the periodic case higher regularity for v for all $p > 1$, namely

$$v \in \begin{cases} W^{2,2}_{\text{loc}}\left(\mathbb{R}^2\right) \cap V_p & \text{for } p \geq 2 \\ W^{2,p}_{\text{loc}}\left(\mathbb{R}^2\right) \cap V_p & \text{for } p \in (1,2) \end{cases} \tag{0.10}$$

which gives the compactness for ∇v, and consequently the existence of weak solutions.

To obtain higher regularity, we will show that there is a $p_0 > 2$ such that

$$\nabla v \in W^{1,p_0}_{\text{loc}}\left(\mathbb{R}^2\right). \tag{0.11}$$

Once having (0.11), we see that ∇v, and thus $D(v)$, is hölderian. In particular, the 'coefficients' $\frac{\partial^2 F(|D(v)|^2)}{\partial p_{ij}\partial p_{kl}}$ are bounded and Hölder continuous for all $p \in (1, \infty)$. Then the standard approach used in the regularity theory of linear elliptic systems gives as much regularity as needed (or as data allow).

The key role in getting (0.11) plays Lemma A formulated below, concerning linear elliptic systems with bounded measurable coefficients and extendable to linear systems of the Stokes type (cf. [10] or [11] for the proofs based on the L^p-estimates for linear systems proved by Bojarski in [2] and Meyers in [7]).

Let $\kappa \in \mathbb{N}$, $\kappa \geq 1$. For $u : \Omega \to \mathbb{R}^2$, let us denote

$$N_i u = \sum_{r=1}^{2} \sum_{|\alpha|=\kappa} a_{ir\alpha} D^\alpha u_r, \quad i = 1, \dots, h.$$

[2]By the standard cancellation we mean the fact that $\int_\Omega w_k \frac{\partial v_i}{\partial x_k} v_i \, dx = 0$ for any w which is divergence-free.

Similarly, denoting $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}$, we put

$$N_{ir}\xi = \sum_{|\alpha|=\kappa} a_{ir\alpha}\xi^\alpha$$

and we assume

$$\operatorname{rang} N\xi = \operatorname{rang}(N_{ir}\xi)_{i=1,\dots,h}^{r=1,2} = 2 \qquad \forall\, \xi \in \mathbb{R}^m i\,.$$

Now consider the equation

$$\int_\Omega A_{ij}(x)N_i u(x)N_j\varphi(x)\,\mathrm{d}x = \int_\Omega g_j N_j\varphi\,\mathrm{d}x, \quad \forall\, \varphi \in W^{\kappa,2}. \tag{0.12}$$

Assume that
(0.13)

$$A_{ij} \in L^\infty(\Omega)\,, A_{ij} = A_{ji} \quad \text{for } i,j = 1,2$$

$$\exists\, \lambda_1, \lambda_2 > 0 \quad \forall\, \eta \in \mathbb{R}^h \text{ and for a.e. } x \in \Omega : \lambda_1 |\eta|^2 \le A_{ij}(x)\eta_i\eta_j \le \lambda_2 |\eta|^2$$

and

$$g \in L^r(\Omega)\,, \quad r > 2. \tag{0.14}$$

Then it holds:

Lemma A. *Let $u \in W^{\kappa,2}(\Omega)$ be a solution of (0.12) with A_{ij} satisfying (0.13). Let $2 \le p \le 2+\rho$. Then there exist $\gamma_1 = \gamma_1(\rho) > 1$ and $\gamma_2 = \gamma_2(\rho) > 1$ such that*

$$\|u\|_{\kappa,p} \le \frac{C}{\lambda_1}\gamma_2^{\,1-\frac{2}{p}}\,\|g\|_p \tag{0.15}$$

for p satisfying

$$p \le 2\left(1 - \lg\left[\frac{1-\frac{\lambda_1}{2\lambda_2}}{1-\frac{\lambda_1}{\lambda_2}}\right]/\lg\gamma_1\right)^{-1}. \tag{0.16}$$

The same result is valid for the generalized Stokes system

$$\int_\Omega A_{ij}(x)N_i u(x)N_j\varphi(x)\,\mathrm{d}x + \int_\Omega \pi \operatorname{div}\varphi = \int_\Omega g_j N_j\varphi\,\mathrm{d}x \quad \forall\, \varphi \in W^{\kappa,2}, \tag{0.17}$$

where the solution $u \in W^{\kappa,2}(\Omega) \cap V_p$. One way how to see that Lemma A is applicable to (0.17) is to introduce the stream function to u and φ, which leads to an elliptic equation of higher order for the stream function to u. An alternative way is to proceed as in the proof of Lemma A (cf. [10] or [11]) using as the starting point $L^{2+\rho}$-estimates ($\rho > 0$) for solutions of the Stokes system instead of $L^{2+\rho}$-estimates for the Laplace equation, see for example [1] for the proof of these estimates.

