On sub-, pseudo- and quasimaximal spaces

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Dedicated to Professor Banaschewski on his seventyfirst birthday.

Abstract. The structure of sub-, pseudo- and quasimaximal spaces is investigated. A method of constructing non-trivial quasimaximal spaces is presented.

Keywords: submaximal, quasimaximal, pseudomaximal, ultrafilter, tensorproduct, measurable cardinal, σ-discrete, tree, Čech-Stone-compactification, retraction
Classification: Primary 54A10; Secondary 54D30

Introduction. A topological space is called submaximal, if every dense set is open. Or equivalently, every subset is the intersection of a closed set and an open set. The reader is expected to be familiar with the basic results about sub- and quasimaximal spaces as they are e.g. presented in Hewitt [Hew43] and Bourbaki [Bou66]. General topological notions are adopted from Engelking [Eng77]. I am assuming **AC**. Recently Arhangel'skii & Collins [A&C95] gave a systematic study of submaximal spaces. Submaximality often comes along with maximal-P spaces (e.g. P = connected, feebly compact, pseudocompact) see Cameron [Cam77] and Raha [Rah71], because it resembles a property of ultrafilters: If $A \subseteq X$ and A intersects every $O \in \mathcal{X} \setminus \{\emptyset\}$, then $A \in \mathcal{X}$. Quasi- and submaximal spaces were introduced by Bourbaki [Bou66]. If without isolated points, Hewitt [Hew43] calls the latter MI-spaces. We begin with a few simple observations. Recall that a space is called T_D, if $cl\{x\} \setminus \{x\}$ is closed for all points $x \in X$. T_D is stronger than T_0.

Definition 1. A topological space $(X, \mathcal{X})$ is called

(a) submaximal, if every dense set is open,
(b) pseudomaximal, if every strictly finer topology adds new isolated points,
(c) quasimaximal, if $(X, \mathcal{X})$ is pseudomaximal and possesses no isolated points.

Lemma 2. *Let $(X, \mathcal{X})$ be a submaximal space. Then*

(a) *$(X, \mathcal{X})$ is T_D,*
(b) *if $x \in X$ is non-isolated, then $cl\{x\} = \{x\}$.*

Proof: (a) $\{x\}$ is open in $cl\{x\}$.
(b) $X \setminus \{x\}$ is dense. □

Grants from the University of Qwa-Qwa are acknowledged.

Remark 3. Hence submaximality alone does not impose strong separation axioms. A trivial ultrafilter on a set provides an example of a submaximal space with a dense singleton.

Lemma 4. *Let $(X, \mathcal{X})$ be a topological space. Then the following statements are equivalent:*

(a) *$(X, \mathcal{X})$ is pseudomaximal;*
(b) *every subspace without new isolated points (i.e. every isolated point of the subspace is isolated in X) is open.*

Proof: Easy. $\square$

Lemma 5. *A subspace $A \subseteq X$ of a pseudomaximal space $(X, \mathcal{X})$ is pseudomaximal.*

Proof: Suppose $B \subseteq A$ does not have new isolated points with respect to A. For every isolated point $i \in B$ choose $U_i \in \mathcal{X}$ such that $A \cap U_i = \{i\}$. Then $\tilde{B} := B \cup \bigcup_i U_i$ has no new isolated points with respect to X: Assume $O_z \cap \tilde{B} = \{z\}$. Then $O_z \cap \bigcup U_i = \{z\}$ and z is isolated in X or $O_z \cap B = \{z\}$ and z belongs to one of the U_i's. Now make use of Lemma 4 and observe $A \cap \tilde{B} = B$. $\square$

Theorem 6. *Let $(X, \mathcal{X})$ be a topological space. Then the following statements are equivalent:*

(a) *$(X, \mathcal{X})$ is pseudomaximal and T_0;*
(b) *$(X, \mathcal{X})$ is submaximal and extremally disconnected.*

Proof:

$\Downarrow$ Let $O \in \mathcal{X}$. clO does not have new isolated points and is therefore open. Let D be dense. Then D contains all isolated points of X. Assume $\{d\} = D \cap O_d$ is a new isolated point of D, where $O_d \in \mathcal{X}$. Since D is dense, $O_d \subseteq cl\{d\}$ and $O_d \setminus \{d\}$ is not open (it even has empty interior). Hence $\exists U_z \in \mathcal{X} : U_z \cap O_d \setminus \{d\} = \{z\}$. It follows $U_z \cap O_d = U_z \cap (O_d \setminus \{d\} \cup \{d\}) = \{z\} \cup U_z \cap \{d\} = \{z, d\}$. Every finite open set in a T_0 space contains an open singleton. $\{d\}$ cannot be open, because d is a *new* isolated point, neither can $\{z\}$ be open.

