Continuous functions between Isbell-Mrówka spaces

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Abstract. Let $\Psi(\Sigma)$ be the Isbell-Mrówka space associated to the MAD-family Σ. We show that if G is a countable subgroup of the group $\mathbf{S}(\omega)$ of all permutations of ω, then there is a MAD-family Σ such that every $f \in G$ can be extended to an autohomeomorphism of $\Psi(\Sigma)$. For a MAD-family Σ, we set $Inv(\Sigma) = \{f \in \mathbf{S}(\omega) : f[A] \in \Sigma$ for all $A \in \Sigma\}$. It is shown that for every $f \in \mathbf{S}(\omega)$ there is a MAD-family Σ such that $f \in Inv(\Sigma)$. As a consequence of this result we have that there is a MAD-family Σ such that $n + A \in \Sigma$ whenever $A \in \Sigma$ and $n < \omega$, where $n + A = \{n + a : a \in A\}$ for $n < \omega$. We also notice that there is no MAD-family Σ such that $n \cdot A \in \Sigma$ whenever $A \in \Sigma$ and $1 \leq n < \omega$, where $n \cdot A = \{n \cdot a : a \in A\}$ for $1 \leq n < \omega$. Several open questions are listed.

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1. Introduction

If X is a set, then $[X]^\omega = \{A \subseteq X : |A| = \omega\}$, and the meaning of $[X]^{<\omega}$ and $[X]^{\leq\omega}$ should be clear. For $A, B \in [\omega]^\omega$, we write $A \subseteq^* B$ if $A - B$ is finite and we write $A =^* B$ if $A \subseteq^* B$ and $B \subseteq^* A$. The Stone-Čech compactification $\beta(\omega)$ of the discrete space ω is identified with the set of all ultrafilters on ω and its remainder $\omega^* = \beta(\omega) - \omega$ is identified with the set of all free ultrafilters on ω. For $A \in [\omega]^\omega$, we write $\widehat{A} = cl_{\beta(\omega)}(A)$ and $A^* = \widehat{A} - A$. Observe that $A =^* B$ iff $A^* = B^*$ for $A, B \in [\omega]^\omega$. For $\mathcal{A} \subseteq [\omega]^\omega$, we define $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$. If $f : \omega \to \omega$ is a function, then $\beta f : \beta(\omega) \to \beta(\omega)$ will stand for the Stone-Čech extension of f. The group of permutations of ω is denoted by $\mathbf{S}(\omega)$, where the operation in $\mathbf{S}(\omega)$ is the usual multiplication of permutations. If $f : \omega \to \omega$ is a function, then f^0 will denote the identity map on ω.

Definition 1.1. *An almost disjoint* (AD) *family of subsets of* ω *is an infinite subset* Σ *of* $[\omega]^\omega$ *such that* $|A \cap B| < \omega$ *whenever* $A, B \in \Sigma$ *and* $A \neq B$. *If* Σ *is an AD-family of subsets of* ω *and it is not a proper subset of any AD-family, then* Σ *is called a maximal almost disjoint* (MAD-) *family.*

It is well-known that there is a MAD-family of cardinality equal to the continuum c (see [GJ, 6Q. 1]) and every MAD-family has cardinality strictly bigger than ω (see [CN, Lemma 12.19]). We remark that if Σ is an AD-family, then Σ^* is a set of pairwise disjoint clopen subsets of ω^* and Σ is a MAD-family iff $\bigcup \Sigma^*$ is a dense subset of ω^*. Conversely, if $\mathcal{O} = \{C_i : i \in I\}$ is a set of pairwise disjoint clopen subsets of ω^* and $\Sigma = \{A_i : i \in I\} \subseteq [\omega]^\omega$ satisfies that $A_i^* = C_i$ for every $i \in I$ and $|A_i \cap B_j| < \omega$ whenever $i, j \in I$ and $i \neq j$, then Σ is an AD-family with $\mathcal{O} = \Sigma^*$. The *almost disjointness number* is $\mathfrak{a} = \min\{|\Sigma| : \Sigma \text{ is a } MAD\text{-family}\}$.

Let Σ be an AD-family. The Isbell-Mrówka space $\Psi(\Sigma)$ associated to Σ is the space whose underlying set is $\omega \cup \Sigma$ and ω is a discrete open subset of $\Psi(\Sigma)$ and a basic open neighborhood of $A \in \Sigma$ has the form $\{A\} \cup E$, where E is a cofinite subset of A. The space $\Psi(\Sigma)$ is a separable, locally compact, zero-dimensional, Tychonoff space for any AD-family Σ. These spaces were discovered independently by J. Isbell and S. Mrówka. It is shown in [Mr] that Σ is a MAD-family if and only if the space $\Psi(\Sigma)$ is pseudocompact. In this article, all the Isbell-Mrówka spaces will be those associated to a MAD-family.

We are primarily concerned with determining when a permutation of ω can be extended to a homeomorphism between two given Isbell-Mrówka spaces. We begin Section 2 with some basic results and we show that if G is a countable subgroup of $\mathbf{S}(\omega)$, then there is a MAD-family Σ such that every element f of G can be extended to an autohomeomorphism of $\Psi(\Sigma)$. We also show here that for every $f \in \mathbf{S}(\omega)$ there is a MAD-family Σ such that $f \in Inv(\Sigma)$, where $Inv(\Sigma) = \{g \in \mathbf{S}(\omega) : g[A] \in \Sigma \text{ for all } A \in \Sigma\}$. Hence, in particular, there is a MAD-family Σ such that $n + A \in \Sigma$ whenever $A \in \Sigma$ and $n < \omega$, where $n + A = \{n + a : a \in A\}$ for $n < \omega$.

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2. Continuous extensions

The following lemma gives a condition for a function $f : \omega \to \omega$ to be extended to a continuous function from $\Psi(\Sigma_0)$ to $\Psi(\Sigma_1)$, where Σ_0 and Σ_1 are MAD-families.