The scheme of the paper is following: Section 1 is devoted to the question of W^{2,p_0}-regularity, $p_0 > 2$, of solution to (0.1)–(0.4) for $p = 2$, where we can apply Lemma A directly as the coefficients are bounded (cf. (0.3)–(0.4)) from above and below. Section 2 deals with the most interesting case $p \in (1,2)$. We construct λ-approximations to (0.1) which are again 'quadratic', so that the corresponding solutions u^{λ}'s are smooth by the results in Section 1. The main task of Section 2 is to obtain this smoothness uniformly for all $\lambda > 0$. Letting $\lambda \to 0$ we finally obtain the W^{2,p_0}-regularity of the solution to the original problem (0.1). Moreover, we show the full regularity result. Section 3 is devoted to the results for $p > 2$. Here we only present theorems and omit the proofs, that are similar to those in Section 2. The detailed proofs for $p > 2$ will be given in a forthcoming paper, where we will analyze the Dirichlet boundary problem. By our preliminary calculations it seems that we will be able to gain the continuity for ∇v 'only' for $p \geq \frac{6}{5}$, in contrary to the periodic problem, where the Hölder continuity of ∇v holds for all $p > 1$, as presented below.

We finish this introductory part by a few words on the physical background of (0.1). The system (0.1) occurs in non-linear fluid mechanics. It describes the steady motion of a class of homogeneous incompressible fluids (having constant density normalized to one). Then v represents the velocity field and π is the pressure. This class of fluids is characterized by the nonlinear dependence of the extra stress tensor on the velocity gradient. The principle of the material frame indifference and the representation of the isotropic tensors of second order in two dimensions lead then to the form

$$T\left(D(v)\right) = 2\mu\left(|D(v)|^2\right) D(v). \tag{0.18}$$

Note that F defined by $F\left(|D(v)|^2\right) = \int_0^{|D(v)|^2} \mu(s)\,\mathrm{d}s$ satisfies (0.2). Setting in addition

$$\mu(s) = 2\mu_0 (1+s)^{\frac{p-2}{2}}, \quad \mu_0 > 0, \tag{0.19}$$

we obtain the classical example of T undergoing (0.2)–(0.4). If $p \neq 2$ then the apparent viscosity μ is a function of $|D(v)|^2$ and the model (0.5)–(0.6) can capture such non-Newtonian phenomena as shear thinning (if $p \in (1,2)$) or shear thickening ($p > 2$), and these models are used in modelling of processes in many branches of science (see [6] for further references).

1. The case $p = 2$

In this section we consider the system (0.1) with assumptions (0.2)–(0.4) for $p = 2$. We have

Theorem 1.1. *Let $f \in V_2^*$ (dual to V_2) and $p = 2$. Then there exists a weak solution $v \in V_2$ and $\pi \in L^2(\Omega)$, $\int_\Omega \pi \,\mathrm{d}x = 0$, such that*

$$\|\nabla u\|_2 + \|\pi\|_2 \le C \|f\|_{V_2^*} . \tag{1.2}$$

If in addition $f \in L^{p_0}(\Omega)$, $p_0 > 2$, then

$$\|v\|_{2,p_0} \le C \left(\|f\|_{V_2^*} + \|f\|_{p_0} \right) . \tag{1.3}$$

Proof: By the classical Leray-Lions theory ($p \ge \frac{3}{2}$) we obtain the existence of a $v \in V_2$ satisfying (0.5) and

$$\|D(v)\|_2 \le C \|f\|_{V_2^*} .$$

It is easy to verify that

$$\|\nabla v\|_2 = \sqrt{2} \, \|D(v)\|_2 \qquad \text{for } v \in V_2 , \tag{1.4}$$

which implies

$$\|\nabla v\|_2 \le C \|f\|_{V_2^*} . \tag{1.5}$$

Defining

$$\langle E, \varphi \rangle \equiv \int_\Omega T_{ij}(D(v)) \, D_{ij}(\varphi) \,\mathrm{d}x + \int_\Omega v_k \frac{\partial v_i}{\partial x_k} \varphi_i \,\mathrm{d}x - \int_\Omega f_i \varphi_i \,\mathrm{d}x$$

for all $\varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ periodic we see from (0.5) and (1.5)

$$\langle E, \varphi \rangle = 0, \quad \forall \, \varphi \in V_2 \quad \text{and} \ \|E\|_{(W^{1,2}(\Omega))^*} \le C < \infty. \tag{1.6}$$

