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Inequalities for surface integrals of non-negative subharmonic functions

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Abstract. Let $\mathcal{H}$ denote the class of positive harmonic functions on a bounded domain Ω in $\mathbb{R}^N$. Let S be a sphere contained in $\overline{\Omega}$, and let σ denote the $(N-1)$-dimensional measure. We give a condition on Ω which guarantees that there exists a constant K, depending only on Ω and S, such that $\int_S u\,d\sigma \leq K \int_{\partial\Omega} u\,d\sigma$ for every $u \in \mathcal{H} \cap C(\overline{\Omega})$. If this inequality holds for every such u, then it also holds for a large class of non-negative subharmonic functions. For certain types of domains explicit values for K are given. In particular the classical value $K = 2$ for convex domains is slightly improved.

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1. Introduction

Throughout this note Ω is a bounded Lipschitz domain in the Euclidean space $\mathbb{R}^N$, where $N \geq 2$. Let σ denote the $(N-1)$-dimensional surface measure and let $\mathcal{S}$ be the class of functions u on $\overline{\Omega}$ that are non-negative and subharmonic on Ω and satisfy

$$u(y) = \limsup_{x \to y, x \in \Omega} u(x) \qquad (y \in \partial\Omega)$$

and

$$0 < \int_{\partial\Omega} u\,d\sigma < +\infty.$$

(Elements of $\mathcal{S}$ may take the value $+\infty$ at points of $\partial\Omega$.) Let $\mathcal{H}$ be the set of functions in $\mathcal{S}$ that are harmonic on Ω. Also let $\mathcal{S}_b$, $\mathcal{H}_b$ denote the sets of bounded elements of $\mathcal{S}, \mathcal{H}$ respectively. A function in $\mathcal{S}$ (respectively $\mathcal{H}$) is called *quasi-bounded* if it is the pointwise limit on Ω of some increasing sequence in $\mathcal{S}_b$ (respectively $\mathcal{H}_b$); we denote the sets of quasi-bounded elements of $\mathcal{S}$ and $\mathcal{H}$ by $\mathcal{S}_{qb}$ and $\mathcal{H}_{qb}$. Let S be a sphere contained in $\overline{\Omega}$. If $\mathcal{E}$ is a subset of $\mathcal{S}$, then we define

$$M(\Omega, S, \mathcal{E}) = \sup\left\{ \int_S u\,d\sigma \Big/ \int_{\partial\Omega} u\,d\sigma : u \in \mathcal{E} \right\}.$$

We shall give estimates for $M(\Omega, S, \mathcal{S}_{qb})$ when Ω satisfies certain geometrical hypotheses.

Many years ago Gabriel [2] and Reuter [8] showed in the cases $N = 2, 3$ respectively that if Ω is convex, then $M(\Omega, S, C(\overline{\Omega}) \cap \mathcal{S}) \leq 2$. More recently Hayman [4] improved this result in the case $N = 2$ by showing that $M(\Omega, S, \mathcal{S}_b) < 2$ when Ω is convex. In fact his proof shows that the constant 2 can be replaced by a smaller constant depending on Ω and S, comparable with that in the following theorem.

We denote the radius of S by ρ.

Theorem 1. *If Ω is convex, then*

$$M(\Omega, S, \mathcal{S}_{qb}) \leq 2(1 - (\rho/d)^N), \tag{1}$$

where d is the diameter of Ω. The result fails if $(\rho/d)^N$ is replaced by $\psi(\rho/d)$ for any function ψ on $(0, 1]$ such that $t^{-N}\psi(t) \to +\infty$ as $t \to 0+$.

In Theorem 1 the class $\mathcal{S}_{qb}$ cannot be replaced by the larger class $\mathcal{S}$. Indeed a very simple example (see Section 2) shows that even when Ω is a ball we have $M(\Omega, S, \mathcal{S}) = +\infty$ when $S \subset \Omega$.

Next we examine the effect of strengthening the convexity hypothesis. Recall that Ω is convex if and only if for each $y \in \partial\Omega$ there exists an open half-space D such that $\Omega \subset D$ and $y \in \partial D$. Modifying this characterization of convexity, we shall say that Ω is *R-convex* if for each $y \in \partial\Omega$ there exists an open ball B of radius R such that $\Omega \subseteq B$ and $y \in \partial B$.

Theorem 2. *If Ω is R-convex, then*

$$M(\Omega, S, \mathcal{S}_{qb}) \leq 2 - \rho/R. \tag{2}$$

Equality is possible in the case where Ω is a ball of radius R.