Lemma 2.1. *Let Σ_0 and Σ_1 be MAD-families and $f : \omega \to \omega$ a finite-to-one function. Then, the following are equivalent:*

(1) *f extends to a continuous function h from $\Psi(\Sigma_0)$ to $\Psi(\Sigma_1)$ with $h[\Sigma_0] \subseteq \Sigma_1$;*
(2) *for every $A \in \Sigma_0$ there is $B \in \Sigma_1$ such that $f[A]^* \subseteq B^*$;*
(3) *for every $A \in \Sigma_0$ there is $B \in \Sigma_1$ such that $f[A] \subseteq^* B$;*
(4) *$\beta f : \beta(\omega) \to \beta(\omega)$ satisfies that for every $A \in \Sigma_0$ there is $B \in \Sigma_1$ such that $\beta f[A^*] \subseteq B^*$.*

Proof: (3) $\Leftrightarrow$ (4) is evident.

(1) $\Rightarrow$ (2). Let $h : \Psi(\Sigma_0) \to \Psi(\Sigma_1)$ be a continuous extension of f and let $A \in \Sigma_0$. Put $B = h(A)$. Since $V = \{B\} \cup B$ is a neighborhood of B, then there is a finite subset F of A such that $\{A\} \cup (A - F) \subseteq h^{-1}(V)$. Hence, $h[A - F] = f[A - F] \subseteq B$ and since F is finite, $f[A] \subseteq^* B$.

(2) $\Leftrightarrow$ (3). This is evident.

(3) $\Rightarrow$ (1). For every $A \in \Sigma_0$, we fix $B_A \in \Sigma_1$ such that $f[A] \subseteq^* B_A$. Then, we define $h : \Psi(\Sigma_0) \to \Psi(\Sigma_1)$ by $h \restriction_\omega = f$ and $h(A) = B_A$ for every $A \in \Sigma_0$. Choose $A \in \Sigma_0$ and let $V = \{B_A\} \cup (B_A - E)$, where E is a finite subset of B_A. Set $F = f[A] - B_A$. Then, F is a finite set and hence $U = \{A\} \cup (A - (f^{-1}(E \cup F)))$ is a neighborhood of A in $\Psi(\Sigma_0)$ and $h[U] \subseteq V$. This shows that h is continuous and extends f. $\square$

If Σ_0 and Σ_1 are *MAD*-families and $f : \omega \to \omega$ is a finite-to-one function that satisfies one of the conditions of Lemma 2.1, then the continuous extension of f will be denoted by $\Psi(f, \Sigma_0, \Sigma_1) : \Psi(\Sigma_0) \to \Psi(\Sigma_1)$, if no confusion arises, then we simply write $\Psi(f)$. If f is finite-to-one, then the symbol $\Psi(f, \Sigma_0, \Sigma_1)$ (or $\Psi(f)$) will also mean that f can be extended to a continuous function from $\Psi(\Sigma_0)$ to $\Psi(\Sigma_1)$. Notice that if $f, g : \omega \to \omega$ are functions, f extends to a continuous function $\Psi(f) : \Psi(\Sigma_0) \to \Psi(\Sigma_1)$ and $\{n < \omega : f(n) \neq g(n)\}$ is finite, then g extends to a continuous function $\Psi(g) : \Psi(\Sigma_0) \to \Psi(\Sigma_1)$ such that $\Psi(f)(A) = \Psi(g)(A)$ for each $A \in \Sigma_0$. If Σ is a *MAD*-family, then $Aut(\Psi(\Sigma))$ will denote the set of all autohomeomorphisms of $\Psi(\Sigma)$ and $\mathbf{S}(\Sigma) = \{f \in \mathbf{S}(\omega) : \Psi(f) \in Aut(\Psi(\Sigma))\}$. Notice that if $\mathbf{S}(\omega)$ is equipped with the topology inherited from the product space ω^ω, then $\mathbf{S}(\Sigma)$ is a dense subgroup of $\mathbf{S}(\omega)$, for every *MAD*-family Σ.

Example 2.2. *There is a MAD-family Σ and a bijection $f : \omega \to \omega$ such that $f[A] = A$ for every $A \in \Sigma$ and f does not have any fixed point. Let $N_0, N_1 \in [\omega]^\omega$ be such that $N_0 \cap N_1 = \emptyset$ and $N_0 \cup N_1 = \omega$. Let Σ_0 be a MAD-family on N_0 and fix a bijection $f : \omega \to \omega$ such that $f[N_0] = f[N_1]$, $f[N_1] = N_0$ and f^2 is the identity map. Then $\Sigma_1 = \{f[A] : A \in \Sigma_0\}$ is a MAD-family on N_1. Now for each $A \in \Sigma_0$ we define $D(A) = A \cup f[A]$. Thus, $\Sigma = \{D(A) : A \in \Sigma_0\}$ is the required MAD-family.*

The following example shows the existence of a *MAD*-family Σ such that for every $f \in \mathbf{S}(\omega)$ without fixed points there is $A \in \Sigma$ with $f[A] \cap A = \emptyset$. We need a lemma which was established by Katětov [Ka] (for a proof see [CN, Lemma 9.1]).

Lemma 2.3. *Let α be a cardinal. If $f : \alpha \to \alpha$ is a function such that $f(\xi) \neq \xi$ for $\xi < \alpha$, then there are subsets A_0, A_1 and A_2 of α such that*

(1) *$\alpha = A_0 \cup A_1 \cup A_2$;*
(2) *$A_i \cap A_j = \emptyset$ for $i, j \leq 2$ and $i \neq j$; and*
(3) *$A_i \cap f[A_i] = \emptyset$ for $i \leq 2$.*

Example 2.4. *It is shown in* [BV] *that for every $p \in \omega^*$ there is an AD-family $\mathcal{A}_p = \{A_C : C \in p\}$ such that $A_C \in [C]^\omega$ for every $C \in p$. We now extend $\mathcal{A}_p$ to a MAD-family Σ_p for every $p \in \omega^*$. Fix $p \in \omega^*$. Let $f \in \mathbf{S}(\omega)$ be without fixed points. It follows from Lemma 2.3, that there is a partition $\{C_0, C_1, C_2\}$ of ω such that $f[C_i] \cap C_i = \emptyset$ for every $i \leq 2$. Since p is an ultrafilter, there is $i \leq 2$ with $C_i \in p$. Then, $A_{C_i} \in [C_i]^\omega$ satisfies that $f[A_{C_i}] \cap A_{C_i} = \emptyset$.*

The following lemma is useful to see when $\Psi(f)$ is a homeomorphism.

