Convergence in compacta and linear Lindelöfness

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Abstract. Let X be a compact Hausdorff space with a point x such that $X \setminus \{x\}$ is linearly Lindelöf. Is then X first countable at x? What if this is true for every x in X? We consider these and some related questions, and obtain partial answers; in particular, we prove that the answer to the second question is "yes" when X is, in addition, ω-monolithic. We also prove that if X is compact, Hausdorff, and $X \setminus \{x\}$ is strongly discretely Lindelöf, for every x in X, then X is first countable. An example of linearly Lindelöf hereditarily realcompact non-Lindelöf space is constructed. Some intriguing open problems are formulated.

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Let X be a space, and P a subset of X. *The Hušek number of P in X*, denoted by $Hus(P,X)$, is the smallest infinite cardinal number τ satisfying the following condition:

(H) for every subset A of $X \setminus P$ such that the cardinality of A is a regular cardinal number which is not less than τ, there is an open neighborhood U of P in X such that $|A \setminus U| = |A|$.

Clearly, $Hus(P,X)$ is always defined. This definition is motivated by the notion of a space *without κ-accessible diagonal* introduced by M. Hušek (see [9] and [10]). The Hušek number is closely related to the pseudocharacter $\psi(P,X)$ of P in X, which is the smallest infinite cardinal number τ such that there exists a family γ of open sets in X such that $\cap\gamma = P$ and $|\gamma| \leq \tau$. Indeed, we obviously have:

Proposition 1. *For any space X and any subset P of X, the Hušek number $Hus(P,X)$ does not exceed the first cardinal number which is greater than $\psi(P,X)$.*

If $P = \{x\}$ for some $x \in X$, we write $Hus(x,X)$ instead of $Hus(\{x\},X)$, and call $Hus(x,X)$ *the Hušek number of X at x*. Of course, *the Hušek number $Hus(X)$* of a space X is the supremum of $Hus(x,X)$, when x runs over X.

The following assertion is obvious:

Proposition 2. *If X is a compact Hausdorff space, and $x \in X$, then* the Hušek number of X *at x is the smallest infinite cardinal number τ satisfying the following condition:*

(NC) *for every infinite subset A of X such that the cardinality $|A|$ of A is a regular cardinal number which is not less than τ, there is a point of complete accumulation in X different from x.*

An infinite subset A of a space X is said to *converge* to a point $x \in X$, if for every open neighborhood U of x, the cardinality of $A \setminus U$ is smaller than the cardinality of A. Clearly, we have:

Proposition 3. *$Hus(x, X)$ is the smallest infinite cardinal number τ such that if an infinite subset $A \subset X$ of regular cardinality converges to x, then the cardinality of A is less than τ.*

It is well known that a space X is compact, if and only if for every infinite subset of X there exists a point of complete accumulation ([1]). On the other hand, the following condition (introduced in [1]):

(CAP) *every uncountable subset A of X of regular cardinality has a point of complete accumulation in X*,

does not characterize Lindelöf spaces. A non-normal Tychonoff space of this kind was constructed by A.S. Mischenko [12]. The spaces satisfying (CAP) were later renamed into *linearly Lindelöf*, or *chain-Lindelöf* spaces (see [13]), since the condition (CAP) turned out to be equivalent to the following requirement: *every open covering γ of X which is a chain* (that is, for any two elements of γ, one is a subset of the other one) *contains a countable subcovering of X.*

All spaces considered are assumed to be T_1. In what follows, τ is an infinite cardinal number, which we also interpret as the first ordinal number of cardinality τ.

Proposition 2 shows that the notion of Hušek number is closely related to the notion of a linearly Lindelöf space. Indeed, we obviously have:

Proposition 4. *Let X be a compact Hausdorff space and $x \in X$. Then $Hus(x, X) \leq \omega_1$, if and only if the space $X \setminus \{x\}$ is linearly Lindelöf.*

One can compare this fact with the following one:

Theorem 5. *If X is a compact Hausdorff space, $x \in X$, and $Hus(x, X) = \omega$, then the point x is isolated in X.*

Proof: Let $Y = X \setminus \{x\}$. Since $Hus(x, X) = \omega$, it follows from Proposition 2 that the space Y is countably compact. On the other hand, Proposition 4 implies that Y is linearly Lindelöf. Every linearly Lindelöf countably compact space is compact. Therefore, Y is compact. Since X is Hausdorff, Y is closed in X. It follows that x is isolated in X. $\square$

Theorem 5 and Proposition 4 motivate the following question:

Question 1. Let X be a compact Hausdorff space, and $x \in X$, such that the space $Y = X \setminus \{x\}$ is linearly Lindelöf (that is, $Hus(x, X) \leq \omega_1$). Is then X first countable at x? Is this true at least under (CH)?