Likewise we can relax the convexity hypothesis. We shall say that Ω satisfies the *exterior R-ball condition* if for each $y \in \partial\Omega$ there exists an open ball B of radius R such that $B \subset \mathbb{R}^N \backslash \overline{\Omega}$ and $y \in \partial B$.

Theorem 3. *If Ω satisfies the exterior R-ball condition, then*

$$M(\Omega, S, \mathcal{S}_{qb}) \leq 2 + \rho/R. \tag{3}$$

If $\epsilon > 0$ and the right-hand side of (3) is replaced by $2 + (1 - \epsilon)\rho/R$, then the result fails.

It is natural to ask for a condition on Ω guaranteeing the finiteness of $M(\Omega, S, \mathcal{S}_{qb})$. Let ω be a bounded domain such that $\mathbb{R}^N \backslash \overline{\omega}$ is connected. We call ω a *Liapunov-Dini domain* if $\partial\omega$ can be covered by finitely many right circular cylinders whose bases have positive distances from $\partial\omega$, and corresponding to each cylinder L there is a C^1 function $\phi : \mathbb{R}^{N-1} \to \mathbb{R}$ and a coordinate system $(y_1, \dots, y_N) = (y', y_N)$ such that

$$L \cap \omega = \{(y', y_N) : y_N > \phi(y')\} \cap L, \quad L \cap \partial\omega = \{(y', y_N) : y_N = \phi(y')\} \cap L$$

and

$$\|\nabla\phi(x') - \nabla\phi(y')\| \leq \delta(\|x' - y'\|) \quad (x', y' \in \mathbb{R}^{N-1})$$

for some increasing continuous function $\delta : [0, +\infty) \to [0, +\infty)$ satisfying

$$\int_0^1 t^{-1}\delta(t)\, dt < +\infty;$$

here ∇ denotes the gradient operator and $\|\cdot\|$ the Euclidean norm. We shall say that Ω satisfies the *exterior Liapunov-Dini* condition if there exists a Liapunov-Dini domain ω such that for each $y \in \partial\Omega$ there is an isometry $T : \mathbb{R}^N \to \mathbb{R}^N$ for which $y \in \overline{T(\omega)} \subset \mathbb{R}^N \backslash \Omega$.

Theorem 4. *If Ω satisfies the exterior Liapunov-Dini condition, then $M(\Omega, S, \mathcal{S}_{qb}) < +\infty$.*

Note that the domain ω given by

$$\omega = \{(x', x_N) : \|x'\| < \sin^p x_N, 0 < x_N < \pi\}$$

is a Liapunov-Dini domain if $0 < p < 1$ but not if $p = 1$. Thus if $\pi/2 < \alpha < \pi$, then the domain

$$\{x \in \mathbb{R}^N : x_N > \|x\| \cos\alpha,\ \|x\| < 1\} \tag{4}$$

just fails to satisfy the exterior Liapunov-Dini condition, and the following example shows that Theorem 4 is close to being sharp.

Example 1. Suppose that $\pi/2 < \alpha < \pi$ and Ω is the domain given by (4). Then $M(\Omega, S, \mathcal{S}_{qb}) = +\infty$ for any sphere $S \subset \overline{\Omega}$.

Acknowledgment. The above definition of the exterior Liapunov-Dini condition is taken from S.J. Gardiner's paper [3]. We are grateful to Dr. Gardiner for drawing this definition to our attention and suggesting its relevance to our investigations.

2. A preliminary result

2.1. The following Proposition simplifies the proofs of Theorems 1–4. Recall our blanket hypothesis that Ω is a Lipschitz domain.

Proposition.

$$M(\Omega, S, \mathcal{S}_{qb}) = M(\Omega, S, C(\overline{\Omega}) \cap \mathcal{H}).$$

To prove the Proposition, suppose that $u \in \mathcal{S}_{qb}$ and let (u_n) be an increasing sequence in $\mathcal{S}_b$ such that $u_n \to u$ on Ω. For each positive integer n define

$$M_n = n + \sup_{\partial\Omega} u_n, \quad f_n(y) = \min\{M_n, u(y)\} \quad (y \in \partial\Omega).$$

Then f_n is upper semi-continuous on $\partial\Omega$ and is therefore equal to the limit of some decreasing sequence in $C(\partial\Omega)$. Hence there exists $g_n \in C(\partial\Omega)$ such that $g_n \geq f_n$ on $\partial\Omega$ and

$$\int_{\partial\Omega} g_n \, d\sigma \leq n^{-1} + \int_{\partial\Omega} f_n \, d\sigma \leq n^{-1} + \int_{\partial\Omega} u \, d\sigma. \tag{5}$$