Lemma 2.5. *Let $\Psi(\Sigma_0)$ and $\Psi(\Sigma_1)$ be MAD-families and $f \in \mathbf{S}(\omega)$. If $\Psi(f) : \Sigma_0 \to \Sigma_1$ is a bijection, then $\Psi(f)$ is a homeomorphism.*

Proof: We shall show that f^{-1} can be extended to a continuous function from $\Psi(\Sigma_1)$ to $\Psi(\Sigma_0)$. In fact, according to Lemma 2.1, it suffices to prove that $f[A] =^* B$ whenever $\Psi(f)(A) = B$ for $A \in \Sigma_0$ and $B \in \Sigma_1$. Indeed, suppose that $\Psi(f)(A) = B$ for $A \in \Sigma_0$ and $B \in \Sigma_1$. By Lemma 2.1, we have $f[A] \subseteq^* B$. Assume that $C = B - f[A]$ is infinite. Then $f^{-1}(C)$ is infinite as well. Hence, there is $D \in \Sigma_0$ such that $f^{-1}(C) \cap D$ is infinite. Since $f[D] \cap B$ is infinite, $\Psi(f)(D) = B$. Thus, $\Psi(f)(A) = \Psi(f)(D)$ and $A \neq D$, which is a contradiction. $\square$

We remark that if $\Psi(f, \Sigma_0, \Sigma_1)$ is a homeomorphism, then $\beta f : \beta(\omega) \to \beta(\omega)$ satisfies that for every $A \in \Sigma_0$ there is $B \in \Sigma_1$ for which $\beta f[A^*] = B^*$. Notice that for an arbitrary homeomorphism $\Psi(f, \Sigma_0, \Sigma_1)$ the following property does not hold in general: for every $A \in \Sigma_0$ there is $B \in \Sigma_1$ such that $f[A] = B$.

Example 2.6. *Let $\{A_n : n < \omega\} \subseteq [\omega]^\omega$ be a partition of ω. For each $n < \omega$, choose $\{a^n_j : j \leq n\} \subseteq A_n$ and $\{b^n_j : j \leq n\} \subseteq A_{n+1} - \{a^{n+1}_j : j \leq n+1\}$. Set $A = \{a^n_j : j \leq n, n < \omega\}$ and $B = \{b^n_j : j \leq n, n < \omega\}$. Then $\mathcal{A} = \{A, B\} \cup \{A_n : n < \omega\}$ is an AD-family. By Zorn's Lemma, we extend $\mathcal{A}$ to a MAD-family Σ so that if $D \in \Sigma - \mathcal{A}$, then $D \cap A = \emptyset = D \cap B$. Now, define $f : \omega \to \omega$ by $f(a^n_j) = b^n_j$ and $f(b^n_j) = a^n_j$ for $j \leq n$ and for $n < \omega$, and $f(k) = k$ if $k \in \omega - (A \cup B)$. Then, we have that $\Psi(f) : \Psi(\Sigma) \to \Psi(\Sigma)$ is a homeomorphism such that $\Psi(f)(D) = D$ for all $D \in \Sigma - \{A, B\}$, $\Psi(f)(A) = B$, $\Psi(f)(B) = A$, $f[A_n] =^* A_n$ and $f[A_n] - A_n = \{a^{n-1}_j : j \leq n-1\} \cup \{b^n_j : j \leq n\}$ for every $1 \leq n < \omega$.*

Let Σ_0 be a *MAD*-family and $\{A_n : n < \omega\} \subseteq \Sigma_0$. Define $B_0 = A_0$ and $B_n = A_n - \bigcup_{m<n} A_m$ for every $0 < n < \omega$. If $\Sigma_1 = (\Sigma_0 - \{A_n : n < \omega\}) \cup \{B_n : n < \omega\}$, then $\{B_n : n < \omega\}$ is pairwise disjoint and $\Psi(\Sigma_0)$ and $\Psi(\Sigma_1)$ are homeomorphic.

Theorem 2.7. *Let Σ_0 and Σ_1 be MAD-families. If $h : \Psi(\Sigma_0) \to \Psi(\Sigma_1)$ is a homeomorphism, then $f = h \mid_\omega$ is a permutation of ω, $h = \Psi(f)$ and for every $A \in \Sigma_0$ there is $B \in \Sigma_1$ such that $f[A] =^* B$ (equivalently, $\beta f[A^*] = B^*$).*

Our next goal is to prove the main theorem of this section. First, we show several preliminary results. We omit the proof of the following easy lemma.

Lemma 2.8. *Let $f \in \mathbf{S}(\omega)$ and $A \in [\omega]^\omega$. Then the following are equivalent:*

(1) *$\{D \in [\omega]^\omega : D = f^k[A]$ for some $k \in \mathbf{Z}\}$ is an AD-family;*
(2) *$\{D \in [\omega]^\omega : D = f^n[A]$ for some $n < \omega\}$ is an AD-family;*
(3) *for every $n < \omega$, either $f^n[A] = A$ or $|A \cap f^n[A]| < \omega$.*

We should remark that for $A \in [\omega]^\omega$ and $f \in S(\omega)$, the condition "for every $n < \omega$, either $f^n[A] =^* A$ or $|A \cap f^n[A]| < \omega$" does not necessarily imply that

"$\{D \in [\omega]^\omega : D = f^k[A]$ for some $k \in \mathbf{Z}\}$ is an AD-family". Indeed, let $A = \omega - \{1\}$ and define $f \in S(\omega)$ by $f(0) = 1$, $f(1) = 0$ and $f(k) = k$ for every $1 < k < \omega$. Then, $f^{2k}[A] = A$ and $f^{2k+1}[A] = f[A] = (A - \{0\}) \cup \{1\}$ for every $k < \omega$.