$\Uparrow$ Let $A \subseteq X$ be without new isolated points. By submaximality, $A \setminus \operatorname{int} A$ is discrete. So $x \in A \setminus \operatorname{int} A$ implies $\exists U_x : U_x \cap A \setminus \operatorname{int} A = \{x\}$. Since A has no new isolated points, $U_x \cap \operatorname{int} A \neq \emptyset$ follows. Hence $clA = cl(\operatorname{int} A)$, which is open, because of extremally disconnectedness and A is open as dense subset of the open set clA. $\square$

Corollary 7 (Bourbaki [Bou66] for T_2)**.** *Let $(X, \mathcal{X})$ be a topological space. Then the following statements are equivalent:*

(a) *X is submaximal, extremally disconnected and without isolated points;*
(b) *X is quasimaximal and T_0.*

Proof:
$\Downarrow$ By Lemma 2, X is T_1. By Theorem 6, X is pseudomaximal.
$\Uparrow$ See Theorem 6. $\square$

Remark 8. (a) A quasimaximal T_0 space is always T_1. The example from Remark 3 is pseudomaximal but not T_1. The two-point indiscrete space is quasimaximal but not submaximal.

(b) More generally, a quasimaximal space can have at most two-point indiscrete subspaces and each of these is open. The set of supersets of a two-point subset together with the empty set provides an example of a quasimaximal topology where the two-point indiscrete subspace is not closed.

(c) Since there are maximal connected T_2 spaces ([GSW77]), there are non-pseudomaximal submaximal spaces without isolated points.

(d) For T_0-spaces, quasimaximal $\Rightarrow$ pseudomaximal $\Rightarrow$ submaximal. None of these implications can be reversed. The Alexandroff compactification $\mathcal{N} \cup \{\infty\}$ of the naturals is submaximal but not pseudomaximal. Refine the filter of finite complements belonging to ∞ to an ultrafilter and the resulting space is pseudomaximal but not quasimaximal.

(e) Recall that a space $(X, \mathcal{X})$ is called (strongly) σ-discrete, if X can be represented as countable union of (closed) discrete subspaces.

Lemma 9. *Let $(X, \mathcal{X})$ be submaximal without isolated points. Then the following statements are equivalent:*

(a) *X is strongly σ-discrete;*
(b) *X is σ-discrete;*
(c) $\exists$ dense $D_n \in \mathcal{X}, n \in \mathbb{N} : \bigcap_{\mathbb{N}} D_n = \emptyset$.

Proof:

$\Downarrow$ Trivial.
$\Uparrow$ Let $A \subseteq X$ be discrete. Since X has no isolated points, $X \setminus A$ is dense and therefore A is closed.
$\Downarrow\Downarrow\Uparrow\Uparrow$ Set $D_n := X \setminus A_n$ and $A_n = X \setminus D_n$. Since $D_n \cap D_m$ is dense for $n, m \in \mathbb{N}$ we may even assume $D_n \subseteq D_m$ for $m \leq n$.

$\square$

Theorem 10. *If every submaximal space is σ-discrete, then there are no measurable cardinals.*

Proof: Let $\mathcal{M}$ be a countable complete ultrafilter on X. Then $(X, \mathcal{M} \cup \{\emptyset\})$ is submaximal (even quasimaximal). Assume there are sets D_n as indicated in Lemma 9(c). Since all D_n are dense, we have $D_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. But $\bigcap D_n = \emptyset$ is a contradiction. $\square$

Remark 11. (a) Theorem 10 answers partly a question of Arhangel'skii & Collins [A&C95].

