Cauchy problem for multidimensional coupled system of nonlinear Schrödinger equation and generalized IMBq equation*

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Dedicated to the memory of Prof. Jan Potoček and Prof. Josef Kolomý.

Abstract. The existence, uniqueness and regularity of the generalized local solution and the classical local solution to the periodic boundary value problem and Cauchy problem for the multidimensional coupled system of a nonlinear complex Schrödinger equation and a generalized IMBq equation

$$i\varepsilon_t + \nabla^2\varepsilon - u\varepsilon = 0,$$

$$u_{tt} - \nabla^2 u - a\nabla^2 u_{tt} = \nabla^2 f(u) + \nabla^2(|\varepsilon|^2)$$

are proved.

Keywords: coupled system of nonlinear Schrödinger equation and generalized IMBq, multidimensional, periodic boundary value problem, Cauchy problem, generalized local solution, classical local solution

Classification: 35L35, 35K55

1. Introduction

In [1], [2] the problems of soliton solutions for the Schrödinger field interacting with the Boussinesq field have been studied and an approximate solution of system

$$i\varepsilon_t + \varepsilon_{xx} - u\varepsilon = 0, \tag{1.1}$$

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\delta}{3}\frac{\partial^4}{\partial x^4}\Big)u - \delta\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial^2}{\partial x^2}(|\varepsilon|^2) \tag{1.2}$$

have been found, where $\delta > 0$ is a constant. The exact soliton solutions of the above system were obtained in [3]–[5] by various techniques. In [2] author suggested that the Boussinesq equation (1.2) (which we call Bq-equation) was replaced by the IBq equation (the improved Bq-equation)

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\delta}{3}\frac{\partial^4}{\partial x^2\partial t^2}\Big)u - \delta\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial^2}{\partial x^2}(|\varepsilon|^2) \tag{1.3}$$

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and the soliton solutions of system (1.1), (1.3) were obtained. A modification of IBq equation analogous to the MKdV equation yields

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^4}{\partial x^2 \partial t^2}\right)u = \frac{\partial^2}{\partial x^2}(u^3), \tag{1.4}$$

which we call the IMBq equation. The IMBq equation with several variables (see [2]) is

$$u_{tt} - \nabla^2 u_{tt} - \nabla^2 u = \nabla^2 u^3. \tag{1.4$'$}$$

In this paper, we study the system of the multidimensional Schrödinger field interacting with real generalized IMBq field

$$i\varepsilon_t + \nabla^2 \varepsilon - u\varepsilon = 0, \tag{1.5}$$

$$u_{tt} - \nabla^2 u - a\nabla^2 u_{tt} = \nabla^2 f(u) + \nabla^2(|\varepsilon|^2), \tag{1.6}$$

where $\varepsilon(x,t)$ denotes complex unknown function of variables $x = (x_1, x_2, \cdots, x_n) \in R^n$ and $t \in R_+, u(x,t)$ denotes a real unknown function of variables x and t, $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right)$, $a > 0$ is a constant, $i = \sqrt{-1}$ and $f(s)$ is a given nonlinear function.

Let $\Omega \subset R^n$ be an n-dimensional cube with width $2D (D > 0)$ in each direction, that is $\overline{\Omega} = \{x = (x_1, x_2, \cdots, x_n) |\ |x_j| \leq D,\ j = 1, 2, \cdots, n\}$, $x + 2D_{e_j}$ denotes $(x_1, \cdots, x_{j-1}, x_j + 2D, x_{j+1}, \cdots, x_n)$ $(j = 1, 2, \cdots, n)$ and $\overline{Q}_T = \{x \in \overline{\Omega}, 0 \leq t \leq T\}$. For the system (1.5), (1.6), we discuss its periodic boundary value problem in the $(n+1)-$dimensional cylindrical domain $\overline{Q}_T$

$$\varepsilon(x,t) = \varepsilon(x + 2D_{e_j}, t),\ u(x,t) = u(x + 2D_{e_j}, t),\ j = 1, 2, \cdots, n, \tag{1.7}$$

$$\varepsilon(x,0) = \varepsilon_0(x),\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x), \tag{1.8}$$

where $\varepsilon_0(x)$, $\varphi(x)$ and $\psi(x)$ are given functions of n-dimensional initial value, satisfying the periodic boundary conditions (1.7). For the system (1.5), (1.6) we also study the Cauchy problem

$$\varepsilon(x,0) = \varepsilon_0(x),\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x), \tag{1.9}$$

where $\varepsilon_0(x)$, $\varphi(x)$ and $\psi(x)$ are given functions defined in R^n.

In Section 2 the existence and uniqueness of the generalized local solution and the classical local solution of the periodic boundary value problem (1.5)–(1.8) are proved. Moreover the regularity of the classical local solution of the problem (1.5)–(1.8) is considered. In Section 3 the existence, uniqueness and regularity of the generalized local solution and the classical local solution of the Cauchy problem (1.5), (1.6), (1.9) are proved.

For simplicity, let $a = 1$ in (1.6). We prove only the existence, uniqueness and regularity of the generalized local solutions and the classical local solution for the 2-dimensional problem, because we can treat the n-dimensional problem by the method of the 2-dimensional case.

2. Periodic boundary value problem (1.5)–(1.8)

We first establish an orthonormal base on $L_2(\Omega)$.

Lemma 2.1 ([6], [7]). *Let $\Omega_1 \subset R^n$ and $\wedge \subset R^m$ be measurable sets. If $\{\varphi_j\}_{j\in J}$ and $\{\chi_k\}_{k\in K}$ (J and K are the index sets) are orthonormal bases in $L_2(\Omega_1)$ and $L_2(\wedge)$ respectively, then the system of functions $\{\varphi_j(x)\chi_k(y)\}_{j\in J,k\in K}$ is an orthonormal base in $L_2(\Omega_1 \times \wedge)$.*

It is clear that Lemma 2.1 is valid for the case of any finite orthonormal bases. Let us consider the eigenvalue problem for the ordinary differential equation

$$\begin{cases} y'' + \lambda y = 0, \\ y(x_1 + 2D) = y(x_1). \end{cases}$$

We can obtain eigenvalues $\lambda_{1l} = \alpha_l^2 = \left(\frac{l\pi}{D}\right)^2$ ($l = 0, 1, \cdots$) and the family of eigenfunctions $\{y_l(x_1)\} = \left\{\frac{1}{\sqrt{2D}}, \frac{1}{\sqrt{D}} \cos \alpha_l x_1, \frac{1}{\sqrt{D}} \sin \alpha_l x_1, l = 1, 2, \cdots\right\}$, which composes an orthonormal base. Similarly, we can obtain eigenvalues $\lambda_{2j} = \beta_j^2 = \left(\frac{j\pi}{D}\right)^2$ ($j = 0, 1, \cdots$) and the family of eigenfunctions $\{z_j(x_2)\} = \left\{\frac{1}{\sqrt{2D}}, \frac{1}{\sqrt{D}} \cos \beta_j x_2, \frac{1}{\sqrt{D}} \sin \beta_j x_2,\ j = 1, 2, \cdots\right\}$ of the periodic boundary value problem $z'' + \lambda z = 0$, $z(x_2 + 2D) = z(x_2)$.$\{z_j(x_2)\}$ composes an orthonormal base. According to Lemma 2.1, the family of functions $\{y_l(x_1)z_j(x_2),\ l, j = 0, 1, \cdots\}$ composes an orthonormal base in $L_2(\Omega)$.

Let $(\varepsilon(x,t), u(x,t))$ be the solution of the problem (1.5)–(1.8), with $\varepsilon(x,t) = \sum_{l,j=0}^{\infty} L_{lj}(t) y_l(x_1) z_j(x_2)$ and $u(x,t) = \sum_{l,j=0}^{\infty} T_{lj}(t) y_l(x_1) z_j(x_2)$. The initial value functions may be expressed $\varepsilon_0(x) = \sum_{l,j=0}^{\infty} \varepsilon_{lj} y_l(x_1) z_j(x_2)$, $\varphi(x) = \sum_{l,j=0}^{\infty} \varphi_{lj} y_l(x_1) z_j(x_2)$ and $\psi(x) = \sum_{l,j=0}^{\infty} \psi_{lj} y_l(x_1) z_j(x_2)$, where ε_{lj} are complex numbers, φ_{lj}, ψ_{lj} are real numbers.
Substituting the expressions of $\varepsilon(x,t)$ and $u(x,t)$ into the system (1.5), (1.6), multiplying both sides of (1.5) and (1.6) by $y_l(x_1)z_j(x_2)$ respectively and integrating over Ω, we get

$$i\dot{L}_{lj} - (\alpha_l^2 + \beta_j^2) L_{lj} - \int_\Omega u\varepsilon y_l(x_1) z_j(x_2)\, dx = 0, \tag{2.1}$$

$$\begin{aligned} &(1 + \alpha_l^2 + \beta_j^2)\ddot{T}_{lj} + (\alpha_l^2 + \beta_j^2) T_{lj} - \int_\Omega (\nabla^2 f(u) + \nabla^2(|\varepsilon|^2)) \cdot \\ &\qquad y_l(x_1) z_j(x_2)\, dx = 0, \qquad l, j = 0, 1, \cdots . \end{aligned} \tag{2.2}$$

Substituting the expressions of $\varepsilon(x,t)$, $u(x,t)$, $\varepsilon_0(x)$, $\varphi(x)$ and $\psi(x)$ into the initial conditions (1.8), we obtain

$$L_{lj}(0) = \varepsilon_{lj}, \quad T_{lj}(0) = \varphi_{lj}, \quad \dot{T}_{lj}(0) = \psi_{lj}, \quad l, j = 0, 1 \cdots , \tag{2.3}$$

where $\dot{T}_{lj}(t) = \frac{d}{dt}T_{lj}(t)$, etc. Hence $L_{lj}(t)$ and $T_{lj}(t)$ satisfy the infinite system of ordinary differential equations (2.1)–(2.3). Let us now prove the existence of the solution for the initial value problem (2.1)–(2.3). For this purpose, we shall first consider the existence of the solution for the following initial value problem of a finite system of ordinary differential equations

$$i\dot{L}_{Nlj} - (\alpha_l^2 + \beta_j^2)L_{Nlj} = \int_\Omega u_N \varepsilon_N y_l(x_1)z_j(x_2)\,dx, \tag{2.4}$$

$$\begin{aligned}(1 + \alpha_l^2 + \beta_j^2)\ddot{T}_{Nlj} &+ (\alpha_l^2 + \beta_j^2)T_{Nlj} \\ &= \int_\Omega (\nabla^2 f(u_N) + \nabla^2(|\varepsilon_N|^2))y_l(x_1)z_j(x_2)\,dx,\end{aligned} \tag{2.5}$$

$$L_{Nlj}(0) = \varepsilon_{lj}, \quad T_{Nlj}(0) = \varphi_{lj}, \quad \dot{T}_{Nlj}(0) = \psi_{lj}, \quad l, j = 0, 1 \cdots, N, \tag{2.6}$$

where $\varepsilon_N(x,t) = \sum_{l,j=0}^N L_{Nlj}(t)y_l(x_1)z_j(x_2)$ and $u_N(x,t) = \sum_{l,j=0}^N T_{Nlj}(t)y_l(x_1)z_j(x_2)$. In order to use Leray-Schauder's fixed-point argument we are also going to consider the following initial value problem of the finite system of ordinary differential equations with the parameter θ

$$i\dot{L}_{Nlj} - (\alpha_l^2 + \beta_j^2)L_{Nlj} = \theta\int_\Omega u_N \varepsilon_N y_l(x_1)z_j(x_2)\,dx, \tag{2.7}$$

$$\begin{aligned}(1 + \alpha_l^2 + \beta_j^2)\ddot{T}_{Nlj} &+ (\alpha_l^2 + \beta_j^2)T_{Nlj} \\ &= \theta\int_\Omega (\nabla^2 f(u_N) + \nabla^2(|\varepsilon_N|^2))y_l(x_1)z_j(x_2)\,dx,\end{aligned} \tag{2.8}$$

$$\begin{aligned}L_{Nlj}(0) &= \theta\varepsilon_{lj}, \quad T_{Nlj}(0) = \theta\varphi_{lj}, \\ \dot{T}_{Nlj}(0) &= \theta\psi_{lj}, \quad l, j = 0, 1 \cdots, N, \ 0 \le \theta \le 1.\end{aligned} \tag{2.9}$$

Lemma 2.2. *Suppose that the following conditions are satisfied.*

1. $f \in C^5$, $|f^{(j)}(s)| \le K_j|s|^{P+(5-j)}$ $(j = 1, 2, 3, 4, 5)$, *where* $K_j (j = 1, 2, 3, 4, 5)$ *are constants,* $p \ge 1$ *is a natural number and* (j) *denotes the order of derivatives.*

2. $(L(t), T(t))$ *is the solution of the system* (2.4), (2.5), *where* $L(t) = (L_{Nlj}(t),\ l, j = 0, 1, \cdots, N)$ *and* $T(t) = (T_{Nlj}(t),\ l, j = 0, 1, \cdots, N)$.