The next result is a direct consequence of Lemma 2.4 (for the details of the proof, we referred the reader to [CN, Theorem 9.2 (a)]).

Lemma 2.9. *If $p \in \beta(\omega)$ and $f : \omega \to \omega$ is a function, then $\beta f(p) = p$ if and only if $\{n < \omega : f(n) = n\} \in p$.*

The following lemma is essentially due to A.I. Baskirov [Ba, Lemma 2].

Lemma 2.10. *Let $f \in \mathbf{S}(\omega)$ be such that f^n has no fixed points for every $1 \leq n < \omega$. Then for every $A \in [\omega]^\omega$ there is $B \in [A]^\omega$ such that $\{f^k[B] : k \in \mathbf{Z}\}$ is an infinite AD-family.*

Baskirov's Lemma may be generalized as follows.

Lemma 2.11. *Let $f \in \mathbf{S}(\omega)$. Then for every $A \in [\omega]^\omega$ there is $B \in [A]^\omega$ such that*

$$\{D \in [\omega]^\omega : D = f^k[B] \ \text{ for some } \ k \in \mathbf{Z}\}$$

is an AD-family and if $f^k[B] \cap B$ is infinite for some $k < \omega$, then $f^k |_B$ is the identity map.

Proof: In virtue of Lemma 2.9 and Lemma 2.10, we may assume that there is $1 \leq n < \omega$ such that $\{k \in A : f^n(k) = k\}$ is infinite. Without loss of generality, we may assume that $f^n |_A$ is the identity map and that n is the least positive integer such that $\{k \in A : f^i(k) = k\}$ is finite for every $1 \leq i < n$. If $n = 1$, then we put $A = B$. Suppose that $1 < n$. Reasoning as in the proof of Lemma 2 of [Ba], for every $1 \leq i < n$ we can find $B_i \in [A]^\omega$ such that $B_{n-1} \subseteq B_{n-2} \subseteq \cdots \subseteq B_1 \subseteq A$ and $f^i[B_i] \cap B_i = \emptyset$ for every $1 \leq i < n$. Then, we put $B = B_{n-1}$. Hence, we have that $\{D \in [\omega]^\omega : D = f^k[B]$ for some $k \in \mathbf{Z}\} = \{f^{1-n}[B], \ldots, f^{-1}[B], B, f[B], \ldots, f^{n-1}[B]\}$. The conclusion follows from Lemma 2.8. $\square$

Lemma 2.12. *Let $\{f_n : n < \omega\}$ be a set of permutations. Then for every $A \in [\omega]^\omega$ there is $B \in [A]^\omega$ such that*

$$\{D^* : D = f_n^k[B] \ \text{ for some } \ n < \omega \ \text{ and for some } \ k \in \mathbf{Z}\}$$

is a set of pairwise disjoint clopen subsets of ω^*. In addition, if there is $m < \omega$ such that f_m^k has no fixed points on A for every $k \in \mathbf{Z}$, then $\{D^* : D = f_n^k[B]$ for some $n < \omega$ and for some $k \in \mathbf{Z}\}$ is infinite.

Proof: Enumerate the set $\{f_n^k \circ f_m^j : (n, m) \in \omega \times \omega, (k, j) \in \mathbf{Z} \times \mathbf{Z}\}$ as $\{g_s : s < \omega\}$. By Lemma 2.11 and by induction, for each $s < \omega$ we may find $B_s \in [A]^\omega$ such that

(1) $B_s \subseteq B_t$ whenever $s < t < \omega$; and

(2) $\{D \in [\omega]^\omega : D = g_s^k[B_s]$ for some $k \in \mathbf{Z}\}$ is an AD-family and if $g_s^k[B_s] \cap B_s$ is infinite for some $k \in \mathbf{Z}$, then $g_s^k\mid_{B_s}$ is the identity map.

Since ω^* is an almost P-space (see [L]), there is $B \in [A]^\omega$ such that $B^* \subseteq \bigcap_{s<\omega} B_s^*$. Fix $(n,m) \in \omega \times \omega$ and $(j,k) \in \mathbf{Z}^2$. Then, we have that $|f_n^k[B]^* \cap f_m^j[B]^*| = |\beta f_n^k[B^*] \cap \beta f_m^j[B^*]| = |B^* \cap \beta(f_n^{-k} \circ f_m^j)[B^*]|$. Choose $t < \omega$ so that $g_t = f_n^{-k} \circ f_m^j$ and consider B_t. If $\beta g_t[B_t^*] \cap B_t^* = \emptyset$, then $\beta g_t[B^*] \cap B^* = \emptyset$ and hence $f_n^k[B]^* \cap f_m^j[B]^* = \emptyset$. Suppose that $\beta g_t[B_t^*] \cap B_t^* \neq \emptyset$. Then $g_t[B_t] \cap B_t$ is infinite. By clause (2), we obtain that $g_t\mid_{B_t}$ is the identity map and since $B \subseteq^* B_t$, we must have that $B^* = \beta g_t[B^*] = \beta(f_n^{-k} \circ f_m^j)[B^*]$; that is, $\beta f_n^k[B^*] = \beta f_m^j[B^*]$.