(b) In the sequel $\mathcal{U}^-(x)$ denotes the filter generated by $\{U_x \setminus \{x\} \mid U_x \in \mathcal{U}(x)\}$, where x is non-isolated and $\mathcal{U}(x)$ is the neighbourhood filter of x.

(c) An open ultrafilter is an ultrafilter with a base of open sets (= open generated ultrafilter). A maximal open filter instead is a maximal filter in $\mathcal{X}$. A filter is called free if it has no adherence points. A filter is non-trivial if it is finer than the filter of finite complements.

Theorem 12. *Let $(X, \mathcal{X})$ be submaximal and let $x \in X$ be non-isolated. Further let $\mathcal{F}$ be a filter with $\mathcal{F} \supseteq \mathcal{U}^-(x)$. Then $\{\operatorname{int} F \mid F \in \mathcal{F}\}$ generates an open filter finer than $\mathcal{F}$.*

PROOF: Let $F \in \mathcal{F}$. $\{x\} \cup X \setminus F$ is not open, because $(U_x \setminus \{x\}) \cap F \neq \emptyset$ for all $U_x \in \mathcal{U}(x)$. Hence $\{x\} \cup X \setminus F$ is not dense. There is $O \in \mathcal{X} \setminus \{\emptyset\}$ with $O \cap (\{x\} \cup X \setminus F) = \emptyset$. This yields $O \cap X \setminus F = \emptyset$ and $O \subseteq F$. Therefore $F \in \mathcal{F} \Rightarrow \operatorname{int} F \neq \emptyset$ and $\emptyset \neq \operatorname{int}(F \cap G) = \operatorname{int} F \cap \operatorname{int} G$ for $F, G \in \mathcal{F}$. □

Corollary 13. *Every ultrafilter finer than $\mathcal{U}^-(x)$ is open generated.* □

Theorem 14. *Let $(X, \mathcal{X})$ be a T_0-space. Then the following statements are equivalent:*

(a) *X is pseudomaximal;*
(b) *$\mathcal{U}^-(x)$ is ultrafilter for every non-isolated $x \in X$.*

PROOF:

$\Downarrow$ Take two different ultrafilters $\mathcal{F}, \mathcal{F}'$ finer than $\mathcal{U}^-(x)$. There are $F \in \mathcal{F}$, $F' \in \mathcal{F}'$ such that $F \cap F' = \emptyset$. By Corollary 13 we may assume that both F, F' are open. By Theorem 6 $clF \cap clF' = \emptyset$, too. But $x \in clF \cap clF'$ is a contradiction. Hence there is exactly one ultrafilter finer than $\mathcal{U}^-(x)$. This implies that $\mathcal{U}^-(x)$ is a prime filter and in turn an ultrafilter itself.

$\Uparrow$ Let $A \subseteq X$ be non-open. We have to show that the subspace A adds an isolated point. Since A is non-open, there is a point $a \in A$ no neighbourhood of which is contained in A. Certainly a is not isolated. Since $\mathcal{U}^-(a)$ is ultrafilter, there is $V \in \mathcal{U}^-(a)$ with $V \subseteq X \setminus A$. $V_a := V \cup \{a\}$ is a neighbourhood of a with $V_a \cap A = \{a\}$. □

Corollary 15. *Let $(X, \mathcal{X})$ be T_0-space. Then the following statements are equivalent:*

(a) *X is quasimaximal;*
(b) *$\mathcal{U}^-(x)$ is ultrafilter for all $x \in X$.* □

Corollary 16. *Let $(X, \mathcal{X})$ be T_0-space such that* $\operatorname{card}(X)$ *is below the first measurable. If X is pseudomaximal, then $\psi(X) \leq \aleph_0$.*

PROOF: If $x \in X$ is non-isolated, choose a countable family $\{G_n \mid n \in \mathbb{N}\} \subset \mathcal{U}^-(x)$ satisfying $\bigcap\{G_n\} = \emptyset$. Then $\{x\} = \bigcap\{G_n \cup \{x\}\}$ is G_δ. □

Remark 17. (a) If $\mathcal{N}$ is the collection of nowhere dense sets of X, then $\{U_x \setminus N \mid U_x \in \mathcal{U}(x) \wedge N \in \mathcal{N}\}$ generates a filter which is only apparently finer than $\mathcal{U}^-(x)$, because in a submaximal space, $N \in \mathcal{N}$ is closed discrete.