Let

$$\begin{aligned}E_N(t) = \sum_{l,j=0}^N (1 + \alpha_l^2 + \beta_j^2 &+ \sum_{\substack{h+m=8 \\ h,m=0,2,4,6,8}} \alpha_l^h \beta_j^m \\ &+2\sum_{\substack{h+m=10 \\ h,m=2,4,6,8}} \alpha_l^h \beta_j^m + \alpha_l^{10} + \beta_j^{10})\dot{T}_{Nlj}^2\end{aligned} \tag{2.10}$$

$$+\sum_{l,j=0}^{N}(1+\alpha_l^2+\beta_j^2+2\sum_{\substack{h+m=10\\h,m=2,4,6,8}}\alpha_l^h\beta_j^m+\alpha_l^{10}+\beta_j^{10})T_{Nlj}^2$$
$$+\sum_{l,j=0}^{N}(1+\sum_{\substack{h+m=10\\h,m=0,2,4,6,8,10}}\alpha_l^h\beta_j^m)|L_{Nlj}|^2+1.$$

Then there is

$$\frac{dE_N(t)}{dt}\le K_6(E_N(t))^{\frac{P+6}{2}}, \tag{2.11}$$

where $K_6>0$ is a constant independent of θ, D and N.

Proof: Multiplying both sides of (2.7) by $(1+\alpha_l^4\beta_j^6+\alpha_l^6\beta_j^4+\alpha_l^2\beta_j^8+\alpha_l^8\beta_j^2+\alpha_l^{10}+\beta_j^{10})\overline{L}_{Nlj}$, summing up the products for $l,j=0,1,\cdots,N$ and taking the imaginary part, we get

$$\begin{aligned}&\frac{d}{dt}\Big\{\sum_{l,j=0}^{N}(1+\sum_{\substack{h+m=10\\h,m=0,2,4,6,8,10}}\alpha_l^h\beta_j^m)|L_{Nlj}|^2\Big\}\\&=2\theta I_m\langle u_N\varepsilon_N,-\sum_{\substack{h+m=10\\h,m=0,2,4,6,8,10}}\varepsilon_{Nx_1^hx_2^m}\rangle,\end{aligned}\tag{2.12}$$

where $\overline{L}_{Nlj}$ is the conjugate number of L_{Nlj} and $\langle u,v\rangle=\int_\Omega u(x)\overline{v}(x)\,dx$.

Multiplying both sides of (2.8) by $(1+\alpha_l^4\beta_j^4+\alpha_l^6\beta_j^2+\alpha_l^2\beta_j^6+\alpha_l^8+\beta_j^8)\dot{T}_{Nlj}$, summing up the products for $l,j=0,1,\cdots,N$, adding $\frac{d}{dt}\sum_{l,j=0}^{N}T_{Nlj}^2$ to both sides of the obtained equation and adding the equation (2.12), we have

$$\begin{aligned}\frac{dE_N(t)}{dt}&=2\theta\langle\nabla^2 f(u_N)+\nabla^2(|\varepsilon_N|^2),\ u_{Nt}+\sum_{\substack{h+m=8\\h,m=0,2,4,6,8}}u_{Nx_1^hx_2^mt}\rangle\\&+\frac{d}{dt}\langle u_N,u_N\rangle+2\theta I_m\langle u_N\varepsilon_N,-\sum_{\substack{h+m=10\\h,m=0,2,4,6,8,10}}\varepsilon_Nx_1^hx_2^m\rangle.\end{aligned}\tag{2.13}$$

$\|\cdot\|_{L_2(\Omega)}=\|\cdot\|$, $\|\cdot\|_{L_\infty(\Omega)}$ and $\|\cdot\|_{H^m(\Omega)}$ denote the norm of the space $L_2(\Omega)$, $L_\infty(\Omega)$ and Sobolev space $H^m(\Omega)$ respectively.

Since by (2.10) we have

$$
\begin{aligned}
E_N(t) = &\sum_{j=1}^{2}\sum_{h=0,1,4,5}\langle u_{Nx_j^h t}, u_{Nx_j^h t}\rangle + \sum_{\substack{h+m=4\\ h,m=1,2,3}}\langle u_{Nx_1^h x_2^m t}, u_{Nx_1^h x_2^m t}\rangle \\
&+2\sum_{\substack{h+m=5\\ h,m=1,2,3,4}}\langle u_{Nx_1^h x_2^m t}, u_{Nx_1^h x_2^m t}\rangle + \langle u_N, u_N\rangle + \sum_{j=1}^{2}\langle u_{Nx_j}, u_{Nx_j}\rangle \\
&+2\sum_{\substack{h+m=5\\ h,m=1,2,3,4}}\langle u_{Nx_1^h x_2^m}, u_{Nx_1^h x_2^m}\rangle + \sum_{j=1}^{2}\langle u_{Nx_j^5}, u_{Nx_j^5}\rangle \\
&+\langle \varepsilon_N, \varepsilon_N\rangle + \sum_{\substack{h+m=5\\ h,m=0,1,2,3,4,5}}\langle \varepsilon_{Nx_1^h x_2^m}, \varepsilon_{Nx_1^h x_2^m}\rangle + 1,
\end{aligned}
\tag{2.14}
$$

by using the Gagliardo-Nirenberg theorem [8] and (2.14) we obtain

$$\|u_N\|_{L_\infty(\Omega)} \le C_1\|u_N\|^{\frac{4}{5}}\|u_N\|^{\frac{1}{5}}_{H^5(\Omega)} \le C_2(E_N(t))^{\frac{1}{2}}, \tag{2.15}$$

$$\|u_{Nx_j}\|_{L_\infty(\Omega)} \le C_3\|u_N\|^{\frac{3}{5}}\|u_N\|^{\frac{2}{5}}_{H^5(\Omega)} \le C_4(E_N(t))^{\frac{1}{2}}, \quad j=1,2, \tag{2.16}$$

$$\|u_{Nx_jx_k}\|, \|u_{Nx_jx_k}\|_{L\infty(\Omega)} \le C_5(E_N(t))^{\frac{1}{2}}, \quad j,k=1,2, \tag{2.17}$$

$$\|u_{Nx_1^j x_2^k}\|, \|u_{Nx_1^j x_2^k}\|_{L\infty(\Omega)} \le C_6(E_N(t))^{\frac{1}{2}}, \quad j+k=3, j,k=0,1,2,3, \tag{2.18}$$

$$\|u_{Nx_1^j x_2^k}\|, \|u_{Nx_1^j x_2^k}\|_{L\infty(\Omega)} \le C_7(E_N(t))^{\frac{1}{2}}, \quad j+k=4, j,k=0,1,2,3,4. \tag{2.19}$$

Similarly we can obtain

$$\|\varepsilon_N\|_{L_\infty(\Omega)}, \|\varepsilon_{Nx_j}\|, \|\varepsilon_{Nx_j}\|_{L_\infty(\Omega)} \le C_8(E_N(t))^{\frac{1}{2}}, \quad j=1,2, \tag{2.20}$$

$$\|\varepsilon_{Nx_jx_k}\|, \|\varepsilon_{Nx_jx_k}\|_{L_\infty(\Omega)} \le C_9(E_N(t))^{\frac{1}{2}}, \quad j,k=1,2, \tag{2.21}$$

$$\|\varepsilon_{Nx_1^j x_2^k}\|, \|\varepsilon_{Nx_1^j x_2^k}\|_{L_\infty(\Omega)} \le C_{10}(E_N(t))^{\frac{1}{2}}, \quad j+k=3, j,k=0,1,2,3, \tag{2.22}$$

$$\|\varepsilon_{Nx_1^j x_2^k}\|, \|\varepsilon_{Nx_1^j x_2^k}\|_{L_\infty(\Omega)} \le C_{11}(E_N(t))^{\frac{1}{2}}, \quad j+k=4, j,k=0,1,2,3,4. \tag{2.23}$$

Using the assumptions of Lemma 2.2, Hölder inequality and the inequalities (2.15)–(2.23) and (2.10), we have

$$
\begin{aligned}
&|\langle \nabla^2 f(u_N), u_{Nt}\rangle| \\
&= |\int_\Omega [f''(u_N)(u_{Nx_1}^2 + u_{Nx_2}^2) + f'(u_N)(u_{Nx_1^2} + u_{Nx_2^2})]u_{Nt}\,dx| \\
&\le C_{12}(E_N(t))^{\frac{p+4}{2}} \sum_{j=1}^{2} \int_\Omega \sum_{j=1}^{2} (|u_{Nx_j}| + |u_{Nx_j^2}|)|u_{Nt}|\,dx \\
&\le C_{12}(E_N(t))^{\frac{p+4}{2}} \sum_{j=1}^{2} (\|u_{Nx_j}\| + \|u_{Nx_j^2}\|)\|u_{Nt}\| \\
&\le C_{13}(E_N(t))^{\frac{p+6}{2}}.
\end{aligned}
\tag{2.24}
$$

By means of integration by parts, Hölder inequality, inequalities (2.15)–(2.19) and the assumptions of the lemma, we have

$$
|\langle \nabla^2 f(u_N), u_{Nx_1^6x_2^2t}\rangle| = \Big|\langle \frac{\partial^3}{\partial x_1^3}\nabla^2 f(u_N), u_{Nx_1^3x_1^2t}\rangle\Big| \le C_{14}(E_N(t))^{\frac{p+6}{2}}. \tag{2.25}
$$

Similarly we have

$$
|\langle \nabla^2 f(u_N), u_{Nx_1^4x_2^4t} + u_{Nx_1^2x_2^6t} + u_{Nx_1^8t} + u_{Nx_2^8t}\rangle| \le C_{15}(E_N(t))^{\frac{p+6}{2}}, \tag{2.26}
$$

$$
|\langle \nabla^2(|\varepsilon_N|^2), u_{Nt} + \sum_{\substack{j+k=8\\ j,k=0,2,4,6,8}} u_{Nx_1^jx_2^kt}\rangle| \le C_{16}(E_N(t))^{\frac{3}{2}}. \tag{2.27}
$$