By de Rham's theorem and Nečas theorem on negative norms (see [8]), we obtain the existence of $\pi \in L^2(\Omega)$ such that

$$\begin{gathered} \int_\Omega \pi \,\mathrm{d}x = 0, \quad \|\pi\|_2 \le C \|f\|_{V_2^*}^2 , \\ \langle E, \varphi \rangle = \int_\Omega \pi \operatorname{div} \varphi \,\mathrm{d}x \quad \forall \, \varphi \in W^{1,2}_{\mathrm{loc}}\left(\mathbb{R}^2\right) \text{periodic.} \end{gathered} \tag{1.7}$$

Due to the periodicity and 'Hilbert's' structure we can now easily apply the standard difference technique to obtain $W^{2,2}$-regularity for v. Indeed, denoting

$$d_h^r v(x) \equiv \frac{v\left(x+he^r\right)-v\left(x\right)}{h}, \quad r=1,2,$$

(e^r denotes unit vector at r-axes direction), we get from (0.5) a similar system for $d_h^r v, r=1,2,$

$$\begin{gathered}\operatorname{div}\left(d_h^r v\right)=0\\ \int_\Omega\left(d_h^r v_k(x)\frac{\partial v_i}{\partial x_k}(x+he^r)+v_k(x)\frac{\partial d_h^r v_i(x)}{\partial x_k}\right)\varphi_i(x)\,\mathrm{d}x\\ +\int_\Omega A_{ij}^{kl}(x)D_{kl}\left(d_h^r v\right)D_{ij}\left(\varphi\right)\,\mathrm{d}x=\int_\Omega d_h^r f_i\varphi_i\,\mathrm{d}x\end{gathered}\tag{1.8}$$

for all $\varphi\in V_2$. In (1.8)

$$A_{ij}^{kl}(x)\equiv\int_0^1\frac{\partial^2 F\left(Dv(x)+s(Dv(x+he^r)-Dv(x))\right)}{\partial p_{ij}\partial p_{kl}}\,\mathrm{d}s.$$

Testing in (1.8) by $d_h^r v\in V_2$ and using (0.3) together with (1.4), we obtain

$$C_1\left\|\nabla d_h^r v\right\|_2^2\le C\left(\|f\|_2^2+\left\|d_h^r v\right\|_4^2\right),\tag{1.9}$$

where we used the fact that $\int_\Omega v_k\frac{\partial}{\partial x_k}\left(d_h^r v_i\right)d_h^r v_i\,\mathrm{d}x=0$ and the term

$$\int_\Omega d_h^r v_k\frac{\partial v_i}{\partial x_k}d_h^r v_i\,\mathrm{d}x$$

was estimated by Hölder's inequality and (1.5). Since in two dimensions

$$\|u\|_4\le\|u\|_2^{1/2}\left\|\nabla u\right\|_2^{1/2},\tag{1.10}$$

we see immediately from (1.9)

$$\left\|\nabla d_h^r v\right\|_2^2\le C\left(\|f\|_2^2+1\right).\tag{1.11}$$

This implies

$$\begin{gathered}\|\nabla\pi\|_2\le C,\\ \left\|\nabla^2 v\right\|_2\le C\left(\|f\|_2+1\right)\end{gathered}\tag{1.12}$$

and the equations $(0.1)_2$ hold a.e. (for $p = 2$!!). Moreover, for $r = 1, 2$, the functions $w^r \equiv \frac{\partial v}{\partial x^r} \in V_2$ and $\sigma^r \equiv \frac{\partial \pi}{\partial x_r} \in L^2(\Omega)$ solve in the weak sense the system

$$\operatorname{div} w^r = 0$$

$$\frac{\partial}{\partial x_r}\left(v_k \frac{\partial v_i}{\partial x_k}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial^2 F(|Dv|^2)}{\partial p_{ij}\partial p_{kl}} D_{kl}\left(w^r\right)\right) + \frac{\partial \sigma^r}{\partial x_i} = \frac{\partial f_i}{\partial x_r}, \quad i = 1, 2. \tag{1.13}$$