Assume that there is $m < \omega$ such that f_m^k has no fixed points on A for every $k \in \mathbf{Z}$. By Lemma 2.10, we may choose $C \in [A]^\omega$ so that $\{f_m^k[C] : k \in \mathbf{Z}\}$ is an infinite AD-family and $B \subseteq^* C$. Hence, $\{f_m^k[B]^* : k \in \mathbf{Z}\}$ is infinite. □

Theorem 2.13. *Let G be a countable subgroup of $\mathbf{S}(\omega)$. Then there is a MAD-family Σ such that*

$$\Psi(f) \in Aut(\Psi(\Sigma)) \quad \textit{for all} \quad f \in G.$$

Proof: Without loss of generality we may assume that there is $h \in G$ such that h^n has no fixed points for every $1 \leq n < \omega$: if such a function h is not in G, then we add one to G. Now, enumerate $[\omega]^\omega$ as $\{A_\xi : \xi < \mathfrak{c}\}$, where A_0 satisfies that $\mathcal{O}_0 = \{D^* : D = f[A_0], f \in G\}$ is an infinite pairwise disjoint set (this is possible because of Lemma 2.12). Notice that if $D^* \in \mathcal{O}_0$, then $\beta f[D^*] \in \mathcal{O}_0$ for ever $f \in G$. Now, we proceed by transfinite induction. Assume that for every $\xi < \lambda < \mathfrak{c}$ we have defined a set $B_\xi \in [\omega]^\omega$ and an infinite set $\mathcal{O}_\xi$ of pairwise disjoint clopen subsets of ω^* such that

(1) for every $\xi < \lambda$, either one of the following conditions holds:
   a. there is $B_\xi \in [A_\xi]^\omega$ such that $\beta f[B_\xi^*] \in \mathcal{O}_\xi$ for all $f \in G$; or
   b. $A_\xi^* \cap D^* \neq \emptyset$ for some $D^* \in \mathcal{O}_\xi$, in this case we have that $B_\xi = B_\zeta$ for some $\zeta < \xi$.

(2) $\mathcal{O}_\xi = \{D^* : D = f[B_\zeta], f \in G$ and $\zeta \leq \xi\}$, for all $\xi < \lambda$.

We should remark that:

(3) $\mathcal{O}_\xi \subseteq \mathcal{O}_\zeta$ whenever $\xi < \zeta < \lambda$;
(4) if $D^* \in \mathcal{O}_\xi$, for some $\xi < \lambda$, then $\beta f[D^*] \in \mathcal{O}_\xi$ for all $f \in G$;
(5) $B_\xi^* \in \mathcal{O}_\xi$ for every $\xi < \lambda$.

Put $\mathcal{O} = \bigcup_{\xi<\lambda} \mathcal{O}_\xi$ and observe that $\mathcal{O}$ is an infinite pairwise disjoint set, by clause (3). We consider two cases:

Case I. Suppose that $D^* \cap \beta f[A_\lambda^*] = \emptyset$ for every $f \in G$ and for every $D^* \in \mathcal{O}$. According to Lemma 2.12, we may find $B_\lambda \in [A_\lambda]^\omega$ such that $\{E^* : E = f[B_\lambda], f \in G\}$ is pairwise disjoint and infinite. Then, we define $\mathcal{O}_\lambda = \bigcup_{\xi<\lambda} \mathcal{O}_\xi \cup \{E^* : E = f[B_\lambda], f \in G\}$. It is not hard to see that $\mathcal{O}_\lambda$ is pairwise disjoint.

Case II. There are $D^* \in \mathcal{O}$ and $f \in G$ such that $D^* \cap \beta f[A^*_\lambda] \neq \emptyset$. Then, we have that $A^*_\lambda \cap \beta f^{-1}(D^*) \neq \emptyset$ and $\beta f^{-1}(D^*) \in \mathcal{O}$. In this case we define $\mathcal{O}_\lambda = \mathcal{O}$ and $B_\lambda = B_\xi$ for some $\xi < \lambda$.

Put $\mathcal{P} = \bigcup_{\xi<\mathfrak{c}} \mathcal{O}_\xi$. We have that $\mathcal{P}$ is a set of pairwise disjoint clopen subsets of ω^*, because of clause (3). Choose $\Sigma \subseteq [\omega]^\omega$ so that $\Sigma^* = \mathcal{P}$ and $|A \cap B| < \omega$ whenever $A, B \in \Sigma$ and $A \neq B$. We have that Σ is an infinite AD-family. By clause (1), we obtain that Σ is a MAD-family. Fix $f \in G$ and $A \in \Sigma$. Then, $A^* \in \mathcal{O}_\lambda$ for some $\lambda < \mathfrak{c}$. By clause (4), we obtain that $\beta f[A^*] \in \mathcal{O}_\lambda$ and hence $\beta f[A^*] = B^*$ for some $B \in \Sigma$. So f extends to a continuous function $\Psi(f) : \Psi(\Sigma) \to \Psi(\Sigma)$, by Lemma 2.1. It remains to show that $\Psi(f)$ is a homeomorphism. In virtue of Lemma 2.5, it suffices to prove that $\Psi(f)$ is a bijection. Indeed, suppose that $\Psi(f)(A) = \Psi(f)(B)$ for $A, B \in \Sigma$. Then, $\beta f[A^*] = \beta f[B^*]$. Hence, $A^* = B^*$ since βf is a homeomorphism. But this is possible only for the case when $A = B$, by the definition of Σ. This shows that $\Psi(f)$ is one-to-one. Let $C \in \Sigma$. Then $C^* = \beta h[B^*_\xi]$ for some $h \in G$ and for some $\xi < \mathfrak{c}$. Hence, $C^* = \beta f[\beta(f^{-1} \circ h)[B^*_\xi]]$. Since $\beta(f^{-1} \circ h)[B^*_\xi] \in \mathcal{O}_\xi \subseteq \mathcal{P}$, $\beta(f^{-1} \circ h)[B^*_\xi] = D^*$ for some $D \in \Sigma$. Hence, $\Psi(f)(D) = C$. Thus, $\Psi(f)$ is a surjection. Therefore, $\Psi(f) \in Aut(\Psi(\Sigma))$. □

In Example 2.6, we saw that there are $f \in S(\omega)$ and a MAD-family Σ such that $\Psi(f) \in Aut(\Psi(\Sigma))$ and $f[A] \notin \Sigma$ for some $A \in \Sigma$.