Integrating by parts and using inequalities (2.15)–(2.23) we get

$$
\begin{aligned}
&|I_m\langle u_N\varepsilon_N, \varepsilon_{Nx_1^4x_2^6}\rangle| = |I_m \int_\Omega (u_N\varepsilon_N)_{x_1^2x_2^3}\overline{\varepsilon}_{Nx_1^2x_2^3}\,dx| \\
&= |I_m \int_\Omega (u_{Nx_1^2x_2^3}\varepsilon_N + 3u_{Nx_1^2x_2^2}\varepsilon_{Nx_2} + 3u_{Nx_1^2x_2}\varepsilon_{Nx_2^2} \\
&+ u_{Nx_1^2}\varepsilon_{Nx_2^3} + 2u_{Nx_1x_2^3}\varepsilon_{Nx_1} + 6u_{Nx_1x_2^2}\varepsilon_{Nx_1x_2} + 6u_{Nx_1x_2}\varepsilon_{Nx_1x_2^2} \\
&+ 2u_{Nx_1}\varepsilon_{Nx_1x_2^3} + u_{Nx_2^3}\varepsilon_{Nx_1^2} + 3u_{Nx_2^2}\varepsilon_{Nx_1^2x_2} \\
&+ 3u_{Nx_2}\varepsilon_{Nx_1^2x_2^2})\overline{\varepsilon}_{Nx_1^2x_2^3}\,dx| \le C_{17}(E_N(t))^{\frac{3}{2}}.
\end{aligned}
\tag{2.28}
$$

Similarly, we have

$$
|I_m\langle u_N\varepsilon_N, \varepsilon_{Nx_1^6x_2^4} + \varepsilon_{Nx_1^8x_2^2} + \varepsilon_{Nx_1^2x_2^8} + \varepsilon_{Nx_1^{10}} + \varepsilon_{Nx_2^{10}}\rangle| \le C_{18}(E_N(t))^{\frac{1}{2}}. \tag{2.29}
$$

Observe that $p \ge 1$ and it follows (2.11) from (2.13) and (2.24)–(2.29). The proof is thus completed. □

It is easy to prove the following lemma by (2.11)

Lemma 2.3. *Under the conditions of Lemma 2.2, if*

$$\begin{aligned}\lim_{N\to\infty} E_N(0) = b = &\sum_{l,j=0}^{\infty}\Big(1+\alpha_l^2+\beta_j^2+\sum_{\substack{h+m=8\\ h,m=0,2,4,6,8}}\alpha_l^h\beta_j^m\\ &+\ 2\sum_{\substack{h+m=10\\ h,m=2,4,6,8}}\alpha_l^h\beta_j^m+\alpha_l^{10}+\beta_j^{10}\Big)\psi_{lj}^2\\ &+\sum_{l,j=0}^{N}\Big(1+\alpha_l^2+\beta_j^2+2\sum_{\substack{h+m=10\\ h,m=2,4,6,8}}\alpha_l^h\beta_j^m+\alpha_l^{10}+\beta_j^{10}\Big)\varphi_{lj}^2\\ &+\sum_{l,j=0}^{N}\Big(1+\sum_{\substack{h+m=10\\ h,m=2,4,6,8}}\alpha_l^h\beta_j^m+\alpha_l^{10}+\beta_j^{10}\Big)|\varepsilon_{lj}|^2+1<\infty,\end{aligned}$$

then $E_N(t)\le b/(1-\frac{K_6(p+4)}{2}b^{(p+4)/2}t)^{2/(p+4)}$ *is uniformly bounded (let M be the bound) and independent of N and D in any closed subinterval* $0\le t\le t_1<t_b$, *where* $t_b=2/[K_6(p+4)b^{\frac{p+4}{2}}]$.

It is easy to prove the following lemma by Lemma 2.3 and Leray-Schauder's fixed-point theorem [9] as in [10].

Lemma 2.4. *Under the conditions of Lemma 2.3, there is a solution of the initial value problem (2.4)–(2.6) of the finite system of ordinary differential equations in* $[0,t_1]$, *here* $0<t_1<t_b$.

From the Ascoli-Arzelà theorem we have

Corollary 2.1. *Under the conditions of Lemma 2.4 and for the sequences* $\{L_{Nlj}\}_{l,j=0}^{N}$ *and* $\{T_{Nlj}\}_{l,j=0}^{N}$ $(N=1,2,\cdots)$ *of the solution for the initial value problem (2.4)–(2.6), there are convergent subsequences* $\{L_{N_slj}\}_{l,j=0}^{N_s}$ *and* $\{T_{N_slj}\}_{l,j=0}^{N_s}$ *respectively. As* $N_s\to\infty$, *then*

$$L_{N_slj}\to L_{lj},\qquad T_{N_slj}\to T_{lj},\qquad \dot T_{N_slj}\to \dot T_{lj}\qquad (l,j=0,1,\cdots)$$

uniformly in $[0,t_1]$.

Lemma 2.5. *Under the conditions of Lemma 2.3, the series* $\sum_{l,j=0}^{\infty}|L_{lj}|^2$, $\sum_{l,j=0}^{\infty}\alpha_l^h\beta_j^m|L_{lj}|^2$ $(h,m=2,4,6,8,h+m=10)$, $\sum_{l,j=0}^{\infty}\alpha_l^{10}|L_{lj}|^2$, $\sum_{l,j=0}^{\infty}\beta_j^{10}|L_{lj}|^2$, $\sum_{l,j=0}^{\infty}\dot T_{lj}^2$, $\sum_{l,j=0}^{\infty}\alpha_l^h\dot T_{lj}^2$ $(h=2,8,10)$, $\sum_{l,j=0}^{\infty}\beta_j^m\dot T_{lj}^2$ $(m=2,8,10)$, $\sum_{l,j=0}^{\infty}\alpha_l^h\beta_l^m\dot T_{lj}^2$ $(h,m=2,4,6,h+l=8)$, $\sum_{l,j=0}^{\infty}\alpha_l^h\beta_j^m\dot T_{lj}^2$ $(h,m=2,4,6,8,\ h+m=10)$, $\sum_{l,j=0}^{\infty}T_{lj}^2$, $\sum_{l,j=0}^{\infty}\alpha_l^hT_{lj}^2$ $(h=2,10)$, $\sum_{l,j=0}^{\infty}\beta_j^mT_{lj}^2$

$(m = 2, 10)$ *and* $\sum_{l,j=0}^{\infty} \alpha_l^h \beta_j^m T_{lj}^2$ $(h, m = 2, 4, 6, 8,\ h + m = 10)$ *are convergent and uniformly bounded* (*let* M *be the bound*) *in* $[0, t_1]$.

Proof: Let us denote $S_J = \sum_{l,j=0}^{J} \alpha_l^4 \beta_j^6 |L_{lj}|^2$ (J is a natural number). Obviously $S_{J+1} > S_J$. Fixing J, taking $N_s(> J)$ sufficiently large and using the Corollary 2.1 we obtain

$$S_J \le \Big| S_J - \sum_{l,j=0}^{J} \alpha_l^4 \beta_j^6 |L_{N_s lj}|^2 \Big| + \sum_{l,j=0}^{N_s} \alpha_l^4 \beta_j^6 |L_{N_s lj}|^2 \le 1 + E_{N_s}(t).$$

Since the functions $E_{N_s}(t)$ are bounded and independent of N_s in $[0, t_1]$, then S_J is bounded. Consequently the series $\sum_{l,j=0}^{\infty} \alpha_l^4 \beta_j^6 |L_{lj}|^2$ is convergent and bounded in $[0, t_1]$. Using the same method we can prove the other conclusions. The lemma is proved. □

Corollary 2.2. *Under the conditions of Lemma* 2.3, *there exists a constant* $M_1 > 0$, *such that*

$$\|\varepsilon\|_{H^5(\Omega)} + \|u\|_{H^5(\Omega)} + \|u_t\|_{H^5(\Omega)} \le M_1,$$
$$\|\varepsilon\|_{C^{3,\lambda}(\Omega)} + \|u\|_{C^{3,\lambda}(\Omega)} + \|u_t\|_{C^{3,\lambda}(\Omega)} \le M_1$$

in $[0, t_1]$, *here* $\varepsilon(x, t) = \sum_{l,j=0}^{\infty} L_{lj}(t) y_l(x_1) z_j(x_2)$ *and* $u(x, t) = \sum_{l,j=0}^{\infty} T_{lj}(t) y_l(x_1) z_j(x_2)$.

Lemma 2.6. *Under the conditions of Lemma* 2.3, $\varepsilon_{N_s} \to \varepsilon$, $\varepsilon_{N_s x_l} \to \varepsilon_{x_l}$ $(l = 1, 2)$, $u_{N_s} \to u$, $u_{N_s x_l} \to u_{x_l}$ $(l = 1, 2)$ *and* $u_{N_s t} \to u_t (N_s \to \infty)$ *uniformly in* Q_{t_1}, *where* $\varepsilon_{N_s}(x, t) = \sum_{l,j=0}^{N_s} L_{N_s lj}(t) y_l(x_1) z_j(x_2)$ *and* $u_{N_s}(x, t) = \sum_{l,j=0}^{N_s} T_{N_s lj}(t) y_l(x_1) z_j(x_2)$.

Proof: Let $L_{N_s lj} \equiv 0$ $(l, j > N_s)$. From Lemma 2.5 it follows that

$$\begin{aligned} |\varepsilon_{N_s} - \varepsilon| \le & \Big| \sum_{l,j=1}^{m} (L_{lj} - L_{N_s lj}) y_l(x_1) z_j(x_2) \Big| \\ & + \Big| \sum_{l,j=m+1}^{\infty} L_{lj} y_l(x_1) z_j(x_2) \Big| + \Big| \sum_{l,j=m+1}^{N_s} L_{N_s lj} y_l(x_1) z_j(x_2) \Big| \\ & \le \frac{1}{D} \sum_{l,j=1}^{m} |L_{lj} - L_{N_s lj}| + \frac{2\sqrt{M}}{\sqrt{D}} \sum_{l,j=m+1}^{\infty} \frac{1}{\alpha_l^2 \beta_j^3}. \end{aligned}$$

Observe Corollary 2.1. We can make the right side of the above inequality small by first choosing m and then choosing N_s, then $\varepsilon_{N_s}(x, t) \to \varepsilon(x, t)$ uniformly in Q_{t_1}, as $N_s \to \infty$. $u_{N_s} \to u$, $u_{N_s x_l} \to u_{x_l}$ $(l = 1, 2)$ and $u_{N_s t} \to u_t$ $(N_s \to \infty)$ are proved in a similar manner. Lemma 2.6 is proved. □

Theorem 2.1. *Under the conditions of Lemma* 2.3, $(L_{lj}(t), T_{lj}(t))$ *is a solution of the initial value problem* (2.1)–(2.3) *in* $0 \le t \le t_1 < t_b$, *where* $L_{lj}(t) = \lim_{N_s\to\infty} L_{N_slj}(t)$, $T_{lj}(t) = \lim_{N_s\to\infty} T_{N_slj}(t)$ *and* $(L_{N_slj}(t), T_{N_slj}(t),\ l,j = 0,1,\cdots,N_s)$ *being solution of the initial value problem* (2.4)–(2.6).