(b) How do quasimaximal spaces look? Let $A \subseteq X$ and take a point $x \in \delta A$ in the boundary of A. Corollary 15 tells us, that neighbourhoods of x either reach out to A or to $X \setminus A$. A picture of this situation suggests that we may call quasimaximal spaces also zipper spaces. This point of view leads to a new characterization of pseudomaximal spaces in Theorem 18. A similar result is already contained in the paper [vDo93] of the late Eric van Douwen for perfectly disconnected (= quasimaximal T_2) spaces.

(c) $\mathcal{U}(x)$ is centered around x. There are other open filters of interest which are not centered: the non-convergent maximal open filters, which we will use to build the Katětov extension. This is done in Theorem 19.

Theorem 18. *Let $(X, \mathcal{X})$ be a T_0-space. Then the following statements are equivalent:*

(a) *$(X, \mathcal{X})$ is pseudomaximal;*
(b) *if $A, B \subseteq X$ are disjoint, then $\forall_{x \in X} x \notin cl_X(A \setminus \{x\}) \cap cl_X(B \setminus \{x\})$.*

Proof: We will make use of Theorem 14.
$\Downarrow$ Of course, an ultrafilter cannot sit simultaneously on two disjoint sets.
$\Uparrow$ Take $x \in X$ and $A \subseteq X$. If x is isolated, the statement is obviously true. Set $B := X \setminus A$ and assume without loss of generality $x \in cl_X(A \setminus \{x\})$. Then $x \notin cl_X(B \setminus \{x\})$, which implies $x \in \mathrm{int}_X(A \cup \{x\})$ and $\mathrm{int}_X(A \cup \{x\}) \setminus \{x\} \subseteq A$. Hence $A \in \mathcal{U}^-(x)$ or $X \setminus A \in \mathcal{U}^-(x)$. □

Theorem 19. *Let $(X, \mathcal{X})$ be a T_0-space and κX its Katětov extension.*

(a) *If X is submaximal, then so is κX.*
(b) *If X is pseudomaximal, then so is κX.*
(c) *If X is quasimaximal, then so is κX.*

Proof: κX is T_0.

(a) Let $D \subseteq \kappa X$ be dense. Since X is open in κX, $D \cap X$ is dense in X and hence open. Since the traces on X of neighbourhood filters $\mathcal{U}(x_\kappa)$ of points $x_\kappa \in \kappa X \setminus X$ are all maximal open filters, $D \cap X \cup \{x_\kappa\} \in \mathcal{U}(x_\kappa)$ for all x_κ. Since $\kappa X \setminus X$ is discrete, $D \subseteq X$ is open.

(b) According to Theorem 6 we have to show that κX is extremally disconnected. Let $O \subseteq \kappa X$ be open. $X \cap cl_\kappa O$ is closed and open in X. On the other hand $x_\kappa \in cl_\kappa O \Rightarrow \forall U_{x_\kappa} : U_{x_\kappa} \cap O \cap X \neq \emptyset \Rightarrow X \cap O \in \mathcal{U}^-(x_\kappa)$, since $\mathcal{U}^-(x_\kappa) = x_\kappa$ is a maximal open filter.

(c) If X is without isolated points, then so is κX. □

Corollary 20. *In a submaximal space, free maximal open filters are ultrafilters.*

Proof: If x_κ is a maximal open filter, then $X \cup \{x_\kappa\}$ is submaximal subspace of κX. An ultrafilter finer than $\mathcal{U}^-(x_\kappa)$ is open generated and must coincide with x_κ. $\square$

Remark 21. (a) All free ultrafilters other than those from Corollary 20 are sitting on nowhere dense sets and do not converge.