Let H_n denote the Perron-Wiener-Brelot (PWB) solution of the Dirichlet problem on Ω with boundary data g_n. (For the PWB approach to the Dirichlet problem we refer to Helms [5, Chapter 8].) Extend H_n to $\overline{\Omega}$ by defining $H_n = g_n$ on $\partial\Omega$. Then, since g_n is continuous on $\partial\Omega$ and Ω is (Dirichlet) regular, $H_n \in C(\overline{\Omega}) \cap \mathcal{H}$. Also, since $u_n \in \mathcal{S}_b$ and $u_n \leq g_n$ on $\partial\Omega$, it follows that u_n belongs to the lower PWB class for the Dirichlet problem with boundary data g_n, and hence $u_n \leq H_n$ on $\overline{\Omega}$. Since $u_n \to u$ on Ω, it follows that $\liminf_{n\to\infty} H_n \geq u$ on Ω. Hence, by Fatou's lemma,

$$\int_{S\cap\Omega} u \, d\sigma \leq \liminf_{n\to\infty} \int_{S\cap\Omega} H_n \, d\sigma.$$

Also, since $H_n = g_n \geq f_n$ on $\partial\Omega$ and (f_n) is increasing to the limit function u on $\partial\Omega$,

$$\int_{S\cap\partial\Omega} u \, d\sigma = \lim_{n\to\infty} \int_{S\cap\partial\Omega} f_n \, d\sigma \leq \liminf_{n\to\infty} \int_{S\cap\partial\Omega} H_n \, d\sigma.$$

Hence

$$\int_S u \, d\sigma \leq \liminf_{n\to\infty} \int_S H_n \, d\sigma. \tag{6}$$

By (5),

$$\limsup_{n\to\infty} \int_{\partial\Omega} H_n \, d\sigma \leq \int_{\partial\Omega} u \, d\sigma. \tag{7}$$

From (6) and (7) we obtain

$$\begin{aligned} \int_S u \, d\sigma \Big/ \int_{\partial\Omega} u \, d\sigma &\leq \liminf_{n\to\infty} \left(\int_S H_n \, d\sigma \Big/ \int_{\partial\Omega} H_n \, d\sigma \right) \\ &\leq M(\Omega, S, C(\overline{\Omega}) \cap \mathcal{H}), \end{aligned}$$

and the Proposition follows.

2.2 Example 2. If Ω is the open unit ball and S is any sphere such that $S \subset \overline{\Omega}$, $S \neq \partial\Omega$, then $M(\Omega, S, \mathcal{S}_{qb}) < 2$ but $M(\Omega, S, \mathcal{S}) = +\infty$.

The inequality in Example 2 is a consequence of Theorem 1. To prove the equality, fix a point y of $\partial\Omega \backslash S$ and define

$$h(x) = (1 - \|x\|^2)\|x - y\|^{-N} \quad (x \in \overline{\Omega}\backslash\{y\}), \quad h(y) = +\infty.$$

(Thus h is proportional to the Poisson kernel of Ω with pole y.) Define a sequence of functions (h_n) on $\overline{\Omega}$ by $h_n = n^{-1} + h$. Clearly $h_n \in \mathcal{H}$ and, writing z for the centre of S, we have

$$\int_S h_n \, d\sigma \Big/ \int_{\partial\Omega} h_n \, d\sigma = n\sigma(S)h_n(z)/\sigma(\partial\Omega) \to +\infty \quad (n \to \infty).$$

3. Proofs of the inequalities in Theorems 1-4

3.1. We need some more notation. The Green function of a Greenian domain ω is denoted by G_ω. The open ball and sphere of centre x and radius r in $\mathbb{R}^N$ are denoted by $B(x,r)$ and $S(x,r)$ respectively, and we write σ_N for $\sigma(S(0,1))$.

Since Ω is assumed to be Lipschitz, it follows from a result of Dahlberg [1] that harmonic measure and surface measure σ on $\partial\Omega$ are mutually equivalent; indeed

$$\frac{d\mu_x}{d\sigma}(y) = k_N \frac{\partial}{\partial n_y} G_\Omega(x,y) \qquad (x \in \Omega), \tag{8}$$

where μ_x denotes harmonic measure on $\partial\Omega$ relative to x, $k_N = (\sigma_N \max\{1, N-2\})^{-1}$, and $\partial/\partial n_y$ denotes differentiation in the direction of the inward normal to $\partial\Omega$ at y (when the normal exists). Let us denote the right-hand side of (8) by $K_\Omega(x,y)$. Then if $h \in C(\overline{\Omega}) \cap \mathcal{H}$, we have

$$h(x) = \int_{\partial\Omega} K_\Omega(x,y)h(y) \, d\sigma(y) \qquad (x \in \Omega).$$

We refer to K_Ω as the *Poisson kernel* of Ω.