PROOF: The functions $L_{N_slj}(t)$ and $T_{N_slj}(t)$ satisfy the system of the Volterra integral equations

$$
\begin{aligned}
iL_{N_slj} = i\varepsilon_{lj} + \int_0^t \{(\alpha_l^2+\beta_j^2)L_{N_slj} \\
+\langle u_{N_s}\varepsilon_{N_s}, y_l(x_1)z_j(x_2)\rangle\}\,d\tau \equiv i\varepsilon_{lj} + H_{N_slj},
\end{aligned}
\tag{2.30}
$$

$$
\begin{aligned}
(1+\alpha_l^2+\beta_j^2)T_{N_slj} = (1+\alpha_l^2+\beta_j^2)(\varphi_{lj}+\psi_{lj}t) \\
-\int_0^t (t-\tau)[(\alpha_l^2+\beta_j^2)T_{N_slj} - \langle \nabla^2 f(u_{N_s}) + \nabla^2(|\varepsilon_{N_s}|^2), y_l(x_1)z_j(x_2)\rangle]\,d\tau \\
\equiv (1+\alpha_l^2+\beta_j^2)(\varphi_{lj}+\psi_{lj}t) - F_{N_slj}, \qquad l,j = 0,1,\cdots,N_s.
\end{aligned}
\tag{2.31}
$$

The aim is to show that the functions $\varepsilon_{lj}(t)$ and $T_{lj}(t)$ satisfy a similar system, indeed, by using of (2.30) and (2.31) we have

$$
\begin{aligned}
|iL_{lj} - i\varepsilon_{lj} - H_{lj}| &\le |iL_{lj} - iL_{N_slj}| + |H_{N_slj} - H_{lj}| \\
&\le |L_{lj} - L_{N_slj}| + (\alpha_l^2+\beta_j^2)t_1\|L_{lj} - L_{N_slj}\|_{C[0,t_1]} \\
&\quad + t_1\|\langle u\varepsilon - u_{N_s}\varepsilon, y_l(x_1)z_j(x_2)\rangle\|_{C[0,t_1]} \\
&\quad + t_1\|\langle u_{N_s}\varepsilon - u_{N_s}\varepsilon_{N_s}, y_l(x_1)z_j(x_2)\rangle\|_{C[0,t_1]},
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
&|(1+\alpha_l^2+\beta_j^2)T_{lj} - (1+\alpha_l^2+\beta_j^2)(\varphi_{lj}+\psi_{lj}t) + F_{lj}| \\
&\quad \le (1+\alpha_l^2+\beta_j^2)|T_{lj} - T_{N_slj}| + |F_{lj} - F_{N_slj}| \\
&\quad \le [1+\alpha_l^2+\beta_j^2 + (\alpha_l^2+\beta_j^2)t_1^2]\|T_{lj} - T_{N_slj}\|_{C[0,t_1]} \\
&\quad + t_1^2[\|\langle f(u) - f(u_{N_s}), y_l'' z_j + y_l z_j''\rangle\|_{C[0,t_1]} \\
&\quad + \|\langle |\varepsilon|^2 - |\varepsilon_{N_s}|^2, y_l'' z_j + y_l z_j''\rangle\|_{C[0,t_1]}\ (l,j = 0,1,\cdots,N_s),
\end{aligned}
\tag{2.33}
$$

where

$$
H_{lj} = \int_0^t \{(\alpha_l^2+\beta_j^2)L_{lj} + \langle u\varepsilon, y_l(x_1)z_j(x_2)\rangle\}\,d\tau,
$$

$$
F_{lj} = \int_0^t (t-\tau)[(\alpha_l^2+\beta_j^2)T_{lj} - \langle \nabla^2 f(u) + \nabla^2(|\varepsilon|^2), y_l(x_1)z_j(x_2)\rangle]\,d\tau.
$$

From Corollary 2.1 and Lemma 2.6 it follows that the right side of (2.32) and (2.33) approaches zero as $N_s \to \infty$. Therefore $(L_{lj}(t), T_{lj}(t))$ is a solution of the integral equations

$$iL_{lj} = i\varepsilon_{lj} + H_{lj},$$
$$(1+\alpha_l^2+\beta_j^2)T_{lj} = (1+\alpha_l^2+\beta_j^2)(\varphi_{lj}+\psi_{lj}t) - F_{lj}, \qquad l,j = 0,1,\cdots.$$

Differentiating the above first formula with respect to t and differentiating the above second formula twice with respect to t we get the conclusion of the theorem. Theorem 2.1 is proved. □

Lemma 2.7 [11]. *Suppose that $H(z_0, z_1, \cdots, z_l)$ is k-times ($k \geq 1$) continuously differentiable with respect to variables $z_0, z_1, \cdots, z_l$ and $z_j(x,t) \in L_\infty(\widetilde{Q}_T) \cap L_2([0,T]; H^k(\Omega))$, $j = 0,1,\cdots,l$. Then we have*

$$\left\| \frac{\partial^{\tilde{k}} H}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \right\|^2_{L_2(\widetilde{\Omega})} \leq C(M,k,l) \sum_{j=1}^{l} \|z_j\|^2_{H^k(\widetilde{\Omega})},$$

where $M = \max_{j=0,1,\cdots,l} \max_{(x,t)\in\widetilde{Q}_T} |z_j(x,t)|$, $\widetilde{Q}_T = \{x = (x_1,\cdots,x_n) \in \widetilde{\Omega} \subset R^n,\ t \in [0,T]\}$, $\widetilde{\Omega}$ is a bounded domain in R^n, $\tilde{k} = (k_1,\cdots,k_n)$, $k_j \geq 0$, $|\tilde{k}| = k = \sum_{j=1}^n k_j$.

Theorem 2.2. *Suppose that the conditions of Lemma 2.3 are satisfied and $\varepsilon_0(x), \varphi(x), \psi(x) \in H^6(\Omega)$. Then there exists a classical solution $\varepsilon(x,t) = \sum_{l,j=0}^\infty L_{lj}(t) y_l(x_1) z_j(x_2)$, $u(x,t) = \sum_{l,j=0}^\infty T_{lj}(t) y_l(x_1) z_j(x_2)$ of the problem (1.5)–(1.8) in Q_{t_1}.*

Proof: It follows from the assumptions

$$\text{(2.34)} \qquad \begin{gathered} \varepsilon_{lj} = \int_\Omega \varepsilon_0(x) y_l(x_1) z_j(x_2)\,dx, \qquad \varphi_{lj} = \int_\Omega \varphi(x) y_l(x_1) z_j(x_2)\,dx, \\ \psi_{lj} = \int_\Omega \psi(x) y_l(x_1) z_j(x_2)\,dx, \qquad l,j = 0,1,\cdots \end{gathered}$$

and they satisfy the condition $b < \infty$. In this case Theorem 2.1 guarantees the existence of a solution of the initial value problem (2.1)–(2.3) in $0 \leq t \leq t_1 < t_b$ and we have

$$\text{(2.35)} \qquad i\dot{L}_{lj} = (\alpha_l^2+\beta_j^2)L_{lj} + \langle u\varepsilon, y_l z_j\rangle,$$

$$\text{(2.36)} \qquad \begin{gathered} (1+\alpha_l^2+\beta_j^2)\ddot{T}_{lj} = -(\alpha_l^2+\beta_j^2)T_{lj} \\ +\langle f(u)+|\varepsilon|^2, y_l'' z_j + y_l z_j''\rangle, \qquad l,j = 0,1,\cdots \end{gathered}$$

in $[0,t_1]$. It follows from the finite system (2.4)–(2.6) of ordinary differential equations that

$$i\dot{L}_{N_slj}=(\alpha_l^2+\beta_j^2)L_{N_slj}+\langle u_{N_s}\varepsilon_{N_s},y_lz_j\rangle, \tag{2.37}$$

$$\begin{aligned}(1+\alpha_l^2+\beta_j^2)\ddot{T}_{N_slj}&=-(\alpha_l^2+\beta_j^2)T_{N_slj}\\&+\langle f(u_{N_s})+|\varepsilon_{N_s}|^2,y_l''z_j+y_lz_j''\rangle,\qquad l,j=0,1,\cdots,N_s.\end{aligned} \tag{2.38}$$

Since L_{N_slj} and T_{N_slj} converge uniformly to L_{lj} and T_{lj} $(l,j=0,1,\cdots)$ in $[0,t_1]$ as $N_s\to\infty$, and ε_{N_s}, u_{N_s}, $u_{N_sx_j}$ $(j=1,2)$ and u_{N_st} converge uniformly to ε, u, u_{x_j} $(j=1,2)$ and u_t respectively in Q_{t_1}, as $N_s\to\infty$, it follows from (2.35), (2.36) and (2.37), (2.38) that $\dot{L}_{N_slj}\to\dot{L}_{lj}$ and $\ddot{T}_{N_slj}\to\ddot{T}_{lj}$ uniformly in $[0,t_1]$ as $N_s\to\infty$.

Multiplying both sides of (2.4) by $(\alpha_l^{12}+\alpha_l^{10}\beta_j^2+\alpha_l^8\beta_j^4+\alpha_l^6\beta_j^6+\alpha_l^4\beta_j^8+\alpha_l^2\beta_j^{10}+\beta_j^{12})\overline{L}_{Nlj}$, summing up the products for $l,j=0,1,\cdots,N$ and taking the imaginary part, we have

$$\begin{aligned}\frac{d}{dt}\Big(\sum_{\substack{h+m=6\\h,m=0,1,\cdots,6}}\|\varepsilon_{Nx_1^hx_2^m}\|^2\Big)&=2I_m\langle\frac{\partial^6}{\partial x_1^6}(u_N\varepsilon_N),\varepsilon_{Nx_1^6}+\varepsilon_{Nx_1^4x_2^2}\\&+\varepsilon_{Nx_1^2x_2^4}+\varepsilon_{Nx_2^6}\rangle+2I_m\langle\frac{\partial^6}{\partial x_2^6}(u_N\varepsilon_N),\varepsilon_{Nx_1^4x_2^2}+\varepsilon_{Nx_1^2x_2^4}+\varepsilon_{Nx_2^6}\rangle.\end{aligned} \tag{2.39}$$

Multiplying both sides of (2.5) by $(\alpha_l^{10}+\alpha_l^2\beta_j^8+\alpha_l^4\beta_j^6+\alpha_l^6\beta_j^4+\alpha_l^8\beta_j^2+\beta_j^{10})\dot{T}_{Nlj}$ and summing up the products for $l,j=0,1,\cdots,N$, we obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\sum_{\substack{h+m=5\\h,m=0,1,\cdots,5}}\|u_{Nx_1^hx_2^mt}\|^2+2\sum_{\substack{h+m=6\\h,m=0,1,\cdots,5}}\|u_{Nx_1^hx_2^mt}\|^2\\&+\|u_{Nx_1^6t}\|^2+\|u_{Nx_2^6t}\|^2+\|u_{Nx_1^6}\|^2+\|u_{Nx_2^6}\|^2\\&+2\sum_{\substack{h+m=6\\h,m=1,2,\cdots,5}}\|u_{Nx_1^hx_2^m}\|^2\Big)\\&\le 2\Big|\langle\frac{\partial^4}{\partial x_1^4}[\nabla^2f(u_N)+\nabla^2(|\varepsilon_N|^2)],u_{Nx_1^6t}+u_{Nx_2^6t}+u_{Nx_1^2x_2^4t}\\&+u_{Nx_1^4x_2^2t}\rangle|+2|\langle\frac{\partial^4}{\partial x_2^4}[\nabla^2f(u_N)+\nabla^2(|\varepsilon_N|^2)],u_{Nx_1^2x_2^4t}+u_{Nx_2^6t}\rangle\Big|.\end{aligned} \tag{2.40}$$

By using the Hölder inequality and Lemma 2.7, we obtain

$$\begin{aligned}&\Big|2\langle\frac{\partial^4}{\partial x_1^4}[\nabla^2f(u_N)+\nabla^2(|\varepsilon_N|^2)],u_{Nx_1^6t}+u_{Nx_2^6t}+u_{Nx_1^2x_2^4t}+u_{Nx_1^4x_2^2t}\rangle\Big|\\&\le C_{19}(\|u_N\|^2_{H^6(\Omega)}+\|u_{Nt}\|^2_{H^6(\Omega)}+\|\varepsilon_N\|^2_{H^6(\Omega)}),\end{aligned} \tag{2.41}$$

$$\begin{aligned}(2.42)\qquad &\Big|2\langle\frac{\partial^4}{\partial x_2^4}[\nabla^2 f(u_N)+\nabla^2(|\varepsilon_N|^2)], u_{Nx_1^2x_2^4t}+u_{Nx_2^6t}\rangle\Big|\\ &\le C_{20}(\|u_N\|^2_{H^6(\Omega)}+\|u_{Nt}\|^2_{H^6(\Omega)}+\|\varepsilon_N\|^2_{H^6(\Omega)}),\end{aligned}$$

$$\begin{aligned}(2.43)\qquad &\Big|2I_m\langle\frac{\partial^6}{\partial x_1^6}(u_N\varepsilon_N), \varepsilon_{Nx_1^6}+\varepsilon_{Nx_1^4x_2^2}+\varepsilon_{Nx_1^2x_2^4}+\varepsilon_{Nx_2^6}\rangle\Big|\\ &\le C_{21}(\|u_N\|^2_{H^6(\Omega)}+\|\varepsilon_N\|^2_{H^6(\Omega)}),\end{aligned}$$

$$\begin{aligned}(2.44)\qquad &\Big|2I_m\langle\frac{\partial^6}{\partial x_1^6}(u_N\varepsilon_N), \varepsilon_{Nx_1^4x_2^2}+\varepsilon_{Nx_1^2x_2^4}+\varepsilon_{Nx_2^6}\rangle\Big|\\ &\le C_{22}(\|u_N\|^2_{H^6(\Omega)}+\|\varepsilon_N\|^2_{H^6(\Omega)}).\end{aligned}$$