If we assume that the condition above is satisfied not at a fixed point x but for all x in X, then a consistent answer is available:

Theorem 6. *If (CH) holds, and X is a compact Hausdorff space such that $X \setminus \{x\}$ is linearly Lindelöf for each $x \in X$ (that is, $Hus(X) \leq \omega_1$), then X is first countable.*

PROOF: Indeed, in view of Propositions 3 and 4, this theorem is contained in the following result of Juhász (see [11]): *under (CH), if X is a compact Hausdorff space, then either X is first countable, or there is a subset A of X such that $|A| = \omega_1$ and A converges to some point x in X.* □

Theorem 6 makes Question 1 especially interesting.

We are going to establish some results in the direction of Question 1 using our recent results in [5].

A space X is *weakly ω-monolithic*, if the closure of any countable discrete subspace of X has a countable network.

Theorem 7. *Let X be a weakly ω-monolithic compact Hausdorff space such that $Hus(X) \leq \omega_1$ (that is, $X \setminus \{x\}$ is linearly Lindelöf for each $x \in X$). Then X is first countable.*

PROOF: First, we need the next result:

Lemma 8. *If X is a compact Hausdorff space such that $X \setminus \{x\}$ is linearly Lindelöf for each $x \in X$, then the tightness of X is countable.*

PROOF OF THE LEMMA: Assume the contrary. Then, according to a fundamental result of Juhász and SzentMiclossy (see [11]), there is a set A of cardinality ω_1 converging to a point $x \in X$, which contradicts Proposition 3. □

We continue the proof of Theorem 7. Let x be any point of X, and $Y = X \setminus \{x\}$. By Lemma 8 and the assumptions, Y is a weakly ω-monolithic linearly Lindelöf space of countable tightness (note that weak ω-monolithicity and countable tightness are both inherited by subspaces). Now we apply the next result from [5]:

Theorem 9. *Every weakly ω-monolithic linearly Lindelöf space of countable tightness is Lindelöf.*

It follows from Theorem 9 that the space Y is Lindelöf. Then Y is σ-compact, since Y is locally compact, which implies that $\{x\}$ is a G_δ-set in X. Therefore, since X is compact and Hausdorff, X is first countable at x. The proof of Theorem 7 is complete. □

A space X is *ω_1-Lindelöf*, if every open covering γ of X such that $|\gamma| \leq \omega_1$ contains a countable subcovering. Every linearly Lindelöf space is ω_1-Lindelöf. The proof of Theorem 7 easily generalizes to the case when $X \setminus \{x\}$ is ω_1-Lindelöf for each $x \in X$.

Recall that a space X is said to be *strongly discretely Lindelöf*, if the closure of every discrete subspace of X is Lindelöf [4]. It is still an open question, whether every strongly discretely Lindelöf Tychonoff space is Lindelöf ([4]). On the other hand, every strongly discretely Lindelöf space is linearly Lindelöf ([4]).

A point x of a space X will be called *a dl-point* (*an ll-point*), if the space $X \setminus \{x\}$ is strongly discretely Lindelöf (linearly Lindelöf). Every dl-point is an ll-point, so the next result would become a corollary from Theorem 6, if we could drop CH in it.

Theorem 10. *Let X be a compact Hausdorff space. Then X is first countable, if and only if every point of X is a dl-point.*

PROOF: Clearly, we only have to prove that if every point of X is a dl-point, then X is first countable. By Lemma 8, the tightness of X is countable. Fix $x \in X$, and put $Y = X \setminus \{x\}$. Then Y is a strongly discretely Lindelöf Tychonoff space of countable tightness. It was shown in [5] that every such space is Lindelöf. Therefore, Y is Lindelöf. Since X is compact and Hausdorff, it follows that X is first countable at x. □

Question 2. Is every locally compact linearly Lindelöf (strongly discretely Lindelöf) Hausdorff space Lindelöf?

Question 3. Is every normal locally compact linearly Lindelöf space Lindelöf?

Since every countably paracompact linearly Lindelöf Tychonoff space is Lindelöf (see [13], [4]), to answer Question 3 in negative, we have to produce an especially nice Dowker space (see [13] about Dowker spaces).