The proofs of the inequalities in Theorems 1-4 are elaborations of the proofs given by Gabriel [2] and Reuter [8]. They depend on the following simple observations. If Ω_o is a Greenian domain containing Ω, then $G_\Omega \le G_{\Omega_o}$ on $\Omega \times \Omega$. Hence if, further, y is a point of $\partial\Omega \cap \partial\Omega_o$ at which $\partial\Omega$ and $\partial\Omega_o$ have a common normal, then

$$K_\Omega(\cdot, y) \le K_{\Omega_o}(\cdot, y)$$

on Ω.

In view of the Proposition, it is enough to prove the inequalities in Theorems 1–4 with $C(\overline{\Omega}) \cap \mathcal{H}$ in place of $\mathcal{S}_{qb}$.

3.2. Of the inequalities we have to prove, (2) in Theorem 2 is technically the simplest, so we begin with it. Let S be the sphere $S(w,\rho)$ and for each $y \in \partial\Omega$ let $B(z_y, R)$ be a ball such that $\Omega \subseteq B(z_y, R)$ and $y \in S(z_y, R)$. For simplicity, we write $B_y = B(z_y, R)$. Note that

$$K_{B_y}(x,y) = \frac{1}{\sigma_N R} \frac{R^2 - \|x - z_y\|^2}{\|x - z_y\|^N} \qquad (x \in B_y). \tag{9}$$

If $h \in C(\overline{\Omega}) \cap \mathcal{H}$, then it follows from the mean value property of harmonic functions and the remarks in Section 3.1 that

$$
\begin{aligned}
\int_S h\,d\sigma &= \sigma_N \rho^{N-1} h(w) \\
&= \sigma_N \rho^{N-1} \int_{\partial\Omega} K_\Omega(w,y) h(y)\,d\sigma(y) \qquad (10)\\
&\le \sigma_N \rho^{N-1} \int_{\partial\Omega} K_{B_y}(w,y) h(y)\,d\sigma(y).
\end{aligned}
$$

Since $\|w - z_y\| \ge \|z_y - y\| - \|w - y\| = R - \|w - y\|$, we obtain from (9) that

$$
K_{B_y}(w,y) \le \frac{1}{\sigma_N R} \frac{(2R - \|w-y\|)}{\|w-y\|^{N-1}} \le \frac{1}{\sigma_N R} \frac{(2R-\rho)}{\rho^{N-1}}.
$$

Hence, by (10),

$$
\int_S h\,d\sigma \le (2 - \rho/R) \int_{\partial\Omega} h\,d\sigma,
$$

as required.

3.3. To prove inequality (1) in Theorem 1, we imitate the proofs of the preceding section, but the role played by the balls B_y is now played by half-balls. Suppose that $y \in \partial\Omega$. Since we now suppose that Ω is convex, there is an open half-space D_y such that $\Omega \subset D_y$ and $y \in \partial D_y$. If d is the diameter of Ω, then Ω is contained in the open half-ball $D_y \cap B(y,d)$, which we denote by β_y. Arguing as before, we find that if $h \in C(\overline{\Omega}) \cap \mathcal{H}$, then

$$
\int_S h\,d\sigma \le \sigma_N \rho^{N-1} \int_{\partial\Omega} K_{\beta_y}(w,y) h(y)\,d\sigma(y), \qquad (11)
$$

where again $S = S(w,\rho)$. We now need to estimate $K_{\beta_y}(w,y)$. We have (see Kuran [6, p. 615])

$$
\begin{aligned}
K_{\beta_y}(w,y) &= \frac{2\mathrm{dist}(w,\partial D_y)}{\sigma_N} \left(\frac{1}{\|w-y\|^N} - \frac{1}{d^N} \right) \\
&\le \frac{2}{\sigma_N} \left(\frac{1}{\|w-y\|^{N-1}} - \frac{\|w-y\|}{d^N} \right) \qquad (12)\\
&\le (2/\sigma_N)(\rho^{1-N} - \rho d^{-N}).
\end{aligned}
$$

It follows from (11) and (12) that

$$
\int_S h\,d\sigma \le 2(1 - (\rho/d)^N) \int_{\partial\Omega} h\,d\sigma,
$$

as required.

3.4. We now turn to the proof of inequality (3) in Theorem 3. Here we cannot make direct use of the mean value property of harmonic functions, since it is not necessarily true that the ball $B(w, \rho)$ bounded by S is contained in Ω. (For instance, Ω might be an annular domain and S a concentric sphere.) The following lemma circumvents this minor difficulty.