For a MAD-family Σ, we set

$$Inv(\Sigma) = \{f \in \mathbf{S}(\omega) : f[A] \in \Sigma \text{ for all } A \in \Sigma\}.$$

Observe that $Inv(\Sigma)$ is a subgroup of $\mathbf{S}(\omega)$ and if $f \in Inv(\Sigma)$, then $\Psi(f) \in Aut(\Psi(\Sigma))$, for every MAD-family Σ. The MAD-family Σ of Example 2.6 satisfies that there is $f \in \mathbf{S}(\omega)$ such that $\Psi(f) \in Aut(\Psi(\Sigma))$ and $f \notin Inv(\Sigma)$. It is not hard to prove that $Inv(\Sigma) \neq S(\omega)$ for every MAD-family Σ (see Theorem 2.19 below). It was shown in Theorem 2.13 that for every countable subgroup G of $\mathbf{S}(\omega)$ there is a MAD-family Σ such that $\Psi(f) \in Aut(\Psi(\Sigma))$ for all $f \in G$. This leads us to ask:

Question 2.14. *If $F \subseteq \mathbf{S}(\omega)$ is countable, does there a MAD-family Σ exist so that $F \subseteq Inv(\Sigma)$?*

Unfortunately, the previous question remains open. If $F = \{f\}$ for $f \in \mathbf{S}(\omega)$, then the answer is in the positive fashion as it is shown in the next theorem.

Theorem 2.15. *For every $f \in \mathbf{S}(\omega)$ there is a MAD-family Σ such that $f \in Inv(\Sigma)$.*

Proof: Fix $f \in \mathbf{S}(\omega)$. We consider two cases:

Case I. There is $1 \leq n < \omega$ such that $\{k < \omega : f^n(k) = k\}$ is infinite. Let n be the least positive integer with this property. If $n = 1$, then we choose a MAD-family Σ_0 of infinite subsets of $F = \{k < \omega : f^n(k) = k\}$ and we define either

$\Sigma = \Sigma_0 \cup \{\omega - F\}$ if $\omega - F$ is infinite or $\Sigma = \Sigma_0$ otherwise. Suppose that $1 < n$. Then, we have that $\{k < \omega : f^i(k) = k\}$ is finite for every $1 \leq i < n$. Following the proof of Lemma 2.11, we may find an infinite subset B of $\{k < \omega : f^n(k) = k\}$ such that

$$\{D \in [\omega]^\omega : D = f^k[B] \text{ for some } k \in \mathbf{Z}\} =$$
$$= \{f^{1-n}[B], \dots, f^{-1}[B], B, \dots, f^{n-1}[B]\}.$$

and $f^i[B] \cap f^j[B] = \emptyset$, whenever $-n < i < j < n$ and $|j - i| < n$. Let Σ_1 be a MAD-family on B. Set $N = \omega - (\bigcup_{k\in\mathbf{Z}} f^k[B])$ and notice that $f^k[N] = N$ for every $k \in \mathbf{Z}$. Define either $\Sigma = \{f^i[A] : A \in \Sigma_1, -n < i < n\} \cup \{N\}$ if N is infinite or $\Sigma = \Sigma_1$ otherwise. Then, we have that Σ is an infinite AD-family on ω. If $C \in [\omega]^\omega$, then either $C \cap N$ is infinite or there is $-n < i < n$ such that $C \cap f^i[B]$ is infinite. Then, $f^{-i}[C] \cap B$ is infinite and hence there is $A \in \Sigma_1$ such that $|A \cap f^{-i}[C] \cap B| = |C \cap f^i[A]| = \omega$. Thus, Σ is a MAD-family and $f \in Inv(\Sigma)$.

Case II. Suppose that $\{k < \omega : f^n(k) = k\}$ is finite for every $1 \leq n < \omega$. In virtue of Lemma 2.9, we have that f^n has no fixed points for every $1 \leq n < \omega$. Now, enumerate $[\omega]^\omega$ as $\{E_\xi : \xi < \mathfrak{c}\}$. We shall proceed by transfinite induction. By Lemma 2.10, choose $A_0 \in [E_0]^\omega$ so that $\{f^k[A_0] : k \in \mathbf{Z}\}$ is an infinite AD-family. Suppose that for every $\xi < \lambda < \mathfrak{c}$ we have defined $A_\xi \in [\omega]^\omega$ such that

(1) $\bigcup_{\zeta<\xi}\{f^k[A_\zeta] : k \in \mathbf{Z}\}$ is an AD-family for every $\xi < \lambda$; and

(2) for every $\xi < \lambda$ there is $k \in \mathbf{Z}$ such that $E_\xi \cap f^k[A_\xi]$ is infinite.

If there are $\xi < \lambda$ and $k \in \mathbf{Z}$ such that $E_\lambda \cap f^k[A_\xi]$ is infinite, then we put $A_\lambda = A_\xi$. Now, Suppose that $|E_\lambda \cap f^k[A_\xi]| < \omega$ for every $\xi < \lambda$ and for every $k \in \mathbf{Z}$. By Lemma 2.10, we may find $A_\lambda \in [E_\lambda]^\omega$ such that $\{f^k[A_\lambda] : k \in \mathbf{Z}\}$ is an infinite AD-family. Let $j, k \in \mathbf{Z}$ and $\xi < \lambda$. Then,

$$|f^j[A_\lambda] \cap f^k[A_\xi]| = |A_\lambda \cap (f^{-j} \circ f^k)[A_\xi]| =$$
$$= |A_\lambda \cap f^{k-j}[A_\xi]| \leq |E_\lambda \cap f^{k-j}[A_\xi]| < \omega.$$

Therefore, $\bigcup_{\zeta\leq\lambda}\{D : D = f^k[A_\zeta], k \in \mathbf{Z}\}$ is an AD-family.