Combining (2.39)–(2.44) and by means of Gronwall's inequality, we get

$$(2.45)\qquad \sum_{\substack{h+m=6\\ h,m=0,1,\cdots,6}}(\|u_{Nx_1^hx_2^m}\|^2+\|u_{Nx_1^hx_2^mt}\|^2+\|\varepsilon_{Nx_1^hx_2^m}\|^2)\le C_{23},\ t\in[0,t_1].$$

Hence, from Corollary 2.2 and (2.45) it follows that

$$(2.46)\qquad \|u_N\|_{H^6(\Omega)}+\|u_{Nt}\|_{H^6(\Omega)}+\|\varepsilon_N\|_{H^6(\Omega)}\le M_2,\qquad t\in[0,t_1].$$

Using the same method as in Lemma 2.5 we can prove that the series $\sum_{l,j=0}^\infty\alpha_l^h\beta_j^m|L_{lj}|^2$, $\sum_{l,j=0}^\infty\alpha_l^h\beta_j^m T_{lj}^2$ and $\sum_{l,j=0}^\infty\alpha_l^h\beta_j^m\dot{T}_{lj}^2$ $(h,m=4,6,8,\ h+m=12)$ are convergent and bounded (let M_3 be the bound) in $[0,t_1]$. Therefore we have

$$|L_{lj}(t)|\le\frac{M_3}{\alpha_l^4\beta_j^2},\qquad |L_{Nlj}(t)|\le\frac{M_3}{\alpha_l^4\beta_j^2},\qquad l,j=1,2,\cdots,\ t\in[0,t_1].$$

From the inequality

$$\begin{aligned}|\varepsilon_{N_sx_1^2}-\varepsilon_{x_1^2}|&\le\Big|\sum_{l,j=0}^m\alpha_l^2(L_{N_slj}-L_{lj})y_lz_j\Big|+\Big|\sum_{l,j=m+1}^\infty\alpha_l^2L_{N_slj}y_lz_j\Big|\\ &\quad+\Big|\sum_{l,j=m+1}^\infty\alpha_l^2L_{lj}y_lz_j\Big|\\ &\le\Big|\sum_{l,j=0}^m\alpha_l^2(L_{N_slj}-L_{lj})y_lz_j\Big|+C_{24}\sum_{l,j=m+1}^\infty\frac{1}{\alpha_l^2\beta_j^2},\end{aligned}$$

it follows that $\varepsilon_{N_s x_1^2} \to \varepsilon_{x_1^2}$ uniformly in Q_{t_1} as $N_s \to \infty$. Using the same method we can prove that $\varepsilon_{N_s x_2^2} \to \varepsilon_{x_2^2}$, $\varepsilon_{N_s x_1 x_2} \to \varepsilon_{x_1 x_2}$, $u_{N_s x_1^2} \to u_{x_1^2}$, $u_{N_s x_1 x_2} \to u_{x_1 x_2}$, $u_{N_s x_2^2} \to u_{x_2^2}$, $u_{N_s x_1 t} \to u_{x_1 t}$, $u_{N_s x_2 t} \to u_{x_2 t}$, $u_{N_s x_1^2 t} \to u_{x_1^2 t}$, $u_{N_s x_2^2 t} \to u_{x_2^2 t}$, and $u_{N_s x_1 x_2 t} \to u_{x_1 x_2 t}$ uniformly in Q_{t_1}, as $N_s \to \infty$.

Similarly, multiplying both sides of (2.4) by $(\alpha_l^2\beta_j^6+\alpha^4\beta_j^4+\alpha_l^6\beta_j^2+\alpha_l^8+\beta_j^8)\overline{\dot{L}}_{Nlj}$, summing up the products for $l,j=0,1,\cdots,N$ and taking the imaginary part, we have

$$
\begin{aligned}
&\sum_{l,j=0}^{N}\Big(\sum_{\substack{h+m=8\\ h,m=0,2,\cdots,8}}\alpha_l^h\beta_j^m\Big)|\dot{L}_{Nlj}|^2\\
&\le \Big|I_m\sum_{l,j=0}^{N}(\alpha_l^{10}+\beta_j^{10}+2\sum_{\substack{h+m=10\\ h,m=2,4,6,8,}}\alpha_l^h\beta_j^m)L_{Nlj}\overline{\dot{L}}_{Nlj}\Big|\\
&\quad+\Big|I_m\int_\Omega u_N\varepsilon_N\sum_{\substack{h+m=8\\ h,m=0,2,4,6,8}}\overline{\varepsilon}_{Nx_1^hx_2^mt}\,dx\Big|.
\end{aligned}\tag{2.47}
$$

Integrating by parts and using Cauchy's inequality and Lemma 2.7, we obtain

$$
\sum_{l,j=0}^{N}\Big(\sum_{\substack{h+m=8\\ h,m=0,2,4,6,8}}\alpha_l^h\beta_j^m\Big)|\dot{L}_{Nlj}|^2\le C_{25},\qquad t\in[0,t_1].\tag{2.48}
$$

Multiplying both sides of (2.5) by $(\alpha_l^2\beta_j^8+\alpha_l^4\beta_j^6+\alpha_l^6\beta_j^4+\alpha_l^8\beta_j^2+\alpha_l^{10}+\beta_j^{10})\ddot{T}_{Nlj}$ and summing up the products for $l,j=0,1,\cdots,N$, we have

$$
\begin{aligned}
&\sum_{l,j=0}^{N}\Big(\sum_{\substack{h+m=10\\ h,m=0,2,4,6,8,10}}\alpha_l^h\beta_j^m+\sum_{\substack{h+m=12\\ h,m=0,2,4,6,8,10,12,}}\alpha_l^h\beta_j^m\Big)\ddot{T}_{Nlj}^2\\
&\le C_{26}\Big[\Big(\sum_{l,j=0}^{N}\sum_{\substack{h+m=12\\ h,m=0,2,4,6,8,10,12}}\alpha_l^h\beta_j^m\Big)T_{Nlj}^2+\|u_N\|_{H^6(\Omega)}^2\\
&\quad+\|\varepsilon_N\|_{H^6(\Omega)}\Big]\le C_{27},\qquad t\in[0,t_1],
\end{aligned}\tag{2.49}
$$

where constant C_{27} is independent of N. From (2.48) and (2.49) it follows that the series $\sum_{l,j=0}^{\infty}\alpha_l^4\beta_j^4|\dot{L}_{lj}|^2$, $\sum_{l,j=0}^{\infty}\alpha_l^8\beta_j^4\ddot{T}_{lj}^2$, $\sum_{l,j=0}^{\infty}\alpha_l^4\beta_j^8\ddot{T}_{lj}^2$ and $\sum_{l,j=0}^{\infty}\alpha_l^6\beta_j^6\ddot{T}_{lj}^2$

are convergent and bounded (let M_4 be the bound) in $[0, t_1]$. Hence

$$|\dot{L}_{lj}(t)| \le \frac{M_4}{\alpha_l^2\beta_j^2}, \qquad |\dot{L}_{Nlj}(t)| \le \frac{M_4}{\alpha_l^2\beta_j^2}, \qquad |\ddot{T}_{lj}(t)| \le \frac{M_4}{\alpha_l^4\beta_j^2},$$
$$|\ddot{T}_{Nlj}(t)| \le \frac{M_4}{\alpha_l^4\beta_j^2}, \quad l, j = 1, 2, \cdots, \ t \in [0, t_1].$$

Using the above method we can prove that $\varepsilon_{N_s t} \to \varepsilon_t$ and $u_{N_s x_1^2 t^2} \to u_{x_1^2 t^2}$ uniformly in Q_{t_1}, as $N_s \to \infty$. Using the same method we can prove that $u_{N_s x_2^2 t^2} \to u_{x_2^2 t^2}$, $u_{N_s t^2} \to u_{t^2}$ uniformly in Q_{t_1}, as $N_s \to \infty$.

Thus if the nonlinear operators defined by (1.5) and (1.6) respectively are applied to $\varepsilon(x,t) = \sum_{l,j=0}^{\infty} L_{lj}(t) y_l(x_1) y_j(x_2)$ and $u(x,t) = \sum_{l,j=0}^{\infty} T_{lj}(t) y_l(x_1) y_j(x_2)$, the results are two functions, say $F(x,t)$ and $G(x,t)$, which are certainly continuous. Thus it is enough to prove $F(x,t) \equiv 0$ and $G(x,t) \equiv 0$, because for all l, j, $\langle F(x,t), y_l(x_1) z_j(x_2)\rangle$ is just the left side of (2.1) and $\langle G(x,t), y_l(x_1) z_j(x_2)\rangle$ is just the left side of (2.2). Therefore $\langle F(x,t), y_l(x_1) y_j(x_2)\rangle \equiv 0$, $\langle G(x,t), y_l(x_1) z_j(x_2)\rangle \equiv 0$. Noticing that $\{y_l(x_1) z_j(x_2), l, j = 0, 1, \cdots\}$ is an orthonormal base, we have $F(x,t) \equiv 0$ and $G(x,t) \equiv 0$. Thus $(\varepsilon(x,t), u(x,t))$ is a classical solution of the problem (1.5)–(1.8), where $\varepsilon(x,t) = \sum_{l,j=0}^{\infty} L_{lj}(t) y_l(x_1) z_j(x_2)$, $u(x,t) = \sum_{l,j=0}^{\infty} T_{lj}(t) y_l(x_1) z_j(x_2)$. Theorem 2.2 is proved. □

It is easy to prove the following theorem as in [10].

Theorem 2.3. *Suppose that $f \in C^3$, $\varepsilon_0(x), \widetilde{\varepsilon}_0(x) \in H^1(\Omega)$, $\varphi(x), \widetilde{\varphi}(x), \psi(x)$, $\widetilde{\psi}(x) \in H^2(\Omega)$. Let $(\varepsilon(x,t), u(x,t))$ and $(\varepsilon_1(x,t), u_1(x,t))$ be two different classical local solutions corresponding to $\varepsilon_0(x)$, $\varphi(x)$, $\psi(x)$ and $\widetilde{\varepsilon}_0(x)$, $\widetilde{\varphi}(x)$, $\widetilde{\psi}(x)$, respectively, of the problem* (1.5)–(1.8) *in Q_{t_1}. Then for any $\varepsilon > 0$ there exists a $\delta > 0$, such that if $\|\varepsilon_0 - \widetilde{\varepsilon}_0\|_{H^1(\Omega)} + \|\varphi - \widetilde{\varphi}\|_{H^2(\Omega)} + \|\psi - \widetilde{\psi}\|_{H^2(\Omega)} < \delta$, we have*

$$\begin{aligned} \|\varepsilon - \varepsilon_1\|_{H^2(\Omega)} + \|\varepsilon_t - \varepsilon_{1t}\|_{H^1(\Omega)} + \|u - u_1\|_{H^2(\Omega)} + \|u_t - u_{1t}\|_{H^2(\Omega)} \\ + \|u_{tt} - u_{2tt}\|_{H^2(\Omega)} < \varepsilon, \qquad 0 \le t \le t_1. \end{aligned} \tag{2.50}$$

Corollary 2.3. *Under the conditions of Theorem* 2.3, *the solution of the problem* (1.5)–(1.8) *in Q_{t_1} is unique.*

Now, we are going to consider the regularity of the solution for problem (1.5)–(1.8).