Lemma. *Let $w, y, z \in \mathbb{R}^N$, $y \neq z$, let $\|y - z\| = R$ and let*

$$H(x) = \frac{\|x - z\|^2 - R^2}{\|x - y\|^N} \qquad (x \in \mathbb{R}^N \backslash \{y\}).$$

If $t > \|w - y\|$, then

$$\int_{S(w,t)} H \, d\sigma = \sigma_N t.$$

It is well known that H is harmonic on $\mathbb{R}^N \backslash \{y\}$. (In fact, modulo a constant factor, H is a harmonic continuation of the Poisson kernel of $B(z, R)$ with pole y.) We write $\mathcal{M}(t)$ for the mean value of H on $S(w, t)$; that is

$$\mathcal{M}(t) = (\sigma_N t^{N-1})^{-1} \int_{S(w,t)} H \, d\sigma.$$

Since H is harmonic on $\mathbb{R}^N \backslash \overline{B(w, \|w - y\|)}$, we have $\mathcal{M}(t) = a\phi_N(t) + b$ when $t > \|w - y\|$, where $\phi_2(t) = \log t$, $\phi_N(t) = t^{2-N}$ $(N \geq 3)$ and a, b are constants. (See e.g. Helms [5, Theorem 4.22].) It is easy to see that $t^{N-2}\mathcal{M}(t) \to 1$ as $t \to +\infty$. Hence $a = 0$, $b = 1$ when $N = 2$, and $a = 1$, $b = 0$ when $N \geq 3$. Thus in all cases $\mathcal{M}(t) = t^{2-N}$ when $t > \|w - y\|$, and this is equivalent to the required result.

Suppose now that the hypotheses of Theorem 3 are satisfied. For each $y \in \partial\Omega$ let $B(z_y, R)$ be a ball such that $B(z_y, R) \subset \mathbb{R}^N \backslash \overline{\Omega}$ and $y \in S(z_y, R)$. Define $A_y = \mathbb{R}^N \backslash \overline{B(z_y, R)}$. Then $\Omega \subset A_y$ and

$$K_{A_y}(x, y) = \frac{1}{\sigma_N R} \frac{\|x - z_y\|^2 - R^2}{\|x - y\|^N} \qquad (x \in A_y)$$

(see e.g. Helms [5, Chapter 9]). It is enough to prove that $M(\Omega, S, C(\overline{\Omega}) \cap \mathcal{H}) \leq 2 + \rho/R$ in the case where $S \subset \Omega$, for a continuity argument will then extend the inequality to the case where $S \subset \overline{\Omega}$, and inequality (3) will follow from the Proposition. Suppose then that $S \subset \Omega$ and $h \in C(\overline{\Omega}) \cap \mathcal{H}$. Then

$$\begin{aligned} \int_S h \, d\sigma &= \int_S \int_{\partial\Omega} K_\Omega(x, y) h(y) \, d\sigma(y) \, d\sigma(x) \\ &\leq \int_{\partial\Omega} \int_S K_{A_y}(x, y) \, d\sigma(x) h(y) \, d\sigma(y) \end{aligned}$$

by Fubini's theorem and the inequality $K_\Omega(x,y) \le K_{A_y}(x,y)$, which holds whenever $x \in \Omega$ and the normal to $\partial\Omega$ at y exists. If we show that

$$\int_S K_{A_y}(x,y)\,d\sigma(x) \le 2 + \rho/R \tag{13}$$

for each $y \in \partial\Omega$, then we shall have

$$\int_S h\,d\sigma \le (2+\rho/R)\int_{\partial\Omega} h\,d\sigma,$$

as required. If $\|w-y\| > \rho$, then by the mean value property of harmonic functions

$$\begin{aligned}\int_S K_{A_y}(x,y)\,d\sigma(x) &\le \sigma_N \rho^{N-1} K_{A_y}(w,y)\\ &\le \frac{\rho^{N-1}}{R}\frac{(\|w-y\|+R)^2 - R^2}{\|w-y\|^N}\\ &= \frac{\rho^{N-1}}{R}\frac{2R+\|w-y\|}{\|w-y\|^{N-1}}\\ &< 2+\rho/R.\end{aligned}$$

If $\|w-y\| < \rho$, then

$$\int_S K_{A_y}(x,y)\,d\sigma(x) = \rho/R,$$

by the Lemma. The case where $\|w-y\| = \rho$ does not arise, since we assume that $S \subset \Omega$. Hence (13) holds for each $y \in \partial\Omega$, and this completes the proof.