Finally, we define $\Sigma = \bigcup_{\xi<\mathfrak{c}}\{D : D = f^k[A_\xi], k \in \mathbf{Z}\}$. It follows from clauses (1) and (2) that Σ is a MAD-family and $f \in Inv(\Sigma)$. □

Corollary 2.16. *There is a MAD-family Σ such that $n+A \in \Sigma$ whenever $A \in \Sigma$ and $n < \omega$, where $n + A = \{n + a : a \in A\}$ for $n < \omega$.*

Proof: Define $\tau : \omega \to \omega$ by $\tau(k) = 1 + k$ for every $k \in \omega$. If $n < \omega$, then $\tau^n(k) = n + k$ for every $k < \omega$. Applying Theorem 2.15, there is a MAD-family Σ such that $\tau^n(A) = n + A \in \Sigma$ for every $n < \omega$ and for every $A \in \Sigma$. □

We shall verify that a MAD-family which is invariant under the multiplication of positive integers does not exist:

Theorem 2.17. *There is no MAD-family Σ such that*

$$n \cdot A \in \Sigma,$$

for every $A \in \Sigma$ and for every $1 \leq n < \omega$, where $n \cdot A = \{n \cdot a : a \in A\}$ for $1 \leq n < \omega$.

PROOF: We define

$$\mathcal{D} = \{D \in [\omega]^\omega : \ |\{d \in D : n \setminus d\}| < \omega \ \text{ for every } \ 1 < n < \omega\}.$$

Suppose that Σ is a MAD-family such that $n \cdot A \in \Sigma$, for every $A \in \Sigma$ and for every $1 \leq n < \omega$. Fix $A \in \Sigma$ and assume that $A \notin \mathcal{D}$. Then, there is $1 < n_0 < \omega$ such that $B_0 = \{a \in A : n_0 \setminus a\}$ is infinite. Choose $C_0 \in [\omega]^\omega$ with $n_0 \cdot C_0 = B_0$. We have that there is $D_0 \in \Sigma$ such that $|D_0 \cap C_0| = \omega$ and so $n_0 \cdot D_0 \cap A$ is infinite. Since $n_0 \cdot D_0 \in \Sigma$, we have $n_0 \cdot D_0 = A$. If $D_0 \notin \mathcal{D}$, by an argument similar to the previous one, we may find $1 < n_1$ and $D_1 \in \Sigma$ such that $n_1 \cdot D_1 = D_0$ and hence $n_0 \cdot n_1 \cdot D_1 = A$. Since every positive natural number has finitely many divisors, there must be $D_r \in \mathcal{D} \cap \Sigma$ and $n_0, \ldots, n_r < \omega$ such that $1 < n_j$ for each $j \leq r$ and $n_0 \cdot \ldots \cdot n_r \cdot D_r = A$. This shows that for every $A \in \Sigma$ either $A \in \mathcal{D}$ or there are $D \in \mathcal{D} \cap \Sigma$ and $1 < n_0 \leq \cdots \leq n_r < \omega$ such that $n_0 \cdot \ldots \cdot n_r \cdot D = A$. Now, enumerate the set of all prime numbers by $\{p_n : n < \omega\}$ and let $P = \{p_0 \cdot \ldots \cdot p_n : n < \omega\}$. It is clear that $|P \cap A| < \omega$ for every $A \in \mathcal{D} \cap \Sigma$. By the maximality of Σ, there is $B \in \Sigma - \mathcal{D}$ such that $P \cap B$ is infinite. We may find $D \in \mathcal{D} \cap \Sigma$ and $1 < n_0 \leq \cdots \leq n_r < \omega$ such that $n_0 \cdot \ldots \cdot n_r \cdot D = B$. Let $N < \omega$ be such that p_n does not divide n_j for every $j \leq r$ and for every $N \leq n < \omega$. Since $P \cap B$ is infinite, the intersection $\{k : p_N \setminus k\} \cap D$ must be infinite, but this is a contradiction. □

We pointed out that $\mathbf{S}(\Sigma)$ is a dense subgroup of $\mathbf{S}(\omega)$ for every MAD-family Σ. This fact may be improved as follows. We need some notation to describe the topology on $S(\omega)$.

If $j < \omega$ and $n < \omega$, then we write $[j, n] = \{f \in \mathbf{S}(\omega) : f(j) = n\}$. We know that $\{[j, n] : (j, n) \in \omega \times \omega\}$ forms a subbase for the topology on $\mathbf{S}(\omega)$ which is considered as a subspace of the product space ω^ω.

Theorem 2.18. *For every MAD-family Σ, we have that $S(\Sigma) - Inv(\Sigma)$ is dense in $S(\omega)$.*

PROOF: Let $V = \bigcap_{j<n}[j, k_j] \neq \emptyset$ be a basic open subset of $S(\omega)$. Fix $A \in \Sigma$, $a \in A - (n \cup \{k_j : j < n\})$ and $b \in \omega - (A \cup n \cup \{k_j : j < n\})$. Define $f : \omega \to \omega$ by $f(j) = k_j$ for every $j < n$, $f(k_j) = j$ for every $j < n$, $f(a) = b$, $f(b) = a$ and $f(k) = k$ for every $k \in \omega - (n \cup \{k_j : j < n\} \cup \{a, b\})$. It is clear that $f \in S(\Sigma)$ and $f[A] =^* A$, but $f[A] \notin \Sigma$, since $f[A] \neq A$. Therefore, $f \in V \cap (S(\Sigma) - Inv(\Sigma))$. □

As a particular case of Theorem 2.18 we have that $S(\Sigma) \neq Inv(\Sigma)$ for every MAD-family Σ.

We do not know whether there is a MAD-family Σ such that $Inv(\Sigma)$ is dense in $\mathbf{S}(\omega)$. Next we present some results related to the density of $Inv(\Sigma)$, in $S(\Sigma)$, and an example of a MAD-family Σ for which $Inv(\Sigma)$ is not dense in $\mathbf{S}(\omega)$.

Let $\mathcal{S} = \{s : s : n \to \omega \text{ is one-to-one}, n < \omega\}$. For $n < \omega$ and a one-to-one function $s : n \to \omega$, $h_s : \omega \to \omega$ will stand for an arbitrary extension of s (i.e., $s \subseteq h_s$), and $\mathcal{H} = \{h_s : s \in \mathcal{S}\}$ will stand for an arbitrary set of these extensions. Two different choices of extensions $h_s's$ will produce two different sets $\mathcal{H}'s$.