(b) Let $(X, \mathcal{X})$ be submaximal without isolated points and let $\mathcal{B}_1, \mathcal{B}_2$ be two free maximal open filters on X. Define a neighbourhood base of ∞ by its trace $\mathcal{B}_1 \cap \mathcal{B}_2$ on X. Then $X \cup \{\infty\}$ is submaximal. This shows that submaximality together with the absence of isolated points does not guarantee the maximality property of Theorem 14.

(c) The existence of free ultrafilters is ensured by AC. Corollary 15 shows that AC can provide many ultrafilters at once which are even interlinked such that the 4th neighbourhood axiom is fulfilled.

(d) The Katětov extension of a discrete space is pseudomaximal.

Theorem 22. *Let $(X, \mathcal{X})$ be a submaximal space. For every non-isolated $x \in X$ select an ultrafilter $P(x) := \mathcal{F}_x \supseteq \mathcal{U}^-(x)$. Then $\mathcal{B}_P := \{\{x\} \cup \operatorname{int}_{\mathcal{X}} F_x \,|\, x \in X \wedge F_x \in P(x)\} \cup \{\{x\} \,|\, x$ is isolated in $(X, \mathcal{X})\}$ is basis of a pseudomaximal topology $\mathcal{X}_P$ on X with the same isolated points as $(X, \mathcal{X})$.*

Proof: To show: $z \in B_1 \cap B_2 \Rightarrow \exists\, B_3 : z \in B_3 \subseteq B_1 \cap B_2$.

Let $z \in (\{x\} \cup \operatorname{int}_{\mathcal{X}} F_x) \cap (\{y\} \cup \operatorname{int}_{\mathcal{X}} F_y)$. There is no problem, if z is isolated in $(X, \mathcal{X})$.

(I) $x = y$: $z = x = y$ poses no problem. Now $z \in \operatorname{int}_{\mathcal{X}} F_x \cap \operatorname{int}_{\mathcal{X}} F_y$ and $\operatorname{int}_{\mathcal{X}} F_x \cap \operatorname{int}_{\mathcal{X}} F_y$ is in $\mathcal{X}$. Hence $\exists\, F_z : \{z\} \cup \operatorname{int}_{\mathcal{X}} F_z \subseteq \operatorname{int}_{\mathcal{X}} F_x \cap \operatorname{int}_{\mathcal{X}} F_y$.

(II) $x \neq y$: w.l.o.g. $z \neq y$. Then $z \in \{x\} \cap \operatorname{int}_{\mathcal{X}} F_y \cup \operatorname{int}_{\mathcal{X}} F_x \cap \operatorname{int}_{\mathcal{X}} F_y$. If $z \in \operatorname{int}_{\mathcal{X}} F_x \cap \operatorname{int}_{\mathcal{X}} F_y$, see (I), otherwise $z = x$ and $\operatorname{int}_{\mathcal{X}} F_y$ is a neighbourhood of x in $(X, \mathcal{X})$. This yields a $G_z \in P(x)$ such that $\{z\} \cup G_z \subseteq \operatorname{int}_{\mathcal{X}} F_x \cap \operatorname{int}_{\mathcal{X}} F_y$.

Assume z is isolated in $\mathcal{X}_P$: $\{z\} = (\{x\} \cup F_x) \cap (\{y\} \cup F_y)$, $x \neq y$. If $z = x$, then F_y is neighbourhood of x, hence $F_x \cap F_y \neq \emptyset$, but $\operatorname{card}(F_x \cap F_y) > 1$. Likewise $z = y$. Therefore $\{x\} \cap F_y = \{y\} \cap F_x = \emptyset$ and $\{z\} = F_x \cap F_y$ shows that $\{z\}$ is already open in $(X, \mathcal{X})$. $(X, \mathcal{X}_P)$ is pseudomaximal by Theorem 14. $\square$

Remark 23. If we are sure that $(X, \mathcal{X}_P)$ contains two non-isolated points with different neighbourhood filters, we may drop the set of isolated points in the definition of $\mathcal{B}_P$ at the beginning. Indeed, assume z is isolated in $(X, \mathcal{X})$. Take two non-isolated points $a, b \in X$ and $F_a \in P(a), F_b \in P(b)$ with $F_a \cap F_b = \emptyset$. Then $\{a, z\} \cup \operatorname{int}_{\mathcal{X}} F_a$, $\{b, z\} \cup \operatorname{int}_{\mathcal{X}} F_b$ are both open in $(X, \mathcal{X}_P)$ and their intersection is $\{z\}$. As the Alexandroff compactification of a discrete space shows the request for two distinct neighbourhood filters is necessary.