As the coefficients $A_{ij}^{kl} \equiv \frac{\partial^2 F(|D(v)|^2)}{\partial p_{ij}\partial p_{kl}}$ satisfy assumption (0.13), $f \in L_{p_0}(\Omega)$ and

$$\left\| v_k \frac{\partial v}{\partial x_k} \right\|_{p_0} \le C \left\|\nabla v\right\|_{p_0} \left\|v\right\|_\infty \le C \left\|v\right\|_{2,2}^2 \le C \left(\|f\|_2^2 + 1\right),$$

we can apply Lemma A that guaranties

$$v \in W^{2,p_0}_{\mathrm{loc}}\left(\mathbb{R}^2\right), \quad \nabla v \in C^{0,\alpha}\left(\bar\Omega\right), \quad \alpha = 1 - \frac{2}{p_0}. \tag{1.14}$$

The proof of Theorem 1.1 is complete. □

Remark 1.15. Let us remark that for the case $p = 2$ we proved that every weak solution has Hölder-continuous first derivatives. Higher regularity is then reduced to a more or less standard application of the technique of the regularity theory for linear elliptic systems. The only term that requires circumspection is the term of the type

$$J \equiv \int_\Omega \frac{\partial^3 F(|D(v)|^2)}{\partial p_{ij}\partial p_{kl}\partial p_{rs}} D_{rs}(\nabla v) D_{kl}(\nabla v) D_{ij}(\nabla^{(2)} v)\, dx,$$

which appears while deriving $W^{3,2}$-regularity. In such a case, we have L^2-norm of the third derivatives of v on the left hand side and we know (see (1.14)) that $v \in W^{2,p_0}_{\mathrm{loc}}\left(\mathbb{R}^2\right)$ for $p_0 > 2$. Write $p_0 = 2 + \rho$. Then the interpolation inequality analogous to (1.10)

$$\|u\|_4 \le \|u\|_{2+\rho}^{\frac{2+\rho}{4}} \|\nabla u\|_2^{\frac{2-\rho}{4}}$$

leads to the following estimate of J (using also (1.14), the Hölder and Young inequalities):

$$\begin{aligned} |J| &\le C \int_\Omega |\nabla^{(2)} v|^2\, |\nabla^{(3)} v| \le C \|\nabla^{(2)} v\|_4^2 \|\nabla^{(3)} v\|_2 \\ &\le \|\nabla^{(2)} v\|_{p_0}^{\frac{2+\rho}{2}} \|\nabla^{(3)} v\|_2^{2-\frac{\rho}{2}} \le \frac{\varepsilon}{2} \|\nabla^{(3)} v\|_2^2 + C. \end{aligned}$$

Choosing $\varepsilon < 2C_1$, we move $\frac{\varepsilon}{2}\|\nabla^{(3)} v\|_2^2$ to the left hand side, and we conclude $W^{3,2}$-regularity for v. The higher regularity is then straightforward.

2. The case $p \in (1,2)$, λ-approximations

In order to prove the W^{2,p_0}-regularity ($p_0 > 2$) for (0.1)–(0.3) for $p \in (1,2)$, we study the following quadratic λ-approximations to (0.1): for $\lambda \in (0,1)$ we consider $(v,\pi) = (v^\lambda, \pi^\lambda)$ as a solution of the system (in $\mathbb{R}^2$)

$$\operatorname{div} v = 0$$

$$v_k \frac{\partial v}{\partial x_k} - \operatorname{div}\left(\left(1+\lambda\,|D(v)|^2\right)^{\frac{2-p}{2}} T\left(D(v)\right)\right) = -\nabla\pi + f, \tag{2.1}$$

where all functions are again periodic, etc. Denoting

$$\tau_{ij}^\lambda \overset{\text{def.}}{\equiv} \left(1+\lambda\,|D(v)|^2\right)^{\frac{2-p}{2}} T_{ij}(D(v)) = 2\left(1+\lambda\,|D(v)|^2\right)^{\frac{2-p}{2}} F'\left(|D(v)|^2\right) D_{ij}(v),$$

we first verify that τ^λ satisfy (0.2)–(0.4) with the quadratic potential

$$F^\lambda\left(|D(v)|^2\right) \equiv \int_0^{|D(v)|^2} (1+\lambda s)^{\frac{2-p}{2}} F'(s)\,\mathrm{d}s. \tag{2.2}$$