Question 4. Let X be a compact Hausdorff space such that $Hus(X) \leq \omega_1$. Is then true that the cardinality of X is not greater than 2^ω?

Note that the answer is "yes" under (CH) (see Theorem 6). Clearly, a positive answer to Question 4 would give a generalization of the theorem on cardinality of first countable compacta in [2].

It is worthwhile to note that compactness in Theorem 7 is needed mainly to prove that the tightness of X is countable. Indeed, we have the following corollary of Theorem 9:

Corollary 11. *If X is a Tychonoff weakly ω-monolithic space of countable tightness, then every ll-point in X is a G_δ in X.*

Under (CH) one can drop in Theorem 9 the assumption that X is weakly ω-monolithic and replace linear Lindelöfness by ω_1-Lindelöfness (see [5]). Therefore, the following is true:

Theorem 12. *Under (CH), if X is a Tychonoff space of countable tightness, and $x \in X$ is such that $X \setminus \{x\}$ is ω_1-Lindelöf, then x is a G_δ in X.*

The questions we consider in this article turn out to be naturally related to the notion of a c-sequential space (see [3]). A space X is called *c-sequential*, if for every closed subspace Y of X and every non-isolated point y of the space Y, there exists a sequence in $Y \setminus \{y\}$ converging to y. Every c-sequential compact Hausdorff space is countably tight (see [11]), and, under Martin's Axiom, sequential (see [3]). We have:

Theorem 13. *Every countably compact Hausdorff space X such that $Hus(X) \leq \omega_1$ is c-sequential.*

PROOF: Let Y be a closed subspace of X and y a non-isolated point in Y. Then the set $A = Y \setminus \{y\}$ is not closed in Y; therefore, the subspace A is not compact. Since every countably compact linearly Lindelöf space is compact, it follows that A is not countably compact. Take any sequence $(y_n : n \in \omega)$ which does not have accumulation points in A. Then $(y_n : n \in \omega)$ converges to y, since Y is countably compact. □

Corollary 14. *Under Martin's Axiom, every compact Hausdorff space X such that $Hus(X) \leq \omega_1$, is sequential.*

Question 5. Is it true in *ZFC* that every compact Hausdorff space X such that $Hus(X) \leq \omega_1$, is sequential?

Example 15. We will construct a consistent example of a linearly Lindelöf hereditarily realcompact non-Lindelöf space H, answering a question in [5]. First, we will describe a linearly Lindelöf Tychonoff non-Lindelöf space X which was introduced independently by R. Buzyakova (see [5]) and G. Gruenhage. Let D be the standard discrete two-point set, with elements 0 and 1, and $\tau = \aleph_\omega$, that is, τ is the first uncountable cardinal number cofinal to ω. Fix a set A of cardinality τ, and consider the product space D^A, with the usual product topology. For a point $x \in D^A$ we denote by A_x the set of all $a \in A$ such that the corresponding coordinate x_a of x is 1. Let X be a subspace of D^A consisting of all points $x \in D^A$ such that $|A_x| < \tau$.

Clearly, X is Tychonoff. It is easy to see that X is pseudocompact, but not compact. Therefore, X is not Lindelöf.

To see that X is linearly Lindelöf, take any uncountable subset B of X such that $|B|$ is a regular cardinal number. Clearly, the weight of X is not greater than τ. Therefore, since τ is not regular, we only have to consider the case when $|B| < \tau$. It is shown in [5] that then there exists a compact subspace K of X such that $|K \cap B| = |B|$. Now it is clear that B has a point of complete accumulation in K. Thus, X is linearly Lindelöf.

Now we will construct the space H; here we use an idea of R. Haydon [7].

Let I be the closed unit interval of the real line, and $Z = X \times I$. We denote by π the natural projection of Z onto I. Let us consider the following condition consistent with *ZFC*:

(A) $2^\omega = 2^\tau$ for $\tau = \aleph_\omega$.

We are going to show that if (A) is satisfied, then there exists a subspace H of Z such that the restriction of π to H is a one-to-one mapping of H, and the other projection maps H onto X.

