On monotone nonlinear variational inequality problems

Ram U. Verma

Abstract. The solvability of a class of monotone nonlinear variational inequality problems in a reflexive Banach space setting is presented.

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1. Introduction

General theory of monotone variational inequalities has been applied to various problems in applied mathematics, physics, engineering sciences, and others. A closely associated notion of the complementarity involves several problems in mathematical programming, game theory, economics, and mechanics. For more details on general variational inequalities, we advise to consult [1], [4]–[14].

Let X be a reflexive real Banach space with dual X^* and $[w, x]$ denote a continuous duality pairing between the elements w in X^* and x in X. Let K be a nonempty closed convex subset of X. Here we present the solvability of a class of monotone nonlinear variational inequality (MNVI) problems: Determine an element x in K for a given w in X^* such that

$$[Sx - Tx - w, v - x] + f(v) - f(x) \geq 0 \quad \text{for all} \quad v \in K, \tag{1.1}$$

where $S, T : K \to X^*$ are nonlinear operators, and $f : X \to (-\infty, +\infty]$ is convex lower semicontinuous functional with $f \not\equiv \infty$. Here S and T are, respectively, p-monotone and p-Lipschitz continuous (or p-Lipschitzian).

Next, we recall some definitions needed for the work at hand.

Definition 1.1. An operator $S : K \to X^*$ is said to be *p-monotone* if, for all $u, v \in K$, there exist constants $r > 0$ and $p > 1$ such that

$$[Su - Sv, u - v] \geq r\|u - v\|^p. \tag{1.2}$$

The inequality (1.2) implies that S is strictly monotone and coercive for $p > 1$, S is strongly monotone for $p = 2$, and S is uniformly monotone for $p \geq 2$.

Definition 1.2. An operator $T : K \to X^*$ is called *p-Lipschitz continuous* (or p-Lipschitzian) if, for all $u, v \in K$, there exist constants $k > 0$ and $p > 1$ such that

$$[Tu - Tv, u - v] \leq k\|u - v\|^p. \tag{1.3}$$

Let us consider an example of p-Lipschitzian operators in the context of generalized pseudocontractions — a mild generalization of the pseudocontractions introduced by Browder and Petryshyn [2] — in a Hilbert space H. Generalized pseudocontractions are more general than Lipschitzian operators and unify certain classes of operators.

Definition 1.3. An operator $T : H \to H$ is said to be a *generalized pseudocontraction* if, for all $u, v \in H$, there exists a constant $k > 0$ such that

$$\|Tu - Tv\|^2 \leq k^2\|u - v\|^2 + \|Tu - Tv - k(u - v)\|^2. \tag{1.4}$$

This is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq k\|x - y\|^2, \tag{1.5}$$

where $T : H \to H$ is 2-Lipschitzian.

Example 1.4 ([JY])**.** Let K be a closed convex subset of a real Hilbert space H, and let $T : K \to K$ be hemicontinuous and 2-Lipschitzian with a constant $0 < k < 1$. Then T has a unique fixed point in K.

Definition 1.5. A multivalued mapping $F : X \to P(X)$ is called the KKM mapping if, for every finite subset $\{u_1, u_2, \ldots, u_n\}$ of X, *conv*$\{u_1, u_2, \ldots, u_n\}$ is contained in $\bigcup\limits_{i=1}^{n} F(u_i)$, where *conv*$\{A\}$ is the convex hull of set A and $P(X)$ denotes the power set of X.

Before we present our main results, we need to recall some auxiliary results [3].