Theorem 24. *Let* $(X, \mathcal{X})$ *be submaximal. Then the following statements are equivalent:*

(a) X *is* σ*-discrete;*
(b) $(X, \mathcal{X}_P)$ (see Theorem 22) *is* σ*-discrete.*

Proof:
$\Downarrow$ $\mathcal{X} \subseteq \mathcal{X}_P$.
$\Uparrow$ Let $X = \bigcup A_n$, where A_n is discrete in $(X, \mathcal{X}_P)$ for all $n \in \mathbb{N}$. Let $I(\mathcal{X})$ denote the set of isolated points of $(X, \mathcal{X})$. Note $I(\mathcal{X}) = I(\mathcal{X}_P)$ by Theorem 22. $(X \setminus A_n) \cup I(\mathcal{X})$ is dense in $(X, \mathcal{X}_P)$ and therefore dense in $(X, \mathcal{X})$. Because $(X, \mathcal{X})$ is submaximal, $X \setminus (X \setminus A_n \cup I(\mathcal{X})) = A_n \cap (X \setminus I(\mathcal{X}))$ is closed discrete. $\bigcup A_n \cap (X \setminus I(\mathcal{X})) = X \setminus I(\mathcal{X})$. Now simply join $I(\mathcal{X})$. $\square$

Remark 25. (a) The quest for σ-discreteness could be reduced to pseudomaximal spaces. Even more, we may assume that all but one term is closed discrete. The exceptional set is the set of isolated points.

(b) In the next Theorem 26, $\beta|X|$ indicates the Čech-Stone-compactification of the discrete space $(X, \mathcal{P}(X))$.

Theorem 26. *Let* $(X, \mathcal{X})$ *be pseudomaximal and* T_2*.* $\exists\, \overline{\ell} : X \to \beta|X|$ *injective and continuous.*

Proof: Take $x \in X$. Define $\overline{\ell}(x) = \mathcal{U}^-(x) \in \beta|X|$ or $\ell(x) = (\{x\})$ if $\{x\}$ is open. Let $\overline{\ell}(x) \in U^*$, U^* basic open set. Then $U \in \overline{\ell}(x)$, $\{x\} \cup U \in \mathcal{U}(x)$, w.l.o.g. U open. Now let $z \in U$, then $U \setminus \{z\} \in \overline{\ell}(z)$, $\overline{\ell}(z) \in (U \setminus \{z\})^* \subseteq U^*$. There is no problem with discrete points. $\square$

Remark 27. (a) If $(X, \mathcal{X})$ is in addition T_3, $\overline{\ell}$ is even embedding. κX provides examples of non-T_3 pseudomaximal spaces.

(b) We learned that regular pseudomaximal spaces $(X, \mathcal{X})$ can be considered as subspaces P of $(\beta|X|)$. There is also a continuous $\underline{\ell} : X \cup P \to P \cong (X, \mathcal{X})$, where $X \cup P$ carries the subspace topology with respect to $\beta|X|$. This will be shown in the next Theorem 28 exploiting the Čech-Stone extension $\beta\underline{\ell}$ of $\underline{\ell}$ and Theorem 18. A closely related result is contained in [vDo93].

Theorem 28. *Let* $(X, \mathcal{X})$ *be* T_3 *and* T_0*. Then* (a) $\Rightarrow$ (b)*, where*

(a) $(X, \mathcal{X}) =: P$ *is pseudomaximal;*
(b) $\beta(X, \mathcal{X}) = \beta(P)$ *is retract of* $\beta|X|$ *under a continuous mapping* $\beta\underline{\ell} : \beta|X| \to \beta(P)$ *where* $\beta\underline{\ell} \circ \beta\overline{\ell} = 1_{\beta(P)}$ *and* $\beta\underline{\ell}/X : |X| \to (X, \mathcal{X})$ *is the identity on* X*.*