Indeed, recalling that $\frac{\partial F(|\eta|^2)}{\partial \eta_{ij}} = 2F'\left(|\eta|^2\right)\eta_{ij}$ we have

$$\frac{\partial^2 F^\lambda\left(|\eta|^2\right)}{\partial\eta_{ij}\partial\eta_{kl}} = \left(1+\lambda\,|\eta|^2\right)^{\frac{2-p}{2}} \frac{\partial^2 F\left(|\eta|^2\right)}{\partial\eta_{ij}\partial\eta_{kl}} + \lambda(2-p)\left(1+\lambda\,|\eta|^2\right)^{-\frac{p}{2}} \frac{\partial F\left(|\eta|^2\right)}{\partial\eta_{ij}}\eta_{kl}$$

and

$$\frac{\partial^2 F^\lambda\left(|\eta|^2\right)}{\partial\eta_{ij}\partial\eta_{kl}}\xi_{ij}\xi_{kl} = \left(1+\lambda\,|\eta|^2\right)^{\frac{2-p}{2}} \frac{\partial^2 F\left(|\eta|^2\right)}{\partial\eta_{ij}\partial\eta_{kl}}\xi_{ij}\xi_{kl} + 2(2-p)\lambda\left(1+\lambda\,|\eta|^2\right)^{-\frac{p}{2}} F'\left(|\eta|^2\right)\eta_{kl}\xi_{kl}\eta_{ij}\xi_{ij}. \tag{2.3}$$

Note that the last term is nonnegative for $p \in (1,2)$. Consequently by (0.3), (0.4) and (0.7)

$$\begin{aligned} &\frac{\partial^2 F^\lambda\left(|\eta|^2\right)}{\partial\eta_{ij}\partial\eta_{kl}}\xi_{ij}\xi_{kl} \ge C_1\lambda^{\frac{2-p}{2}}\,|\xi|^2 \overset{\text{def.}}{\equiv} \tilde{C}_1(\lambda)\,|\xi|^2 \\ &\left|\frac{\partial^2 F^\lambda\left(|\eta|^2\right)}{\partial\eta_{ij}\partial\eta_{kl}}\right| \le C_2 + C_5. \end{aligned} \tag{2.4}$$

Thus we can apply Theorem 1.1 and we obtain for each $\lambda > 0$ the existence of $p^\lambda > 2$ and v^λ such that

$$v^\lambda \in W^{2,p^\lambda}_{\mathrm{loc}}\left(\mathbb{R}^2\right) \cap C^1(\bar\Omega).$$

In the rest of this section we will show that (for certain $p_0 > 2$) we have

$$\left\|v^\lambda\right\|_{2,p_0} \le K \text{ uniformly with respect to } \lambda > 0. \tag{2.5}$$

Assume for a while that (2.5) holds. Then letting $\lambda \to 0$ we can find a sequence $\lambda_n \to 0$ such that $v^n \equiv v^{\lambda_n}$ satisfy

$$\begin{aligned} v^n &\rightharpoonup v \quad \text{weakly in } W^{2,p_0}(\Omega)\\ v^n &\to v \quad \text{strongly in } W^{1,2}(\Omega). \end{aligned}$$

Then it is trivial to see that v satisfies (0.5) and $v \in W^{2,p_0}(\Omega)$ (cf. (2.5)). Then $\nabla v \in C^{0,\alpha}\left(\bar\Omega\right)$, and we have

Theorem 2.6. *Let $p \in (1,2)$ and $f \in L^{p'}(\Omega)$. Then there exists a solution v, π to (0.1)–(0.4) such that*

$$v \in W^{2,p'}_{\mathrm{loc}}\left(\mathbb{R}^2\right) \cap C^{1,\alpha}\left(\bar\Omega\right),$$
$$\pi \in W^{1,p'}_{\mathrm{loc}}\left(\mathbb{R}^2\right), \qquad \int_\Omega \pi \,\mathrm{d}x = 0.$$

The direct consequence of Theorem 2.6 (see also Remark 1.15) is the following

Theorem 2.7. *Let $p \in (1,2)$, $k \in \mathbb{N}$, $k \ge 1$. Assume $F \in C^{k+2}\left(\mathbb{R}^+_0\right)$ and $f \in W^{k,2}(\Omega)$. Then*

$$v \in W^{k+2,2}_{\mathrm{loc}}\left(\mathbb{R}^2\right).$$

In particular, if $F \in C^\infty\left(\mathbb{R}^+_0\right)$ and $f \in C^\infty\left(\bar\Omega\right)$, then $v \in C^\infty\left(\bar\Omega\right)$.