3.5. We come now to the proof of Theorem 4. Arguing as in the proof of Theorem 3, we see that it is enough to prove that $M(\Omega, S, C(\overline{\Omega}) \cap \mathcal{H}) < +\infty$ when Ω satisfies the exterior Liapunov-Dini condition and $S \subset \Omega$. Suppose then that $S = S(w,\rho) \subset \Omega$ and let ω be a Liapunov-Dini domain with the property that for each $y \in \partial\Omega$ there is an isometry T_y such that $y \in \overline{T_y(\omega)} \subset \mathbb{R}^N\backslash\Omega$. Define a function g on $\mathbb{R}^N\backslash\overline{\omega}$ as follows: in the case $N=2$, let g be the Green function of $\mathbb{R}^2\backslash\overline{\omega}$ with pole at ∞; in the case $N \ge 3$, let g be the solution of the Dirichlet problem on $\mathbb{R}^N\backslash\overline{\omega}$ with boundary data 0 on $\partial\omega$ and 1 at ∞. Then g is positive and harmonic on $\mathbb{R}^N\backslash\overline{\omega}$ and vanishes continuously on $\partial\omega$. Let ρ_1, ρ_2 be such that $0 < \rho_1 < \rho < \rho_2$ and the annular compact set $A = \{x : \rho_1 \le \|x-w\| \le \rho_2\}$ is contained in Ω. Let $c = \operatorname{dist}(\partial A, \partial\Omega)$ and let $C_1 = \inf\{g(x) : \operatorname{dist}(x,\partial\omega) \ge c\}$. Then $C_1 > 0$. If $z \in \partial A$ and $y \in \partial\Omega$, then

$$\operatorname{dist}(T_y^{-1}(z), \partial\omega) = \operatorname{dist}(z, T_y(\partial\omega)) \ge c.$$

Hence $g \circ T_y^{-1}(z) \ge C_1$. Now G_Ω has a finite supremum, C_2 say on $S \times \partial A$. For each $x \in S$ and each $y \in \partial\Omega$, the function $C_2 g \circ T_y^{-1} - C_1 G(x,\cdot)$ is harmonic on

$\Omega\backslash A$ and has non-negative limit at each point of $\partial\Omega\cup\partial A$. Hence, by the minimum principle,

$$G_\Omega(x,\cdot) \le (C_2/C_1) g \circ T_y^{-1}$$

on $\Omega\backslash A$. By a result of Widman [9], there is a positive constant C_3 such that

$$\frac{\partial g}{\partial \nu_y}(y) \le C_3 \qquad (y \in \partial\omega),$$

where $\partial/\partial\nu_y$ denotes differentiation in the direction of the inward normal to $R^N\backslash\overline{\omega}$ at y. It follows that

$$K_\Omega(x,y) = k_N \frac{\partial}{\partial n_y} G_\Omega(x,y) \le k_N C_2 C_3 / C_1 = C, \text{ say},$$

when $x \in S$ and y is a point of $\partial\Omega$ at which $\partial\Omega$ has a normal (and $\partial/\partial n_y$ denotes differentiation in the direction of the inward normal to $\partial\Omega$ at y). Hence if $h \in C(\overline{\Omega}) \cap \mathcal{H}$, then

$$\begin{aligned} \int_S h\,d\sigma &= \int_S \int_{\partial\Omega} K(x,y) h(y)\,d\sigma(y)\,d\sigma(x) \\ &\le \int_S C \int_{\partial\Omega} h(y)\,d\sigma(y)\,d\sigma(x) \\ &= C\sigma_N \rho^{N-1} \int_{\partial\Omega} h\,d\sigma, \end{aligned}$$

so that $M(\Omega, S, C(\overline{\Omega}) \cap \mathcal{H}) \le C\sigma_N\rho^{N-1}$.

4. Examples

4.1. Examples 3, 4, 5 below demonstrate the assertions of sharpness in Theorems 1, 2, 3 respectively.

Example 3. Let Ω be an open half-ball of diameter d. If $0 < \rho < d/4$, then there is a sphere S of radius ρ contained in Ω such that

$$M(\Omega, S, \mathcal{S}_{qb}) \ge 2(1 - 2^N(\rho/d)^N). \tag{14}$$

Without loss of generality, let

$$\Omega = \{x \in \mathbb{R}^N : \|x\| < d/2, x_N > 0\}$$

and let

$$\tau = \{x \in \partial\Omega : x_N = 0\}, \quad \nu = \{x \in \partial\Omega : x_N > 0\}.$$

For each positive integer n, define

$$h_n(x) = (x_N + \frac{d}{2n}) \left\{ \left(x_1^2 + \ldots + x_{N-1}^2 + (x_N + \frac{d}{2n})^2 \right)^{-N/2} - \left(\frac{d}{2} + \frac{d}{2n} \right)^{-N} \right\}.$$