Proof:
$\Downarrow$ Define $\underline{\ell} : X \cup P \to (X, \mathcal{X})$ by $\underline{\ell}(x) := x$ if x is isolated and $\underline{\ell}(\mathcal{U}^-(x)) := x$ if x is not isolated. $\underline{\ell}$ is continuous: Take $U \in \mathcal{U}(x)$. There is $V \in \mathcal{U}(x)$ such that $clV \subseteq U$. $O := V \cup \{\mathcal{U}^-(y) \,|\, V \in \mathcal{U}^-(y)\}$ is an open set in $X \cup P$ containing

$\mathcal{U}^-(x)$. $V \in \mathcal{U}^-(y)$ implies $y \in clV$, hence $\underline{\ell}[O] \subseteq clV \subseteq U$. Observe that P is C^*-embedded in $\beta|X|$.

□

Theorem 29. *Let $(X, \mathcal{X})$ be a topological space. Then the following statements are equivalent:*

(a) *X is submaximal;*
(b) *every ultrafilter on X is open or closed;*
(c) *for all non-isolated $x \in X$, $\mathcal{U}^-(x)$ is the intersection of all its finer open ultrafilters.*

Proof:

$\Downarrow$ Let $\mathcal{U}$ be ultrafilter on X. Either all $F \in \mathcal{U}$ have non-empty interior (then $\mathcal{U}$ is open), or for one $G \in \mathcal{U} : \operatorname{int} G = \emptyset$. Then G is closed discrete and $\mathcal{U}$ is closed.

$\Downarrow$ Every filter is the intersection of its finer ultrafilters. If $\mathcal{U} \supseteq \mathcal{U}^-(x)$ would be closed, then $x \in \bigcap cl\mathcal{U} = \bigcap \mathcal{U} = \emptyset$, a contradiction.

$\Uparrow\Uparrow$ Let $D \subseteq X$ be dense. D contains all isolated points. Let $x \in X$ and $\mathcal{F} \supseteq \mathcal{U}^-(x)$ be open ultrafilter. Then $\forall F \in \mathcal{F} : F \cap D \neq \emptyset$ and $D \in \mathcal{F}$. Hence $D \in \bigcap\{\mathcal{F}\} = \mathcal{U}^-(x)$. $\exists U_x : U_x \setminus \{x\} \subseteq D$. If $x \in D$ then $U_x \subseteq D$ and D is open, because x was arbitrary.

□

Corollary 30. *Let $(X, \mathcal{X})$ be a topological space such that X is finite. Then the following statements are equivalent:*

(a) *X is submaximal;*
(b) *every point $x \in X$ is open or closed.* □

Definition 31. A map $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ between topological spaces is called almost open, if $A \subseteq X$, $\operatorname{int} A \neq \emptyset$ implies $\operatorname{int}(f[A]) \neq \emptyset$.

Lemma 32. *Let $f : (X, \mathcal{X}) \to (Y, \mathcal{Y})$ be a continuous map between topological spaces. Then the following statements are equivalent:*

(a) *f is almost open;*
(b) *the inverse image under f of a dense set is dense.* □

Corollary 33. *Let $(X, \mathcal{X})$ be T_0. Then the following statements are equivalent:*

(a) *$(X, \mathcal{X})$ is submaximal;*
(b) *$(X, \mathcal{X})$ is almost open quotient of a topological sum of pseudomaximal T_0 spaces (and therefore also quotient of a single pseudomaximal T_0-space).*

Proof:

$\Downarrow$ Let Ξ be the set of all maps ξ from X into the set of all ultrafilters on X such that $\xi(x)$ is an ultrafilter converging to x. As in the proof of Theorem 22 we can construct for each selection $\xi \in \Xi$ a pseudomaximal space X_ξ on X. Then

$(X, \mathcal{X})$ is quotient of the sum $\coprod_{\xi \in \Xi} X_\xi$, which corresponds to the intersection of the topologies on X_ξ. The quotient is almost open by Corollary 13. Observe now that a sum of pseudomaximal spaces is pseudomaximal.