Proof of Theorem 2.6: As described above, it remains to show (2.5) for v^λ being the solution of (2.1). Let us first note that by testing $(2.1)_2$ by $v = v^\lambda$ we obtain, with help of (0.6), the uniform estimate

$$\left\|D(v^\lambda)\right\|_p \le C < \infty\,. \tag{2.8}$$

This helps us to show another uniform estimate, namely

$$\int_\Omega \left(1 + \left|D(v^\lambda)\right|^2\right)^{\frac{p-2}{2}} \left|D(\nabla v^\lambda)\right|^2 \mathrm{d}x \le C < \infty. \tag{2.9}$$

Indeed, multiplying $(2.1)_2$ scalarly by $-\Delta v^\lambda$ (recall that for λ fixed we have enough regularity and $-\Delta v^\lambda$ is divergence-free and periodic), integrating over Ω, performing integration by parts and using the notation (2.2), we obtain

$$\int_\Omega \frac{\partial^2 F^\lambda\left(\left|D(v^\lambda)\right|^2\right)}{\partial p_{ij}\partial p_{kl}} D_{ij}\left(\frac{\partial v^\lambda}{\partial x_s}\right) D_{kl}\left(\frac{\partial v^\lambda}{\partial x_s}\right) \mathrm{d}x = \int_\Omega f\Delta v^\lambda \,\mathrm{d}x - \int_\Omega \frac{\partial v_k^\lambda}{\partial x_s}\frac{\partial v_i^\lambda}{\partial x_k}\frac{\partial v_i^\lambda}{\partial x_s}\,\mathrm{d}x. \tag{2.10}$$

Using the fact $\frac{\partial v_1^\lambda}{\partial x_1} = -\frac{\partial v_2^\lambda}{\partial x_2}$ one can check that the last term vanishes. Using (0.3) and $\left(1+\lambda\left|\eta\right|^2\right)^{\frac{2-p}{2}} \geq 1$ in (2.4) leads to

$$\frac{\partial^2 F^\lambda\left(|\eta|^2\right)}{\partial\eta_{ij}\partial\eta_{kl}}\xi_{ij}\xi_{kl} \geq C_1\left(1+|\eta|^2\right)^{\frac{p-2}{2}}|\xi|^2. \tag{2.11}$$

Then from (2.10)

$$C_1\int_\Omega\left(1+\left|D(v^\lambda)\right|^2\right)^{\frac{p-2}{2}}\left|D(\nabla v^\lambda)\right|^2 \mathrm{d}x \leq \left\|\nabla^2 v^\lambda\right\|_p \|f\|_{p'}. \tag{2.12}$$

Now we will show that $\left\|\nabla^2 v^\lambda\right\|_p$ is controlled by the integral on the left-hand side of (2.12). With help of Hölder's inequality we have

$$\left\|D\left(\nabla v^\lambda\right)\right\|_p^2 \leq \left(1+\left\|D\left(v^\lambda\right)\right\|_p\right)^{2-p}\int_\Omega\left(1+\left|D(v^\lambda)\right|^2\right)^{\frac{p-2}{2}}\left|D(\nabla v^\lambda)\right|^2 \mathrm{d}x. \tag{2.13}$$

The algebraic identity

$$\frac{\partial^2 v_i}{\partial x_j\partial x_k} = \frac{\partial D_{ik}(v)}{\partial x_j} + \frac{\partial D_{ij}(v)}{\partial x_k} - \frac{\partial D_{jk}(v)}{\partial x_i} \tag{2.14}$$

together with (2.13) and (2.8) give

$$\left\|\nabla^2 v^\lambda\right\|_p^2 \leq C\int_\Omega\left(1+\left|D(v^\lambda)\right|^2\right)^{\frac{p-2}{2}}\left|D(\nabla v^\lambda)\right|^2 \mathrm{d}x. \tag{2.15}$$