Lemma 2.19. *Let Σ be a MAD-family. Then $Inv(\Sigma)$ is dense in $\mathbf{S}(\omega)$ if and only if there is a set $\mathcal{H}$ of extensions such that $\mathcal{H} \subseteq Inv(\Sigma)$.*

Proof: Necessity. Suppose that $Inv(\Sigma)$ is dense in $\mathbf{S}(\omega)$ and fix $s \in \mathcal{S}$. We have that the domain of s is equal to n for some $n < \omega$. Consider the basic open $V = \bigcap_{j<n+1}[j, s(j)]$. We have that $V \cap \mathbf{S}(\omega) \neq \emptyset$. By assumption, there is $h \in V \cap Inv(\Sigma)$. It is evident that h extends s. Thus, the set $\mathcal{H}$ satisfies the conditions.

Sufficiency. Suppose that $\mathcal{H} \subseteq Inv(\Sigma)$ and let $V = \bigcap_{j<n}[j, k_j]$ be a basic nonempty open set of $\mathbf{S}(\omega)$. Notice that $k_i \neq k_j$ provided that $i < j < n$. Define $s : n \to \omega$ by $s(j) = k_j$ for every $j < n$. Then, we have that $s \in \mathcal{S}$. By hypothesis, there is $h_s \in \mathcal{H} \subseteq Inv(\Sigma)$ which extends s. Therefore, $h_s \in V \cap Inv(\Sigma)$. This shows that $Inv(\Sigma)$ is dense in $\mathbf{S}(\omega)$. $\square$

We remark that if the condition of Question 2.14 holds for some of the countable sets $\mathcal{H}$, then there is a MAD-family Σ such that $Inv(\Sigma)$ is dense in $\mathbf{S}(\omega)$.

Definition 2.20. *Let Σ be a MAD-family. We say that a finite set $\{a_0, \dots, a_n\}$ of positive integers generates Σ if*

$$\{a_0, \dots, a_n\} \cap A \neq \emptyset \quad \textit{for all} \quad A \in \Sigma.$$

Theorem 2.21. *If Σ is a MAD-family generated by a finite set $\{a_0, \dots, a_n\}$ of positive integers, then $Inv(\Sigma)$ is not dense in $\mathbf{S}(\omega)$.*

Proof: Suppose that $Inv(\Sigma)$ is dense in $\mathbf{S}(\omega)$. According to Lemma 2.19, there is a set $\mathcal{H} = \{h_s : s \in \mathcal{S}\}$ of extensions such that $\mathcal{H} \subseteq Inv(\Sigma)$. Since $Inv(\Sigma)$ is a subgroup of $\mathbf{S}(\omega)$, $h_s^{-1} \in Inv(\Sigma)$ for all $s \in \mathcal{S}$. Fix $A \in \Sigma$. We have that $\omega - A$ is infinite. Choose a one-to-one function $s : m \to \omega$, where $m = \max\{a_j : j \leq n\} + 1$, so that $s(a_j) \in \omega - A$ for every $j \leq n$. Consider $h_s \in \mathcal{H}$. Since $h_s^{-1}(A) \in \Sigma$ there is $i \leq n$ such that $a_i \in h_s^{-1}(A)$ and hence $h_s(a_i) = s(a_i) \in A$, but this is a contradiction. $\square$

As a direct application of Theorem 2.21, we have that if Σ is a MAD-family, then $\Delta = \{A \cup \{0\} : A \in \Sigma\}$ is also a MAD-family such that $Inv(\Delta)$ is not dense in $\mathbf{S}(\omega)$.

The proof of the next result is straightforward.

Theorem 2.22. *Let Σ be a MAD-family generated by the finite set $\{a_0, \dots, a_n\}$. If for every $F \in [\omega - \{a_0, \dots, a_n\}]^{<\omega}$ there is $A \in \Sigma$ such that $A \cap F = \emptyset$, then f is a permutation of $\{a_0, \dots, a_n\}$ for every $f \in Inv(\Sigma)$.*

Let Σ be a *MAD*-family generated by the set $\{a_0, \dots, a_n\}$. We have that $|\Sigma| > \omega$ and hence Σ can be enumerated as $\{A_\xi : \xi < \alpha\}$, where α is an uncountable cardinal number. Enumerate $[\omega - \{a_0, \dots, a_n\}]^{<\omega}$ as $\{F_n : n < \omega\}$. Define $B_n = A_n - F_n$ for each $n < \omega$ and $B_\xi = A_\xi$ for every $\omega \leq \xi < \alpha$. Then, $\{B_\xi : \xi < \alpha\}$ is a *MAD*-family satisfying the conditions of Theorem 2.22.

Theorem 2.23. *If Σ is a MAD-family satisfying that there is $A \in \Sigma$ such that*

(1) *$A \cap B \neq \emptyset$ for every $B \in \Sigma$; and*
(2) *for every $B \in \Sigma - \{A\}$ there is $C \in \Sigma$ such that $B \cap C = \emptyset$,*

then $f[A] = A$ for every $f \in Inv(\Sigma)$.

Proof: Let $f \in Inv(\Sigma)$. If $f^{-1}(A) \neq A$, then, by clause (2), there is $C \in \Sigma$ such that $f^{-1}(A) \cap C = \emptyset$ and hence $A \cap f[C] = \emptyset$, which is a contradiction to clause (1). Therefore, $f^{-1}(A) = A$ and hence $f[A] = A$. □

Let $\{A, B, C\}$ be a partition of ω in three infinite subsets. Fix a_0 and a_1 two different points of A. Let Σ_0 and Σ_1 be *MAD*-families on B and C, respectively. Then,

$$\Sigma = \{D \cup \{a_0\} : D \in \Sigma_0\} \cup \{D \cup \{a_1\} : D \in \Sigma_1\} \cup \{A\}$$

is a *MAD*-family on ω that satisfies the conditions of Theorem 2.23.

Question 2.24. *Is there a MAD-family Σ such that $Inv(\Sigma)$ is dense in $S(\Sigma)$?*

Question 2.25. *Is there a MAD-family Σ such that $Inv(\Sigma)$ is closed in $S(\Sigma)$?*

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