Then $h_n \in C(\overline{\Omega}) \cap \mathcal{H}$. Clearly $h_n \to 0$ uniformly on ν as $n \to \infty$. Hence

$$\int_\nu h_n \, d\sigma \to 0 \qquad (n \to \infty). \tag{15}$$

Also,

$$\begin{aligned}
\int_\tau h_n \, d\sigma &= \frac{d}{2n} \int_0^{d/2} \{(t^2 + (\frac{d}{2n})^2)^{-N/2} - (d/2)^{-N}(1 + \frac{1}{n})^{-N}\} \sigma_{N-1} t^{N-2} \, dt \\
&= \frac{d\sigma_{N-1}}{2n} \int_0^{d/2} t^{N-2}(t^2 + (\frac{d}{2n})^2)^{-N/2} \, dt + o(1) \\
&= \sigma_{N-1} \int_0^n s^{N-2}(s^2 + 1)^{-N/2} ds + o(1) \qquad (16) \\
&\to \sigma_{N-1} \sqrt{\pi} \frac{\Gamma((N-1)/2)}{2\Gamma(N/2)} \quad (n \to \infty).
\end{aligned}$$

If S is the sphere of centre $(0, \ldots, 0, \rho)$ and radius ρ, then

$$\begin{aligned}
\int_S h_n \, d\sigma &= \sigma_N \rho^{N-1} h_n(0, \ldots, 0, \rho) \\
&= \sigma_N \rho^{N-1} (\rho + \frac{d}{2n}) \{(\rho + \frac{d}{2n})^{-N} - (\frac{d}{2} + \frac{d}{2n})^{-N}\} \qquad (17) \\
&\to \sigma_N \rho^N \{\rho^{-N} - (d/2)^{-N}\} \quad (n \to \infty).
\end{aligned}$$

Recalling that $\sigma_N = N\pi^{N/2}/\Gamma(N/2+1)$, we obtain from (15), (16) and (17) after some simplification that

$$\lim_{n\to\infty} \left(\int_S h_n \, d\sigma \Big/ \int_{\partial\Omega} h_n \, d\sigma \right) = 2(1 - 2^N (\rho/d)^N),$$

and this establishes (14).

4.2 Example 4. If $\Omega = B(0, R)$ and $0 < \rho < R$, then there exists a sphere S of radius ρ contained in Ω such that

$$M(\Omega, S, \mathcal{S}_{qb}) = 2 - \rho/R.$$

Let $y^{(n)} = (0, \ldots, 0, R + 1/n)$ and define

$$h_n(x) = ((R + 1/n)^2 - \|x\|^2)\|x - y^{(n)}\|^{-N}$$

for each positive integer n. Then $h_n \in C(\overline{\Omega}) \cap \mathcal{H}$ and

$$\int_{\partial\Omega} h_n \, d\sigma = \sigma_N R^{N-1} h_n(0) = \sigma_N R^{N-1}(R + 1/n)^{2-N}.$$

If S is the sphere of centre $(0, \dots, 0, R - \rho)$ and radius ρ, then

$$\int_S h_n \, d\sigma = \sigma_N \rho^{N-1} h_n(0, \dots, 0, R - \rho) = \sigma_N \rho^{N-1} \frac{(R + \frac{1}{n})^2 - (R - \rho)^2}{(\rho + 1/n)^N}.$$

Hence

$$\lim_{n\to\infty} \left(\int_S h_n \, d\sigma \middle/ \int_{\partial\Omega} h_n \, d\sigma \right) = 2 - \rho/R.$$

Thus $M(\Omega, S, \mathcal{S}_{qb}) \geq 2 - \rho/R$. The reverse inequality follows from Theorem 2.

4.3 Example 5. Let

$$\Omega = \{x \in \mathbb{R}^N : R < \|x\| < 2R\}.$$

If $\epsilon > 0$ and ρ is sufficiently small, then there exists a sphere S of radius ρ contained in $\overline{\Omega}$ such that

$$M(S, \Omega, \mathcal{S}_{qb}) > 2 + (1 - \epsilon)\rho/R. \tag{18}$$

For each integer $n > 1/R$ let $y^{(n)} = (0, \dots, 0, R - 1/n)$ and define

$$p_n(x) = (\|x\|^2 - (R - 1/n)^2)\|x - y^{(n)}\|^{-N}.$$

Let q_n be the solution of the Dirichlet problem on Ω with boundary data 0 on $S(0, R)$ and p_n on $S(0, 2R)$. Define $h_n = p_n - q_n$ on Ω and extend h_n continuously to $\overline{\Omega}$. Then $h_n \in C(\overline{\Omega}) \cap \mathcal{H}$ and