Lemma 2.8. *If $f(s) \in C^k(R)$ and $\varepsilon_0(x), \varphi(x), \psi(x) \in H^k(\Omega)$ ($k \ge 7$, the case of $k \le 6$ was proved), then*

$$\|\varepsilon_N\|_{H^k(\Omega)} + \|u_N\|_{H^k(\Omega)} + \|u_{Nt}\|_{H^k(\Omega)} \le M_5, \qquad t \in [0, t_1], \tag{2.51}$$

where M_5 is a constant independent of N.

PROOF: Multiplying both sides of (2.4) by $\sum_{\substack{h+m=2k \\ h,m=0,2,\cdots,2k}} \alpha_l^h \beta_j^m \overline{L}_{Nlj}(t)$, summing up the products for $l,j=0,1,\cdots,N$ and taking the imaginary part, we get

$$
\begin{aligned}
&\frac{d}{dt}\Big\{ \sum_{l,j=0}^{N} \sum_{\substack{h+m=2k \\ h,m=0,2,\cdots,2k}} \alpha_l^h \beta_j^m |L_{Nlj}|^2 \Big\} \\
&= 2I_m \int_\Omega (u_N \varepsilon_N)(-1)^k \sum_{\substack{h+m=2k \\ h,m=0,2,\cdots,2k}} \overline{\varepsilon}_{Nx_1^h x_2^m}\, dx.
\end{aligned}
\tag{2.52}
$$

Multiplying both sides of (2.5) by $\sum_{\substack{h+m=2(k-1) \\ h,m=0,2,\cdots,2(k-1)}} \alpha_l^h \beta_j^m \dot{T}_{Nlj}$ and summing up the products for $l,j=0,1,\cdots,N$, we get

$$
\begin{aligned}
&\frac{d}{dt}\Big\{ \sum_{l,j=0}^{N} \Big[\sum_{\substack{h+m=2(k-1) \\ h,m=0,2,\cdots,2(k-1)}} (\alpha_l^h \beta_j^m + \alpha_l^{h+2} \beta_j^m + \alpha_l^h \beta_j^{m+2}) \Big] \dot{T}_{Nlj}^2 \\
&+ \sum_{l,j=0}^{N} \Big[\sum_{\substack{h+m=2(k-1) \\ h,m=0,2,\cdots,2(k-1)}} (\alpha_l^{h+2} \beta_j^m + \alpha_l^h \beta_j^{m+2}) \Big] T_{Nlj}^2 \Big\} \\
&= 2\int_\Omega [\nabla^2 f(u_N) + \nabla^2(|\varepsilon_N|^2)] \cdot \\
&\sum_{l,j=0}^{N} \sum_{\substack{h+m=2(k-1) \\ h,m=0,2,\cdots,2(k-1)}} \alpha_l^h \beta_j^m \dot{T}_{Nlj} y_l(x_1) z_j(x_2)\, dx.
\end{aligned}
\tag{2.53}
$$

Combining (2.52) and (2.53) and integrating by parts with respect to x, we immediately get

(2.54)

$$
\begin{aligned}
&\frac{d}{dt}\Big\{ \sum_{\substack{h+m=k-1 \\ h,m=0,1,\cdots,(k-1)}} \|u_{Nx_1^h x_2^m t}\|^2 + 2 \sum_{\substack{h+m=k \\ h,m=1,2,\cdots,(k-1)}} \|u_{Nx_1^h x_2^m t}\|^2 \\
&+ \sum_{j=1}^{2} (\|u_{Nx_j^k t}\|^2 + \|u_{Nx_j^k}\|^2) + 2 \sum_{\substack{h+m=k \\ h,m=1,2,\cdots,(k-1)}} \|u_{Nx_1^h x_2^m}\|^2
\end{aligned}
$$

$$+ \sum_{\substack{h+m=k\\ h,m=0,1,\cdots,k}} \|\varepsilon_{Nx_1^h x_2^m}\|^2 \Big\}$$

$$\le 2\Big|\langle \frac{\partial^{k-2}}{\partial x_1^{k-2}}[\nabla^2 f(u_N)+\nabla^2(|\varepsilon_N|^2)], \sum^{*} u_{Nx_1^h x_2^m t}\rangle\Big|$$

$$+ 2\Big|\langle \frac{\partial^{k-2}}{\partial x_2^{k-2}}[\nabla^2 f(u_N)+\nabla^2(|\varepsilon_N|^2)] \sum^{**} u_{Nx_1^h x_2^m t}\rangle\Big|$$

$$+ 2\Big| I_m \int_\Omega \sum_{\substack{h+m=k\\ h,m=0,1,\cdots,k}} (u_N\varepsilon_N)_{x_1^h x_2^m} \sum_{\substack{h+m=k\\ h,m=0,1,\cdots,k}} \overline{\varepsilon}_{Nx_1^h x_2^m}\, dx\Big|,$$

where $\sum^* = \sum_{h+m=k}$, $h = k, k-2, \cdots, k-2\big(\frac{k-1}{2}\big)$, when k is an odd number; $h = k, k-2, \cdots, k-2\big(\frac{k}{2}\big)$, when k is an even number; $\sum^{**} = \sum_{h+m=k}$, $m = k, k-2, \cdots, k-2\big(\frac{k-3}{2}\big)$, when k is an odd number; $m = k, k-2, \cdots, k-2\big(\frac{k-4}{2}\big)$, when k is an even number.
By using Hölder's inequality and Lemma 2.7, we obtain

$$\begin{aligned} &\Big|2\langle \frac{\partial^{k-2}}{\partial x_1^{k-2}}[\nabla^2 f(u_N)+\nabla^2(|\varepsilon_N|^2)],\ \textstyle\sum^{*} u_{Nx_1^h x_2^m t}\rangle\Big| \\ &\le C_{28}(\|u_N\|^2_{H^k(\Omega)} + \|u_{Nt}\|^2_{H^k(\Omega)} + \|\varepsilon_N\|^2_{H^k(\Omega)}), \end{aligned} \tag{2.55}$$

$$\begin{aligned} &\Big|2\langle \frac{\partial^{k-2}}{\partial x_2^{k-2}}[\nabla^2 f(u_N)+\nabla^2(|\varepsilon_N|^2)],\ \textstyle\sum^{**} u_{Nx_1^h x_2^m t}\rangle\Big| \\ &\le C_{29}(\|u_N\|^2_{H^k(\Omega)} + \|u_{Nt}\|^2_{H^k(\Omega)} + \|\varepsilon_N\|^2_{H^k(\Omega)}), \end{aligned} \tag{2.56}$$

$$\begin{aligned} &\Big|2 I_m \int_\Omega \sum_{\substack{h+m=k\\ h,m=0,1,\cdots,k}} (u_N\varepsilon_N)_{x_1^h x_2^m} \sum_{\substack{h+m=k\\ h,m=0,1,\cdots,k}} \overline{\varepsilon}_{Nx_1^h x_2^m}\, dx\Big| \\ &\le C_{30}(\|u_N\|^2_{H^k(\Omega)} + \|\varepsilon_N\|^2_{H^k(\Omega)}). \end{aligned} \tag{2.57}$$

Substituting (2.55)–(2.57) into (2.54) and by means of Gronwall's inequality, we get (2.51). Lemma 2.8 is proved. □

Lemma 2.9. *Under the conditions of Lemma 2.8, if $k = 2r + \xi_r$ $(\xi_r \ge 0,\ r \ge 1)$, then there are estimates*

$$\sup_{0\le t\le t_1} \|\varepsilon_{Nt^\beta}\|_{H^{k-2\beta}(\Omega)} \le C_{31}, \qquad \beta = 1, 2, \cdots, r, \tag{2.58}$$

$$\sup_{0\le t\le t_1} \|u_{Nt^{\beta+1}}\|_{H^{k-2(\beta-1)}(\Omega)} \le C_{32}, \qquad \beta = 1, 2, \cdots, r, \tag{2.59}$$

where constants C_{31} and C_{32} are independent of N.

PROOF: By mathematical induction we prove that (2.58) and (2.59) hold. Multiplying both sides of (2.4) by $\overline{\dot{L}}_{Nlj}(t)+\sum\limits_{\substack{h+m=2\xi_1\\ h,m=0,2,\cdots,2\xi_1}}\alpha_l^h\beta_j^m\overline{\dot{L}}_{Nlj}$, summing up the products for $l,j=0,1,\cdots,N$ and taking the imaginary part we have

$$
\begin{aligned}
\|\varepsilon_{Nt}\|^2+\sum_{\substack{h+m=\xi_1\\ h,m=0,1,\cdots,\xi_1}}\|\varepsilon_{Nx_1^hx_2^mt}\|^2 &= I_m\sum_{l,j=0}^{N}(\alpha_l^2+\beta_j^2)L_{Nlj}\overline{\dot{L}}_{Nlj}\\
&+I_m\int_\Omega u_N\varepsilon_N\overline{\varepsilon}_{Nt}\,dx\\
&+I_m\sum_{l,j=0}^{N}\Big\{\sum_{\substack{h+m=2\xi_1\\ h,m=0,2,\cdots,2\xi_1}}(\alpha_l^{h+2}\beta_j^m+\alpha_l^h\beta_j^{m+2})\Big\}L_{Nlj}\overline{\dot{L}}_{Nlj}\\
&+I_m\int_\Omega u_N\varepsilon_N(-1)^{\xi_1}\sum_{\substack{h+m=2\xi_1\\ h,m=0,2,\cdots,2\xi_1}}\overline{\varepsilon}_{Nx_1^hx_2^mt}\,dx.
\end{aligned}\tag{2.60}
$$

Integrating by parts with respect to x and using Cauchy's inequality, Lemma 2.7 and (2.51), from (2.60) we can obtain (2.58) for $\beta=1$.

Multiplying both sides of (2.5) be $\ddot{T}_{Nlj}(t)+\sum\limits_{\substack{h+m=2(k-1)\\ h,m=0,2,\cdots,2(k-1)}}\alpha_l^h\beta_j^m\ddot{T}_{Nlj}(t)$, summing up the products for $l,j=0,1,\cdots,N$ we obtain

(2.61)

$$
\begin{aligned}
&\|u_{Nt^2}\|^2+\|\nabla u_{Nt^2}\|^2+\sum_{\substack{h+m=k-1\\ h,m=0,1,\cdots,k-1}}\|u_{Nx_1^hx_2^mt^2}\|^2\\
&+2\sum_{\substack{h+m=k\\ h,m=0,1,\cdots,k}}\|u_{Nx_1^hx_2^mt^2}\|^2+\sum_{j=1}^{2}\|u_{Nx_j^kt^2}\|^2\\
&=-\sum_{l,j=0}^{N}\Big\{(\alpha_l^2+\beta_j^2)T_{Nlj}\Big(\ddot{T}_{Nlj}+\sum_{\substack{h+m=2(k-1)\\ h,m=0,2,\cdots,2(k-1)}}\alpha_l^h\beta_j^m\ddot{T}_{Nlj}\Big)\Big\}\\
&+\int_\Omega(\nabla^2 f(u_N)+\nabla^2(|\varepsilon_N|^2))\Big(u_{Nt^2}+(-1)^{k-1}\sum_{\substack{h+m=k-1\\ h,m=0,1,\cdots,k-1}}u_{Nx_1^hx_2^mt^2}\Big)\,dx.
\end{aligned}
$$

Integrating by parts with respect to x and using Cauchy's inequality, Lemma 2.7 and (2.51), from (2.61) we can obtain (2.59) for $\beta=1$.