Lemma 1.6 ([3, Theorem 4])**.** *Let Y be a convex set in a topological vector space X, and let K be a nonempty subset of Y. For all $x \in K$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \ldots, x_n\}$ of K is contained in the corresponding union $\bigcup\limits_{i=1}^{n} F(x_i)$. If there is a nonempty subset K_0 of K such that the intersection $\bigcap_{x\in K_0} F(x)$ is compact and K_0 is contained in a compact convex subset of Y, then $\bigcap\limits_{x\in K} F(x) \neq \emptyset$.*

Lemma 1.7 ([3, Corollary 1]). *Let K be a nonempty set in a topological vector space X. Let $F : K \to P(K)$ be a KKM mapping from K into the power set of K. If $F(u)$ is closed in X for all $u \in K$ and is compact for at least one $u \in K$, then $\bigcap_{u\in K} F(u) \neq \emptyset$.*

We note that in Lemma 1.6 the hypothesis "$\bigcap_{x\in K_0} F(x)$ is compact" does not rule out the possibility that it may be empty. However, the conclusion "$\bigcap_{x\in K} F(x) \neq \emptyset$" does imply that $\bigcap_{x\in K_0} F(x)$ is nonempty. The compactness condition in Lemma 1.7 is relaxed in Lemma 1.6.

2. The main results

Theorem 2.1. *Let K be a convex subset of a reflexive real Banach space X with dual X^* and $0 \in K$. Let $S : K \to X^*$ be hemicontinuous and p-monotone and let $T : K \to X^*$ be hemicontinuous and p-Lipschitz continuous. Let us further assume that $f : K \to (-\infty, \infty]$ is a convex functional with $f(0) = 0$, $f(u) > 0$ and $f \not\equiv \infty$. Then, for a given $w \in X^*$, an element u in K is a solution of the MNVI problem*

$$[Su - Tu - w, v - u] + f(v) - f(u) \geq 0 \quad \text{for all} \quad v \in K \tag{2.1}$$

iff u is a solution of a new MNVI problem

$$[Sv - Tv - w, v - u] + f(v) - f(u) \geq c\|v - u\|^p \quad \text{for all} \quad v \in K, \tag{2.2}$$

where $c = r - k > 0$ and $p > 1$. Here r is the p-monotonicity constant of S and k is the p-Lipschitz continuity constant of T.

When S and T are monotone and antimonotone, respectively, and $w = 0$, Theorem 2.1 reduces to [8, Lemma 1].

Corollary 2.2. *Let K be a nonempty convex subset of X and let $S : K \to X^*$ and $T : K \to X^*$ both be hemicontinuous, and be monotone and antimonotone, respectively. Let f be convex with $f \not\equiv \infty$. Then the following variational inequality problems are equivalent:*

$$u \in K : [Su - Tu, v - u] + f(v) - f(u) \geq 0 \quad \text{for all} \quad v \in K; \tag{2.3}$$

$$u \in K : [Sv - Tv, v - u] + f(v) - f(u) \geq 0 \quad \text{for all} \quad v \in K. \tag{2.4}$$

For $T = 0$ and f an indicator functional (that is, $f = 0$ on K and $f = \infty$ off K), Theorem 2.1 reduces to

Corollary 2.3. *Let K be a nonempty closed convex subset of a reflexive real Banach space X with dual X^* and let $S : K \to X^*$ be hemicontinuous and p-monotone. Then the MNVI problem*

$$u \in K : [Su - w, v - u] \geq 0 \ \text{ for all } \ v \in K, \tag{2.5}$$

has a unique solution iff the MNVI problem

$$u \in K : [Sv - w, v - u] \geq r\|v - u\|^p \ \text{ for all } \ v \in K, \tag{2.6}$$

has a unique solution for each $w \in X^*$.

Proof of Theorem 2.1: Suppose that (2.1) holds. Since S is p-monotone and T is p-Lipschitz continuous, this implies that

$$[(S-T)v - (S-T)u, v-u] \geq c\|v-u\|^p$$

or

$$\begin{aligned}[(S-T)v, v-u] &\geq c\|v-u\|^p + [(S-T)u, v-u] \\ &\geq c\|v-u\|^p + [w, v-u] + f(u) - f(v).\end{aligned}$$