We shall do this by transfinite recursion along the ordinal number 2^ω. In the construction τ stands for $\aleph_\omega$. Let $\mathcal{P}$ be the family of all compact subspaces K of Z such that $\pi(K)$ is uncountable. The weight of Z is $\tau = \aleph_\omega$; therefore, the cardinality of $\mathcal{P}$ is not greater than 2^τ (see [8]). From (A) it follows that $|\mathcal{P}| \leq 2^\omega$. Thus, we can represent $\mathcal{P}$ in the form:

$$\mathcal{P} = \{K_\alpha : \alpha < 2^\omega\}.$$

Assume now that for some $\alpha < 2^\omega$ and every $\beta < \alpha$ a point $z_\beta = (x_\beta, y_\beta) \in K_\beta$ is already chosen. Note that $|\pi(K)| = 2^\omega$ for every $K \in \mathcal{P}$. Indeed, if $K \in \mathcal{P}$, then $\pi(K)$ is an uncountable compact subspace of the interval I; therefore $|\pi(K)| = 2^\omega$. Thus, $|\pi(K_\alpha)| = 2^\omega$. We put $M_\alpha = \{y_\beta : \beta < \alpha\}$. Since $|M_\alpha| < 2^\omega$, we can choose a point $z_\alpha = (x_\alpha, y_\alpha) \in K_\alpha$ such that $y_\alpha \in \pi(K_\alpha) \setminus M_\alpha$.

Let H be the set of the points of the transfinite sequence $\{(x_\alpha, y_\alpha) : \alpha < 2^\omega\}$ defined in this way. Clearly, the restriction of π to H is a one-to-one continuous mapping of H into I. Therefore the space H is hereditarily realcompact (see [6]).

Let us show that H is linearly Lindelöf. Let A be any uncountable subset of H of regular cardinality. If $|A| > \aleph_\omega$, then, since the weight of H is $\aleph_\omega$, there exists a point of complete accumulation for A in H.

It remains to consider the case when $|A| < \aleph_\omega$. Let p be the projection mapping of Z onto X. There are two possibilities:

Case 1. $|p(A)| < |A|$. Then, since $|A|$ is regular, there is $x \in X$ such that $|(\{x\} \times I) \cap A| = |A|$.

Case 2. $|p(A)| = |A|$. Since $|p(A)| < \aleph_\omega$ and $|p(A)|$ is regular, there is a compact subspace B of X such that $|B \cap p(A)| = |p(A)| = |A|$. Then $B \times I$ is a compact subspace of Z such that $|(B \times I) \cap A| = |A|$.

Thus, in any case there exists a compact subspace F of Z such that $|F \cap A| = |A|$. Since $A \subset H$, π is one-to-one on A; therefore, the set $L_1 = \pi(F \cap A)$ is an uncountable subset of I. Let C be the set of all points of complete accumulation of L_1 in I. Clearly, C is an uncountable compact subspace of I, and $C \subset \pi(F)$.

Take any $c \in C$, and consider $S = \{z \in F : \pi(z) = c\}$. Since the restriction of π to F is a perfect mapping, at least one point of the set S must be a point of complete accumulation of the set $A_1 = A \cap F$. Therefore, the set K of all points of complete accumulation of A_1 in F satisfies the following condition:

$$\pi(K) = C.$$

It follows that $\pi(K)$ is uncountable. Clearly, K is a compact subset of Z. Thus, $K \in \mathcal{P}$. Then, by the definition of H, $H \cap K \neq \emptyset$, and any point of $H \cap K$ is a point of complete accumulation of A. Therefore, H is linearly Lindelöf. The space X is a continuous image of H, since $\{x\} \times I \in \mathcal{P}$. Since X is not Lindelöf, it follows that H is not Lindelöf.

It is easy to modify H in Example 15 in such a way that the image of H under the projection π would coincide with I (just fix $a \in X$ and add to H the subset E of $\{a\} \times I$ such that $\pi(E) = I \setminus \pi(H)$). Therefore, we have:

Proposition 16. *Under the assumption* (A)*, there is a Tychonoff topology* $\mathcal{T}$ *on the closed unit interval* I *such that the space* $(I, \mathcal{T})$ *is Tychonoff, hereditarily realcompact, linearly Lindelöf and not Lindelöf.*

Note that we cannot prove Proposition 16 in ZFC, since under (CH) every linearly Lindelöf topology on I containing the usual topology of I is obviously Lindelöf.

Question 6. Can one add to the properties of $(I, \mathcal{T})$ in Proposition 16 local compactness (and local metrizability or, at least, first countability)?

Question 7. Is every locally compact ω_1-Lindelöf Hausdorff space of countable tightness Lindelöf (in ZFC)?

Note that if there were a consistent example of a non-metrizable compact Hausdorff space without ω_1-accessible diagonal (that is, with a small diagonal) — which would provide an answer to a famous problem of M. Hušek (see [9]), — then the answer to Question 7 would be in negative.

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