Assume that (2.58) and (2.59) hold for β. We are going to prove that (2.58) and (2.59) hold for $\beta+1$. Differentiating (2.4) β times with respect to t, multiplying both sides of the obtained equations by $\overline{L}_{Nljt^{\beta+1}}+\sum_{\substack{h+m=2\xi_{\beta+1}\\ h,m=0,2,\cdots,2\xi_{\beta+1}}}\alpha_l^h\beta_j^m\overline{L}_{Nljt^{\beta+1}}$, summing up the products for $l,j=0,1,\cdots,N$, integrating by parts and taking the imaginary part, we get

$$\|\varepsilon_{Nt^{\beta+1}}\|^2+\sum_{\substack{h+m=\xi_{\beta+1}\\ h,m=0,1,\cdots,\xi_{\beta+1}}}\|\varepsilon_{Nx_1^hx_2^mt^{\beta+1}}\|^2 \tag{2.62}$$
$$\leq\Big|I_m\sum_{l,j=0}^N\Big\{(\alpha_l^2+\beta_j^2)L_{Nljt^\beta}\Big(\overline{L}_{Nljt^{\beta+1}}+\sum_{\substack{h+m=2\xi_{\beta+1}\\ h,m=0,2,\cdots,2\xi_{\beta+1}}}\alpha_l^h\beta_j^m\overline{L}_{Nljt^{\beta+1}}\Big)\Big\}\Big|$$
$$+\Big|I_m\int_\Omega(u_N\varepsilon_N)_{t^\beta}\Big\{\overline{\varepsilon}_{Nt^{\beta+1}}+(-1)^{\xi_{\beta+1}}\sum_{\substack{h+m=2\xi_{\beta+1}\\ h,m=0,2,\cdots,2\xi_{\beta+1}}}\overline{\varepsilon}_{Nx_1^hx_2^mt^{\beta+1}}\Big\}\,dx\Big|.$$

Integrating by parts with respect to x and using Cauchy's inequality, Lemma 2.7, (2.51) and the assumptions of the mathematical induction, we obtain (2.58) for $\beta+1$.

Differentiating (2.5) β times with respect to t, multiplying both sides of the obtained equations by $T_{Nljt^{\beta+2}}+\sum_{\substack{h+m=2(\xi_{\beta-1})\\ h,m=0,2,\cdots,2(\xi_{\beta-1})}}\alpha_l^h\beta_j^mT_{Nljt^{\beta+2}}$, summing up the products for $l,j=0,1,\cdots,N$ and integrating by parts, we obtain

$$\|u_{Nt^{\beta+2}}\|^2+\|\nabla u_{Nt^{\beta+2}}\|^2+\sum_{\substack{h+m=\xi_\beta-1\\ h,m=0,1,\cdots,\xi_\beta-1}}\|u_{Nx_1^hx_2^mt^{\beta+2}}\|^2 \tag{2.63}$$
$$+2\sum_{\substack{h+m=\xi_\beta\\ h,m=1,2,\cdots,\xi_\beta}}\|u_{Nx_1^hx_2^mt^{\beta+2}}\|^2+\sum_{j=1}^2\|u_{Nx_j^{\xi_\beta}t^{\beta+2}}\|^2$$
$$=-\sum_{l,j=0}^N\Big\{(\alpha_l^2+\beta_j^2)T_{Nljt^\beta}\Big(T_{Nljt^{\beta+2}}+\sum_{\substack{h+m=2(\xi_\beta-1)\\ h,m=0,2,\cdots,2(\xi_\beta-1)}}\alpha_l^h\beta_j^mT_{Nljt^{\beta+2}}\Big)\Big\}$$
$$+\int_\Omega(\nabla^2f(u_N)+\nabla^2(|\varepsilon_N|^2))_{t^\beta}\Big(u_{Nt^{\beta+2}}$$
$$+(-1)^{\xi_\beta-1}\sum_{\substack{h+m=2(\xi_\beta-1)\\ h,m=0,2,\cdots,2(\xi_\beta-1)}}u_{Nx_1^hx_2^mt^{\beta+2}}\Big)\,dx.$$

Using the same method as above, from (2.63) we can obtain (2.59) for $\beta+1$. Lemma 2.9 is proved. □

It is easy to prove that the following theorem is valid by Lemmas 2.8, 2.9 and the compactness theorem.

Theorem 2.4. *Suppose that* $f(s) \in C^k(R)$, $|f^{(j)}(s)| \le K_j|s|^{p+(5-j)}$ $(j = 1,2,3,4,\ 5,\ p \ge 1)$ *and* $\varepsilon_0(x), \varphi(x), \psi(x) \in H^k(\Omega)$. *If* $k = 2r + \xi_r$ $(\xi_r \ge 0, r \ge 1)$, *then the solution* $(\varepsilon(x,t), u(x,t))$ *of the problem* (1.5)–(1.8) *has generalized derivatives* $D_x^\alpha D_t^\beta \varepsilon$ $(0 \le |\alpha| + 2\beta \le k)$, $D_x^\alpha D_t^\beta u$ $(0 \le |\alpha| \le k,\ \beta = 0,1;\ 0 \le |\alpha| + 2(\beta-2) \le k,\ \beta = 2,3,\cdots,r)$ *and continuous derivatives* $D_x^\alpha D_t^\beta \varepsilon$ $(0 \le |\alpha| + 2(\beta+1) \le k-2)$ *and* $D_x^\alpha D_t^\beta u$ $(0 \le |\alpha| \le k-2,\ \beta = 0,1;\ 0 \le |\alpha| + 2(\beta-1) \le k-2,\ \beta = 2,3,\cdots,r)$, *where* $D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$, $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \ge 0$ $(i = 1,2)$, $|\alpha| = \alpha_1 + \alpha_2$; $D_t = \frac{\partial}{\partial t}$, $D_t^\beta = \frac{\partial^\beta}{\partial t^\beta}$.

Remark 2.1. Obviously, after obtaining the integral estimates (2.51), (2.58) and (2.59) of the approximate solution for the problem (1.5)–(1.8), we can obtain the existence and regularity of the generalized local solution (when $k \ge 5$) and the classical local solution (when $k \ge 6$) for the problem (1.5)–(1.8) by the compactness theorem, too.

3. Cauchy problem (1.5), (1.6), (1.9)

In this section, we are going to consider the Cauchy problem (1.5), (1.6), (1.9).

Theorem 3.1. *Suppose that the condition* (1) *of Lemma 2.2 is satisfied and* $\varepsilon_0(x), \varphi(x), \psi(x) \in H^k(R^2)$. *If* $k \ge 5$, *then there exists a unique generalized local solution* $(\varepsilon(x,t), u(x,t))$ *of the Cauchy problem* (1.5), (1.6), (1.9) *in* $[0,t^*] \times R^2$, *where* $0 < t^* < t_{b^*}$,

$$
\begin{aligned}
b^* = {} & \|\psi\|^2 + \|\psi_{x_1}\|^2 + \|\psi_{x_2}\|^2 + \sum_{\substack{h+m=4\\ h,m=0,1,2,3,4}} \|\psi_{x_1^h x_2^m}\|^2 \\
& + 2\sum_{\substack{h+m=5\\ h,m=1,2,3,4}} \|\psi_{x_1^h x_2^m}\|^2 + \|\psi_{x_1^5}\|^2 + \|\psi_{x_2^5}\|^2 + \|\varphi\|^2 + \|\varphi_{x_1}\|^2 + \|\varphi_{x_2}\|^2 \\
& + 2\sum_{\substack{h+m=5\\ h,m=1,2,3,4}} \|\varphi_{x_1^h x_2^m}\|^2 + \|\varphi_{x_1^5}\|^2 + \|\varphi_{x_2^5}\|^2 + \|\varepsilon_0\|^2 \\
& + \sum_{\substack{h+m=5\\ h,m=1,2,3,4}} \|\varepsilon_{0x_1^h x_2^m}\|^2 + \|\varepsilon_{0x_1^5}\|^2 + \|\varepsilon_{0x_2^5}\|^2 + 1,
\end{aligned}
$$

$t_{b^*} = 2/[K_6(p+4)b^{*(p+4)/2}]$ *and* $\|\cdot\| = \|\cdot\|_{L_2(R^2)}$. *If* $k \ge 6$, *then there exists a unique classical local solution* $(\varepsilon(x,t), u(x,t))$ *of the Cauchy problem* (1.5), (1.6), (1.9) *in* $[0, t^*] \times R^2$. *The solution* $(\varepsilon(x,t), u(x,t))$ *of the Cauchy problem* (1.5), (1.6), (1.9) *has the same regularities as in Theorem* 2.4.

Proof: Let us take a real sequence $\{D_s\}$ $(D_s > 1)$ such that $D_s \to \infty$, as $s \to \infty$. For every s, let us construct periodic functions $\varepsilon_{0s}(x)$, $\varphi_s(x)$ and $\psi_s(x)$ with period $2D_s$, such that

(1) $\varepsilon_{0s}, \varphi_s, \psi_s \in H^k(\Omega_s)$, where $\overline{\Omega}_s = \{x \in (x_1, x_2) \mid |x_j| \le D_s, j = 1, 2\}$,
(2) $\varepsilon_{0s}(x) = \varepsilon_0(x)$, $\varphi_s(x) = \varphi(x)$, $\psi_s(x) = \psi(x)$, as $x \in [-(D_s-1), D_s-1] \times [-(D_s-1), D_s-1] = \widetilde{\Omega}_s$.

Then

$$\begin{aligned}
\|\varepsilon_{0sx_1^h x_2^m}\|_{L_2(\widetilde{\Omega}_s)} &\le \|\varepsilon_{0x_1^h x_2^m}\|, \quad (h+m=k,\ h,m=0,1,\cdots,k),\\
\|\varphi_{sx_1^h x_2^m}\|_{L_2(\widetilde{\Omega}_s)} &\le \|\varphi_{x_1^h x_2^m}\|, \quad (h+m=k,\ h,m=0,1,\cdots,k),\\
\|\psi_{sx_1^h x_2^m}\|_{L_2(\widetilde{\Omega}_s)} &\le \|\psi_{x_1^h x_2^m}\|, \quad (h+m=k,\ h,m=0,1,\cdots,k).
\end{aligned}$$

We consider the following periodic boundary value problem

$$i\varepsilon_t + \nabla^2\varepsilon - u\varepsilon = 0, \tag{3.1}$$

$$u_{tt} - \nabla^2 u - \nabla^2 u_{tt} = \nabla^2 f(u) + \nabla^2(|\varepsilon|^2), \tag{3.2}$$

$$\varepsilon(x,t) = \varepsilon(x + 2D_{se_j}, t), u(x,t) = u(x + 2D_{se_j}, t), \qquad j = 1, 2, \tag{3.3}$$

$$\varepsilon(x,0) = \varepsilon_0(x), u(x,0) = \varphi(x), u_t(x,0) = \psi(x). \tag{3.4}$$

Let $\{y_l(x_1)\}$ be an orthonormal base of eigenfunctions of the following boundary value problem of an ordinary differential equation

$$y'' + \lambda y = 0, \qquad y(x_1) = y(x_1 + 2D_s),$$

corresponding to eigenvalue $\lambda_{1l} = \alpha_l^2 = \left(\frac{l\pi}{D_s}\right)^2$, $(l = 0, 1, \cdots)$ and $\{z_j(x_2)\}$ be an orthonormal base of eigenfunctions of the following boundary value problem

$$z'' + \lambda z = 0, \qquad z(x_2) = z(x_2 + 2D_s),$$

corresponding to eigenvalue $\lambda_{2j} = \beta_j^2 = \left(\frac{j\pi}{D_s}\right)^2$, $(j = 0, 1, \cdots)$. According to Lemma 2.1, the family of functions $\{y_l(x_1)z_j(x_2), l, j = 0, 1, \cdots\}$ composes an orthonormal base in $L_2(\overline{\Omega}_s)$.