Hence, from (2.12) and (2.15) we have (2.9) and

$$\left\|\nabla^2 v^\lambda\right\|_p^2 \leq C < \infty. \tag{2.16}$$

Now we are ready to prove (2.5). Similarly to the analysis in Section 1, we have for $w_r = \frac{\partial v^\lambda}{\partial x_r}, r = 1, 2$, the equation

$$\int_\Omega \frac{\partial^2 F^\lambda\left(\left|D(v^\lambda)\right|^2\right)}{\partial p_{ij}\partial p_{kl}} D_{kl}(w_r) D_{ij}(\phi)\,\mathrm{d}x = \int_\Omega G^\lambda_\alpha \frac{\partial \phi_\alpha}{\partial x_r}\,\mathrm{d}x, \tag{2.17}$$

valid for all ϕ smooth, $\operatorname{div}\phi = 0$. Here

$$G^\lambda \equiv v^\lambda_k \frac{\partial v^\lambda}{\partial x_k} - f.$$

It follows from (2.16) and the assumption on f the existence of $q > 2$ such that

$$\left\|G^\lambda\right\|_q \le C < \infty.$$

For fixed $\lambda > 0$ all assumption of Lemma A are verified. Denote $V = V(\lambda) \equiv \sup_{x\in\bar\Omega}\left(1 + \left|D(v^\lambda(x))\right|^2\right)^{\frac{1}{2}}$. Then $(2.4)_2$ and (2.11) imply

$$C_1 V^{p-2}\,|\xi|^2 \le \frac{\partial^2 F^\lambda\left(\left|D(v^\lambda)\right|^2\right)}{\partial \eta_{ij}\partial \eta_{kl}} \xi_{ij}\xi_{kl} \le (C_2 + C_5)\,|\xi|^2 \tag{2.18}$$

valid for all $\lambda > 0$. By Lemma A

$$\|w_r\|_{1,q} \le K V^{2-p}, \quad r = 1, 2, \tag{2.19}$$

with

$$q \le 2\left(1 - \lg\left[\frac{1 - \frac{1}{2}\frac{C_1}{C_2}V^{p-2}}{1 - \frac{C_1}{C_2}V^{p-2}}\right] / \lg\gamma_3\right)^{-1}, \quad \gamma_3 > 1\,, \tag{2.20}$$

which gives

$$q = 2\left(1 - \delta\lg\left[\frac{1 - \frac{1}{2}\frac{C_1}{C_2}V^{p-2}}{1 - \frac{C_1}{C_2}V^{p-2}}\right] / \lg\gamma_3\right)^{-1} \qquad \text{for a certain } \delta \in (0, 1]\,.$$

This implies

$$1 - \frac{2}{q} = \frac{\delta}{\lg\gamma_3}\lg\left(\frac{1 - \frac{1}{2}\frac{C_1}{C_2}V^{p-2}}{1 - \frac{C_1}{C_2}V^{p-2}}\right) = \frac{\delta}{\lg\gamma_3}\lg\left(1 + \frac{\frac{1}{2}\frac{C_1}{C_2}V^{p-2}}{1 - \frac{C_1}{C_2}V^{p-2}}\right) \ge$$
$$\begin{cases} \delta\frac{\lg 2}{\lg\gamma_3} & \text{if } \frac{3}{2}\frac{C_1}{C_2}V^{p-2} \ge 1, \\ \delta\frac{1}{4\lg\gamma_3}\frac{C_1}{C_2}V^{p-2} & \text{if } \frac{3}{2}\frac{C_1}{C_2}V^{p-2} < 1, \end{cases}$$

as $\lg(1+x) \geq \frac{x}{2}$ on $[0,1]$. Thus setting $L \equiv \min\left(\frac{\lg 2}{\lg \gamma_3}, \frac{C_1}{4C_2 \lg \gamma_3}\right)$ yields $(1 \geq V^{p-2})$

$$1 - \frac{2}{q} \geq LV^{p-2}\delta \equiv M\delta. \tag{2.21}$$

Then $q > \frac{2}{1-M\delta} > 2$. (Of course, we restrict only on such q's that $W^{2,p}(\Omega) \hookrightarrow W^{1,q}(\Omega)$.) Let us finally denote $\theta \equiv \left(1 + \left|D(v^\lambda)\right|^2\right)^{\frac{1}{2}}$. By (2.8) and (2.9)