$$\int_{\partial\Omega} h_n \, d\sigma = \int_{S(0,R)} p_n \, d\sigma = \sigma_N R,$$

by the Lemma. If S is the sphere of centre $(0, \dots, 0, R + \rho)$ and radius $\rho < R/2$, then

$$\begin{aligned} \int_S h_n \, d\sigma &= \sigma_N \rho^{N-1}(p_n - q_n)(0, \dots, 0, R + \rho) \\ &\geq \sigma_N \rho^{N-1} \left(\frac{2R + \rho - \frac{1}{n}}{(\rho + \frac{1}{n})^{N-1}} - A_n \right), \end{aligned}$$

where $A_n = \sup_S q_n$. Hence

$$\liminf_{n\to\infty} \left(\int_S h_n \, d\sigma \middle/ \int_{\partial\Omega} h_n \, d\sigma \right) \geq 2 + (\rho/R)(1 - \rho^{N-2} \limsup A_n). \tag{19}$$

On $S(0,2R)$, we have

$$p_n \leq (3R - \frac{1}{n})(R + \frac{1}{n})^{1-N} < 3R^{2-N}. \tag{20}$$

Hence $q_n < 3R^{2-N}$ on Ω, and in the case $N \geq 3$ it follows that the expression on the right-hand side of (19) exceeds $2 + (1-\epsilon)(\rho/R)$ if ρ is sufficiently small. In the case $N = 2$, q_n has boundary values 0 on $S(0,R)$ and boundary values less than 3 on $S(0,2R)$, by (20). Hence by the maximum principle,

$$q_n(x) \leq (3/\log 2)\log(\|x\|/R) \qquad (x \in \Omega),$$

so that $A_n \leq (3/\log 2)\log(1+2\rho/R)$, and again the right-hand side of (19) exceeds $2 + (1-\epsilon)\rho/R$ if ρ is small enough. It follows in all cases that (18) holds if ρ is small enough.

4.4. Finally, we verify Example 1. There exist a number $p \in (0,1)$ and a non-negative continuous function f on $S(0,1) \cap \partial\Omega$ such that the function h defined by

$$h(x) = (\|x\|^{-p-N+2} - \|x\|^p)f(x/\|x\|)$$

is harmonic on the cone $\Omega_o = \{x : x_N > \|x\| \cos\alpha\}$, tends to 0 at each point of $\partial\Omega_o \backslash \{0\}$, and is positive on Ω. (See e.g. Kuran [7, Theorems 1, 2].) We normalize so that $\sup f = 1$. For each positive integer n let $y^{(n)} = (0, \ldots, 0, -1/n)$ and define

$$h_n(x) = h(x - y^{(n)}) + (1 + \frac{1}{n})^p - (1 + \frac{1}{n})^{-p-N+2}.$$

Then $h_n \in C(\overline{\Omega}) \cap \mathcal{H}$. We shall show that

$$\int_{\partial\Omega} h_n \, d\sigma \to 0 \qquad (n \to \infty). \tag{21}$$

Clearly $h_n \to 0$ uniformly on $S(0,1) \cap \partial\Omega$, so

$$\int_{S(0,1)\cap\partial\Omega} h_n \, d\sigma \to 0 \qquad (n \to \infty). \tag{22}$$

If $x \in \partial\Omega \backslash S(0,1)$, then

$$\|x - y^{(n)}\| \geq \min\{n^{-1}\sin\alpha, \|x\| - n^{-1}\}$$

and hence

$$\|x - y^{(n)}\| \geq \|x\| \sin\alpha/(1 + \sin\alpha),$$

so that

$$h_n(x) \leq A\|x\|^{-p-N+2} + 2^p \qquad (x \in \partial\Omega \backslash S(0,1)),$$

where $A = (1 + \operatorname{cosec} \alpha)^{p+N-2}$. Since $p < 1$, the function $\|x\|^{-p-N+2}$ is integrable on $\partial\Omega \backslash S(0,1)$. Since also $h_n \to 0$ pointwise on $\partial\Omega\backslash\{0\}$, we obtain by dominated convergence that

$$\int_{\partial\Omega\backslash S(0,1)} h_n \, d\sigma \to 0 \qquad (n \to \infty). \tag{23}$$

From (22) and (23) we obtain (21). If $S = S(w, \rho) \subset \overline{\Omega}$, then

$$\int_S h_n \, d\sigma = \sigma_N \rho^{N-1} h_n(w) \to \sigma_N \rho^{N-1} h(w) > 0 \qquad (n \to \infty).$$

Hence $M(\Omega, S, \mathcal{S}_{qb}) = +\infty$.

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