This implies that

$$[(S-T)v - w, v-u] + f(v) - f(u) \geq c\|v-u\|^p.$$

Conversely, if (2.2) holds, then by choosing an element v with $f(v) < +\infty$, we find that $f(u)$ is finite. Let x be an element of K such that $v_t = (1-t)u + tx$ satisfies (2.2) for $0 < t < 1$. Then, it follows that $v_t - u = t(x-u)$ and, as a result, we find that

$$[(S-T)v_t - w, v_t - u] + f(v_t) - f(u) \geq c\|v_t - u\|^p$$

or

$$t[(S-T)v_t - w, x-u] + f((1-t)u + tx) - f(u) \geq c\|v_t - u\|^p.$$

Since f is convex, this implies that

$$\begin{aligned}t[(S-T)v_t - w, x-u] &\geq c\|v_t - u\|^p + f(u) - (1-t)f(u) - tf(x) \\ &= c\|t(x-u)\|^p + t(f(u) - f(x)).\end{aligned}$$

Thus, given that $t > 0$, we find

$$[(S-T)v_t - w, x-u] + f(x) - f(u) \geq ct^{p-1}\|x-u\|^p.$$

Since the hemicontinuity of S and T implies the hemicontinuity of $S-T$, we find that $(S-T)v_t$ converges weakly to $(S-T)u$ in X^* as $t \to 0$. Hence, we obtain

$$[(S-T)u - w, x-u] + f(x) - f(u) \geq 0 \ \text{ for all } \ x \in K,$$

that is, the variational inequality (2.1) holds. □

Theorem 2.4. *Let K be a nonempty closed convex subset of a reflexive real Banach space X with $0 \in K$. Let $S : K \to X^*$ be hemicontinuous and p-monotone with constant $r > 0$, $T : K \to X^*$ be hemicontinuous and p-Lipschitz continuous with constant $k > 0$, and $f : X \to (-\infty, +\infty]$ be convex lower semicontinuous with $f \not\equiv \infty$. Then the MNVI problem*

$$u \in K : [Su - Tu - w, v - u] + f(v) - f(u) \geq 0 \text{ for all } v \in K \tag{2.7}$$

has a unique solution for each $w \in X^*$.

For $w = 0$, S strictly monotone, T strictly antimonotone, and K bounded, Theorem 2.4 reduces to [8, Theorem 3].

Corollary 2.5. *Let K be a nonempty bounded closed convex subset of X, and $S, T : K \to X^*$ both be hemicontinuous and be strictly monotone and antimonotone, respectively. Let $f : X \to (-\infty, +\infty]$ be convex lower semicontinuous with $f \not\equiv \infty$. Then the variational inequality problem*

$$u \in K : [Su - Tu, v - u] + f(v) - f(u) \geq 0 \text{ for all } v \in K \tag{2.8}$$

has a unique solution.

When $T = 0$ and f is an indicator functional on K (that is, $f = 0$ on K and $f = \infty$ off K), Theorem 2.4 reduces to [5, Theorem 2].

Corollary 2.6. *Let X be a reflexive real Banach space with dual X^* and K be a nonempty closed convex subset X. Let $S : K \to X^*$ be hemicontinuous and p-monotone. Then the variational inequality problem*

$$u \in K : [Su - w, v - u] \geq 0 \text{ for all } v \in K \tag{2.9}$$

has a unique solution for each $w \in X^*$.

Proof of Theorem 2.4: We first prove the existence of the solution of the MNVI problem (2.7). Let us define the multivalued mappings F, $G : K \to P(K)$ by

$$F(v) = \{u \in K : [Su - Tu - w, v - u] + f(v) - f(u) \geq 0\} \text{ for all } v \in K$$

and

$$G(v) = \{u \in K : [Sv - Tv - w, v - u] + f(v) - f(u) \geq c\|v - u\|^p\} \text{ for all } v \in K,$$

respectively. We show by a contradiction approach that F is a KKM mapping. Assume $\{v_1, v_2, \ldots, v_n\}$ is in K, $\sum_{i=1}^{n} t_i = 1$, $t_i > 0$ and $v = \sum_{i=1}^{n} t_i v_i$ is not in $\bigcup_{i=1}^{n} F(v_i)$. Then for $u = v$,