Suppose that the approximate solution of the problem (3.1)–(3.4) is $(\varepsilon_{Ns}(x,t) = \sum_{l,j=0}^{N_s} L_{N_s lj}(t) y_l(x_1) z_j(x_2)$, $u_{N_s}(x,t) = \sum_{l,j=0}^{N_s} T_{N_s lj}(t) y_l(x_1) z_j(x_2))$, where $L_{N_s lj}(t)$ and $T_{N_s lj}(t)$ are the coefficients to be determined. The coefficients should satisfy the initial value problem

$$\langle i\varepsilon_{N_s t} + \nabla^2\varepsilon_{N_s} - u_{N_s}\varepsilon_{N_s}, y_l z_j\rangle = 0, \tag{3.5}$$

$$\langle u_{N_s tt} - \nabla^2 u_{N_s} - \nabla^2 u_{N_s tt} - (\nabla^2 f(u_{N_s}) + \nabla^2(|\varepsilon_{N_s}|^2), y_l z_j\rangle = 0, \tag{3.6}$$

$$L_{N_s lj}(0) = \langle \varepsilon_{0s}, y_l z_j\rangle,\ T_{N_s lj}(0) = \langle \varphi_s, y_l z_j\rangle,$$

$$\dot{T}_{N_s lj} = \langle \psi_s, y_l z_j\rangle, \qquad l, j = 0, 1, \cdots, N_s. \tag{3.7}$$

Let

$$E_{N_s}(t)=\sum_{l,j=0}^{N_s}[(1+\alpha_l^2+\beta_j^2+\sum_{\substack{h+m=8\\ h,m=0,2,4,6,8}}\alpha_l^h\beta_j^m+2\sum_{\substack{h+m=10\\ h,m=2,4,6,8}}\alpha_l^h\beta_j^m$$

$$+\alpha_l^{10}+\beta_j^{10})\dot{T}_{N_slj}^2+(1+\alpha_l^2+\beta_j^2+2\sum_{\substack{h+m=10\\ h,m=2,4,6,8}}\alpha_l^h\beta_j^m+\alpha_l^{10}+\beta_j^{10})T_{N_slj}^2$$

$$+(1+\sum_{\substack{h+m=10\\ h,m=2,4,6,8}}\alpha_l^h\beta_j^m+\alpha_l^{10}+\beta_j^{10})|L_{N_slj}|^2]+1.$$

It follows that for any s, $\lim_{N_s\to\infty}E_{N_s}(0)=b_s<b^*$. Hence $t_{b^*}\leq t_{b_s}$. By the same method as above for obtaining the estimates (2.51), (2.58) and (2.59) and by imbedding theorem we have

(3.8) $\|\varepsilon_{N_s}\|_{H^k(\Omega_s)}+\|u_{N_s}\|_{H^k(\Omega_s)}+\|u_{N_st}\|_{H^k(\Omega_s)}\leq C_{33}, \qquad t\in[0,t_{b^*}],$

(3.9) $\|\varepsilon_{N_s}\|_{C^{k-2,\lambda}(\Omega_s)}+\|u_{N_s}\|_{C^{k-2,\lambda}(\Omega_s)}+\|u_{N_st}\|_{C^{k-2,\lambda}(\Omega_s)}\leq C_{34}, \qquad t\in[0,t_{b^*}],$

(3.10) $\|\varepsilon_{N_st^\beta}\|_{H^{k-2\beta}(\Omega_s)}\leq C_{35}, \qquad \beta=1,2,\cdots,r, \qquad t\in[0,t_{b^*}],$

(3.11) $\|\varepsilon_{N_st^\beta}\|_{C^{k-2(\beta+1),\lambda}(\Omega_s)}\leq C_{36}, \qquad \beta=1,2,\cdots,r, \qquad t\in[0,t_{b^*}],$

(3.12) $\|u_{N_st^{\beta+1}}\|_{H^{k-2(\beta-1)}(\Omega_s)}\leq C_{37}, \qquad \beta=1,2,\cdots,r, \qquad t\in[0,t_{b^*}],$

(3.13) $\|u_{N_st^{\beta+1}}\|_{C^{k-2\beta,\lambda}(\Omega_s)}\leq C_{38}, \qquad \beta=1,2,\cdots,r, \qquad t\in[0,t_{b^*}],$

where constants C_j (j = 33–38) are independent of N_s and D_s, $0<\lambda<1$, $k=2r+\xi_r$ ($\xi\geq 0, r\geq 1$).

If $k=5$, then from (3.8)–(3.13) we obtain

$$\|\varepsilon_{N_s}\|_{H^5(\Omega_s)}+\|\varepsilon_{N_st}\|_{H^3(\Omega_s)}+\|u_{N_s}\|_{H^5(\Omega_s)}+\|u_{N_st}\|_{H^5(\Omega_s)} + \|u_{N_st^2}\|_{H^5(\Omega_s)}\leq C_{39} \qquad t\in[0,t_{b^*}], \tag{3.14}$$

$$\|\varepsilon_{N_s}\|_{C^{3,\lambda}(\Omega_s)}+\|\varepsilon_{N_st}\|_{C^{1,\lambda}(\Omega_s)}+\|u_{N_s}\|_{C^{3,\lambda}(\Omega_s)}+\|u_{N_st}\|_{C^{3,\lambda}(\Omega_s)} + \|u_{N_st^2}\|_{H^{3,\lambda}(\Omega_s)}\leq C_{40} \qquad t\in[0,t_{b^*}], \tag{3.15}$$

where the constants C_{39}, C_{40} are independent of N_s and D_s. By virtue of (3.15) and the Ascoli-Arzelà theorem, we can select from $(\{\varepsilon_{N_s}\},\{u_{N_s}\})$ a subsequence, still denoted by $(\{\varepsilon_{N_s}\},\{u_{N_s}\})$, such that when $N_s\to\infty$, the subsequences $\{D_x^\alpha\varepsilon_{N_s}\}$ $(0\leq|\alpha|\leq 1)$, $\{D_x^\alpha u_{N_s}\}$ $(0\leq|\alpha|\leq 3)$ and $\{D_x^\alpha u_{N_st}\}$ $(0\leq|\alpha|\leq 3)$ uniformly converge to the limit functions $\{D_x^\alpha\varepsilon_s\}$ $(0\leq|\alpha|\leq 1)$, $\{D_x^\alpha u_s\}$ $(0\leq|\alpha|\leq 3)$ and $\{D_x^\alpha u_{st}\}$ $(0\leq|\alpha|\leq 3)$ in $\Omega_s\times[0,t_{b^*}]$ respectively.

The estimate (3.14) still holds for the above subsequence $(\{\varepsilon_{N_s}\}, \{u_{N_s}\})$. Hence we can select from $(\{\varepsilon_{N_s}\}, \{u_{N_s}\})$ a subsequence, still denoted by $(\{\varepsilon_{N_s}\}, \{u_{N_s}\})$, such that when $N_s \to \infty$, the subsequences $\{D_x^\alpha \varepsilon_{N_s}\}$ $(0 \le |\alpha| \le 5)$ and $\{D_x^\alpha u_{N_s}\}$ $(0 \le |\alpha| \le 5)$ in $L_\infty([0, t_{b^*}]; L_2(\Omega_s))$ weakly star converge to limit functions $\{D_x^\alpha \varepsilon_s\}$ $(0 \le |\alpha| \le 5)$ and $\{D_x^\alpha u_s\}$ $(0 \le |\alpha| \le 5)$ respectively; the subsequences $\{D_x^\alpha \varepsilon_{N_s t}\}$ $(0 \le |\alpha| \le 3)$ and $\{D_x^\alpha u_{N_s t}\}$ $(0 \le |\alpha| \le 5)$ in $L_\infty([0, t_{b^*}]); L_2(\Omega_s))$ weakly star converge to limit functions $\{D_x^\alpha \varepsilon_{st}\}$ $(0 \le |\alpha| \le 3)$ and $\{D_x^\alpha u_{st}\}$ $(0 \le |\alpha| \le 5)$ respectively and the subsequences $\{D_x^\alpha u_{N_s tt}\}$ $(0 \le |\alpha| \le 5)$ in $L_\infty([0, t_{b^*}]; L_2(\Omega_s))$ weakly star converge to limit functions $\{D_x^\alpha u_{stt}\}$ $(0 \le |\alpha| \le 5)$ respectively. From a corollary of the resonance theorem ([12]) it follows that the estimates (3.8)–(3.13) still hold for $(\{\varepsilon_s\}, \{u_s\})$, which is a generalized local solution of the problem (3.1)–(3.4). Using the Ascoli-Arzelà theorem we can select a subsequence of $(\{\varepsilon_s\}, \{u_s\})$, still denoted by $(\{\varepsilon_s\}, \{u_s\})$, such that when $s \to \infty$ the subsequences $\{D_x^\alpha \varepsilon_s\}$ $(0 \le |\alpha| \le 1)$, $\{D_x^\alpha u_s\}$ $(0 \le |\alpha| \le 3)$ and $\{D_x^\alpha u_{st}\}$ $(0 \le |\alpha| \le 3)$ in any domain $\{-L \le x_1,\ x_2 \le L,\ 0 \le t \le t_{b^*}\}$ $(L > 0)$ uniformly converge to limit functions $D_x^\alpha \varepsilon$ $(0 \le |\alpha| \le 1)$, $D_x^\alpha u(0 \le |\alpha| \le 3)$ and $D_x^\alpha u_t$ $(0 \le |\alpha| \le 3)$ respectively. From (3.8)–(3.13) of $k = 5$ it follows that when $s \to \infty$ the subsequences $\{D_x^\alpha \varepsilon_s\}$ $(0 \le |\alpha| \le 5)$ and $\{D_x^\alpha u_s\}$ $(0 \le |\alpha| \le 5)$ in $L_\infty([0, t_{b^*}];$ $L_2(-L, L) \times (-L, L))$ weakly star converge to limit functions $D_x^\alpha \varepsilon$ $(0 \le |\alpha| \le 5)$ and $D_x^\alpha u$ $(0 \le |\alpha| \le 5)$ respectively, the subsequences $\{D_x^\alpha \varepsilon_{st}\}$ $(0 \le |\alpha| \le 3)$ and $\{D_x^\alpha u_{st}\}$ $(0 \le |\alpha| \le 5)$ in $L_\infty([0, t_{b^*}];\ L_2((-L, L) \times (-L, L))$ weakly star converge to limit functions $D_x^\alpha \varepsilon_t$ $(0 \le |\alpha| \le 3)$ and $D_x^\alpha u_t$ $(0 \le |\alpha| \le 5)$ respectively and the subsequences $\{D_x^\alpha u_{stt}\}$ $(0 \le |\alpha| \le 5)$ in $L_\infty([0, t_{b^*}];\ L_2((-L, L) \times (-L, L))$ weakly star converge to limit functions $D_x^\alpha u_{tt}$ $(0 \le |\alpha| \le 5)$ respectively. Therefore if $k \ge 5$, then there exists a generalized local solution $(\varepsilon(x,t), u(x,t))$ of the problem (1.5), (1.6), (1.9).

Similarly, we can prove that if $k \ge 6$, then there exists a classical local solution $(\varepsilon(x,t), u(x,t))$ of the problem (1.5), (1.6), (1.9). The solution has the same regularities in Theorem 2.4.

Obviously, the generalized local solution or the classical local solution of the problem (1.5), (1.6), (1.9) is unique. The proof is complete. $\square$

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