$$\left\|\theta^{\frac{p}{2}}\right\|_{1,2} < K, \tag{2.22}$$

and by (2.19)

$$\left\|\theta^{\frac{p}{2}}\right\|_{1,q} < KV^{2-p}. \tag{2.23}$$

For $p_0 \in (2, q)$, the interpolation inequality implies

$$\left\|\theta^{\frac{p}{2}}\right\|_{1,p_0} \leq \left\|\theta^{\frac{p}{2}}\right\|_{1,2}^{1-a} \left\|\theta^{\frac{p}{2}}\right\|_{1,q}^{a} \leq KV^{a(2-p)},$$

where a is given by $\frac{1}{p_0} = \frac{1-a}{2} + \frac{a}{q}$. Then it holds due to (2.21)

$$1 - \frac{2}{p_0} = 1 - 2\left(\frac{a}{q} + \frac{1-a}{2}\right) = a\left(1 - \frac{2}{q}\right) \geq M\delta a = LV^{p-2}\delta a,$$

and from the Morrey imbedding inequality (cf. [13, p. 58])

$$\begin{aligned} V^{\frac{p}{2}} = \left\|\theta^{\frac{p}{2}}\right\|_{C(\bar{\Omega})} &\leq K\left(\frac{p_0 - 1}{p_0 - 2}\right)^{1-\frac{1}{p_0}} V^{a(2-p)} \\ &\leq K\left(\frac{1}{1 - \frac{2}{p_0}}\right)^{1-\frac{1}{q}} V^{a(2-p)} \leq \tilde{K}V^{(2-p)(a+1-\frac{1}{q})}\delta^{1-\frac{1}{q}}. \end{aligned}$$

We need $(2-p)(a+1-\frac{1}{q}) < \frac{p}{2}$, which can be rewritten as

$$p > \frac{2(a+1-\frac{1}{q})}{a + \frac{3}{2} - \frac{1}{q}} \overset{a\to 0+}{\to} \frac{2(q-1)}{\frac{3}{2}q - 1} \overset{q\to 2}{\to} 1.$$

Therefore, for given $p \in (1,2)$ we find q_0 so that $\frac{2(q_0-1)}{\frac{3}{2}q_0 - 1} \leq \frac{1+p}{2}$. Taking $q < q_0$ in such a way that (2.20) holds, then we can fix a so that

$$p > \frac{2(a+1-\frac{1}{q})}{a + \frac{3}{2} - \frac{1}{q}}.$$

Under these circumstances, $\left\|\theta^{\frac{p}{2}}\right\|_{C(\bar{\Omega})} = \left\|\left(1 + \left|D(v^\lambda)\right|^2\right)^{\frac{p}{4}}\right\|_{C(\bar{\Omega})} \leq K$ which implies (2.3) from (2.23). The proof of Theorem 2.5 is complete. □

3. The case $p > 2$

Using the analogous regularization we can repeat the procedures of Section 2 for $p \in (2,3)$ under some restriction on the constants C_1, C_2, C_5, compare also with [9]. By a slightly different regularization based on changing the values of F outside the ball with a radius R of a suitable functional F_R with quadratic growth (see [5]) and using the bootstrap arguments for higher p we obtain the following

Theorem 3.1. *Let $r, p \in (2,\infty)$ and $f \in L^r(\Omega)$. Then there exists a solution v, π to* (0.1)–(0.4) *such that*

$$v \in W^{2,r}_{\mathrm{loc}}\left(\mathbb{R}^2\right) \cap C^{1,\alpha}\left(\bar{\Omega}\right),$$
$$\pi \in W^{1,r}_{\mathrm{loc}}\left(\mathbb{R}^2\right), \quad \int_\Omega \pi \, \mathrm{d}x = 0.$$

The direct consequence of Theorem 2.6 is the following

Theorem 2.7. *Let $p \in (2,\infty)$, $k \in \mathbb{N}$, $k \geq 1$. Assume $F \in C^{k+2}\left(\mathbb{R}_0^+\right)$ and $f \in W^{k,2}(\Omega)$. Then*

$$v \in W^{k+2,2}_{\mathrm{loc}}\left(\mathbb{R}^2\right).$$

In particular, if $F \in C^\infty\left(\mathbb{R}_0^+\right)$ and $f \in C^\infty\left(\bar{\Omega}\right)$, then $v \in C^\infty\left(\bar{\Omega}\right)$.

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(*Received* January 24, 1997)