$\Uparrow$ Let $q : (Y, \mathcal{Y}) \to (X, \mathcal{X})$ be almost open quotient. If $D \subseteq X$ is dense in X, then $q^{-1}[D]$ is dense in Y by Lemma 32. Hence $q^{-1}[D]$ is open, because Y is submaximal and D is open, because q is quotient. □

Remark 34. (a) Corollary 33 is not true for plain quotient maps, because the quotient of submaximal spaces is not necessarily submaximal, see [A&C95].

(b) I am turning now to the problem of constructing quasimaximal spaces, since so far only trivial or elusive examples (provided by AC) are known, see [C&G96]. The close connection to ultrafilters (see Theorem 14) suggest a construction similar to the tensor product in ultrafilter theory. An attempt at infinite products leads to zero-dimensional quasimaximal spaces on a countable tree. I believe that the most fascinating quasimaximal spaces are not zero-dimensional (i.e. not regular).

Definition 35. Let $(X, \mathcal{X})$, $(Y, \mathcal{Y})$ be two quasimaximal T_0 spaces. The tensor product $X \otimes Y$ of $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ is defined by the following neighbourhood system of points $(x, y) \in X \times Y$: $\mathcal{U}(x, y) = \{\{(x, y)\} \cup W \mid W \in \mathcal{U}^-(x) \otimes \mathcal{U}^-(y)\}$, where $W \in \mathcal{U}^-(x) \otimes \mathcal{U}^-(y) \Leftrightarrow \{a \mid \{b \mid (a, b) \in W\} \in \mathcal{U}^-(y)\} \in \mathcal{U}^-(x)$.

Theorem 36. *$X \otimes Y$ is a quasimaximal topological space.*

Proof: First we have to verify the 4th neighbourhood axiom. Let $p_X : X \times Y \to X$ be the projection and $W \in \mathcal{U}(x, y)$. There is $U_x \subseteq p_X[W]$, $U_x \in \mathcal{U}(x)$, such that $p_X[W] \in \mathcal{U}(r)$ for all $r \in U_x$. For all $r \in U_x$ there is $V_{r,y} \subseteq \{b \mid (r, b) \in W\}$, $V_{r,y} \in \mathcal{U}^-(y)$, such that $\{b \mid (r, b) \in W\} \in \mathcal{U}(z)$ for all $z \in V_{r,y}$. Define $\tilde{W} \subseteq W$ by $\tilde{W} = \bigcup\{\{r\} \times V_{r,y} \mid r \in U_x\}$. Then $\tilde{W} \in \mathcal{U}(x, y)$ and $W \in \mathcal{U}(s, t)$ for all $(s, t) \in \tilde{W}$. $X \otimes Y$ is quasimaximal by Corollary 15. □

Definition 37. Set $E := \{f \mid f : \omega_0 \to \omega_0\}$ and $T := \{f/n \mid n < \omega_0 \wedge f \in E\}$. I am going to define basic neighbourhoods of $g/m \in T$. Let $\mathcal{U}$ be a free ultrafilter on ω_0 and select an assignment $\xi : T \to \omega_0^*$. A basic neighbourhood of g/m is determined by $\psi \in \prod_{t \in T} \xi(t)$ and $U \in \mathcal{U}$: $U(g/m, \psi, U) := \{f/n \mid n \in U \wedge f/n \supseteq g/m \wedge \forall k \geq m : f(k) \in \psi(f/k)\} \cup \{g/m\}$.

Theorem 38. *T with the topology from Definition 37 is quasimaximal.*

Proof: Note $T = \bigcup_n \omega_0^n$. The trace of $U(g/m, \psi, \omega_0)$ on ω_0^n, $n > m$, for varying ψ is the tensor product of $n - m$ free ultrafilters with the fixed ultrafilter $(\{g/m\})$. Therefore $\mathcal{U}^-(g/m)$ is the ultrafiltered sum of ultrafilters and ultrafilter itself. By Corollary 15, T is quasimaximal. □

Remark 39. (a) By choosing ξ properly, we can make T rigid or homogeneous. (b) If $\mathcal{U}$ is a fixed ultrafilter $(\{k\})$ on ω_0, the construction still works, but all points in levels above k are isolated.

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(*Received* June 16, 1997)