$$[Su - Tu - w, v_i - u] < f(u) - f(v_i) \text{ for any } i = 1, \ldots, n.$$

Thus, we find

$$\begin{aligned}0 = [Su - Tu - w, v - u] &= [Su - Tu - w, \sum_{i=1}^{n} t_i v_i - u] \\ &= \sum_{i=1}^{n} t_i [Su - Tu - w, v_i - u] < \sum_{i=1}^{n} t_i (f(u) - f(v_i)) \\ &= f(u) - \sum_{i=1}^{n} t_i f(v_i) \leq f(u) - f(\sum_{i=1}^{n} t_i v_i) \\ &= f(u) - f(v) = 0,\end{aligned}$$

a contradiction. This implies that $conv\{v_1, v_2, \ldots, v_n\}$ is contained in $\bigcup_{i=1}^{n} F(v_i)$.

Next, to show $F(v) \subset G(v)$ for all $v \in K$, let u belong to $F(v)$. Then using the p-monotonicity of S and p-Lipschitz continuity of T, we obtain

$$[(S-T)v - (S-T)u, v - u] \geq c\|v - u\|^p.$$

Thus,

$$\begin{aligned}[(S-T)v, v-u] &\geq c\|v-u\|^p + [(S-T)u, v-u] \quad \text{or} \\ [(S-T)v - w, v-u] &\geq c\|v-u\|^p + [(S-T)u - w, v-u] \\ &\geq c\|v-u\|^p + f(u) - f(v) \quad \text{or} \\ [(S-T)v - w, v-u] + f(v) - f(u) &\geq c\|v-u\|^p \quad \text{for all} \quad v \in K.\end{aligned}$$

This implies that u belongs to $G(v)$ and, consequently, G is a KKM mapping on K. Hence, by Theorem 2.1, we find $\bigcap_{v \in K} F(v) = \bigcap_{v \in K} G(v)$.

Since f is lower semicontinuous and the duality pairing $[\cdot, \cdot]$ is continuous, it follows that $G(v)$ is closed for all $v \in K$. Clearly, K is a weakly compact set in X with weak topology and, as a result, $G(v)$ is weakly compact in K since $G(v)$ is contained in K for each $v \in K$. Now, by Lemma 1.7, we find

$$\bigcap_{v \in K} F(v) = \bigcap_{v \in K} G(v) \neq \emptyset.$$

Hence, there exists an element u_0 in K such that

$$[Su_0 - Tu_0 - w, v - u_0] + f(v) - f(u_0) \geq 0 \quad \text{for all} \quad v \in K.$$

To show the uniqueness of the solution, let x_1, x_2 be two solutions of the MNVI problem (2.7), that is,

$$[Sx_1 - Tx_1 - w, v - x_1] + f(v) - f(x_1) \geq 0 \quad \text{for all} \quad v \in K, \tag{2.10}$$

and

$$[Sx_2 - Tx_2 - w, v - x_2] + f(v) - f(x_2) \geq 0 \quad \text{for all} \quad v \in K. \tag{2.11}$$

Setting $v = x_2$ in (2.10) and $v = x_1$ in (2.11), and adding, we obtain

$$-[Sx_1 - Tx_1 - w, x_1 - x_2] + [Sx_2 - Tx_2 - w, x_1 - x_2] \geq 0,$$

or

$$-[Sx_1 - Sx_2, x_1 - x_2] + [Tx_1 - Tx_2, x_1 - x_2] \geq 0,$$

or

$$[Sx_1 - Sx_2, x_1 - x_2] \leq [Tx_1 - Tx_2, x_1 - x_2].$$

Since S is p-monotone with constant $r > 0$ and T is p-Lipschitz continuous with constant $k > 0$, this implies that

$$r\|x_1 - x_2\|^p \leq [Sx_1 - Sx_2, x_1 - x_2] \leq [Tx_1 - Tx_2, x_1 - x_2] \leq k\|x_1 - x_2\|^p.$$

It follows that

$$(r - k)\|x_1 - x_2\|^p \leq 0.$$

Since $r - k > 0$, we find that $x_1 = x_2$. This completes the proof. □

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International Publications, 12046 Coed Drive, Orlando, Florida 32826, USA

Istituto per la Ricerca di Base, Division of Mathematics, I-86075 Monteroduni (IS), Molise, Italy

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