Comment.Math.Univ.Carolin. 39,1 (1998)7–13

The Re-nonnegative definite solutions to the matrix equation $AXB = C$

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Abstract. An $n \times n$ complex matrix A is called Re-nonnegative definite (Re-nnd) if the real part of x^*Ax is nonnegative for every complex n-vector x. In this paper criteria for a partitioned matrix to be Re-nnd are given. A necessary and sufficient condition for the existence of and an expression for the Re-nnd solutions of the matrix equation $AXB = C$ are presented.

Keywords: Re-nonnegative define matrix, matrix equation, generalized singular value decomposition

Classification: 15A24, 15A57

In 1996, Lei Wu and Bryan Cain [1] defined a Re-nonnegative definite (Re-nnd) matrix (that is, $A \in \mathbb{C}^{n\times n}$ is called Re-nnd if $Re[x^*Ax] \geq 0$ for every nonzero x in $\mathbb{C}^{n\times 1}$), presented a criterion for Re-nndness, and solved the matrix inverse problem: Given complex matrices X and B, find the set of all complex Re-nnd matrices A such that $AX = B$. It is well known that the matrix equation

$$AXB = C, \tag{1}$$

where A, B, C are given and X is unknown, is very important; it was investigated by C.G. Khatri and S.K. Mitra [2], K.E. Chu [3], A.D. Porter and N. Mousouris [4], D. Hua [5], Q.W. Wang [6]–[8] and others. In this paper we extend the results of [1], give criteria for 2×2 and 3×3 partitioned matrices to be Re-nnd, derive a necessary and sufficient condition for the existence of and an expression for Re-nnd solutions of the equation (1). Throughout this paper, $\mathbb{C}$, $\mathbb{C}^{m\times n}$, $\mathbb{C}_r^{m\times n}$, GL_n, E^n will represent the complex field, the set of all $m\times n$ matrices over $\mathbb{C}$, the set of all matrices in $\mathbb{C}^{m\times n}$ with rank r, the set of all $n \times n$ invertible matrices and the set of all $n\times n$ Re-nnd matrices, respectively. A^*, rank A, $Re[b]$ and I_i will denote the conjugate transpose of a complex matrix A, the rank of A, the real part of a complex number b, and $i \times i$ identity matrix, respectively. $H(A) = \frac{1}{2}(A^* + A)$, $P^{-*} = (P^*)^{-1} = (P^{-1})^*$.

Project supported by NSF of Shandong & China.

2. Criteria for partitioned matrices to be Re-nnd

In this section, we improve a result concerning Re-nndness, and give a criterion for 3×3 matrix to be Re-nnd.

Lemma 1 ([1]). *$A \in E^n$ iff $H(A)$ is nonnegative definite (abbreviated nnd).*

Extending Lemma 2 in [1], we have the following

Lemma 2. *Let a Hermitian matrix A be partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are Hermitian submatrices. Then the following conditions are equivalent:

(i) *A is nnd;*

(ii) $\operatorname{rank}[A_{11}, A_{12}] = \operatorname{rank} A_{11}$, *both A_{11} and $A_{22} - U^* A_{11} U$ are nnd where U is an arbitrary but fixed solution of the matrix equation $A_{11}X = A_{12}$ for X;*

(iii) $\operatorname{rank}[A_{12}^*, A_{22}] = \operatorname{rank} A_{22}$, *both A_{22} and $A_{11} - U^* A_{22} U$ are nnd where U is an arbitrary but fixed solution of $A_{22}X = A_{12}^*$ for X.*

Theorem 1. *Suppose*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{C}^{n \times n}$$

where $A_{ii} \in \mathbb{C}^{n_i \times n_i}$ $(n_1 + n_2 = n)$. Then the following statements are equivalent:

(i) *$A \in E^n$;*

(ii) $\operatorname{rank}(A_{11} + A_{11}^*) = \operatorname{rank}[A_{11} + A_{11}^*, A_{12} + A_{21}^*]$, *both A_{11} and $A_{22} - U^* A_{11} U$ are Re-nnd, where U is an arbitrary but fixed solution of the matrix equation*

$$(A_{11} + A_{11}^*)X = A_{12} + A_{21}^*$$

for X;

(iii) $\operatorname{rank}(A_{22} + A_{22}^*) = \operatorname{rank}(A_{12}^* + A_{21}, A_{22} + A_{22}^*)$, *both A_{22} and $A_{11} - U^* A_{22} U$ are Re-nnd, where U is an arbitrary solution of the matrix equation*

$$(A_{22} + A_{22}^*)X = A_{12}^* + A_{21}$$

for X.

Proof: Note that

$$2H(A) = \begin{pmatrix} A_{11} + A_{11}^* & A_{12} + A_{21}^* \\ A_{21} + A_{12}^* & A_{22} + A_{22}^* \end{pmatrix},$$

$2H(A_{22}) = A_{22} + A_{22}^*$, $2H(A_{11} - U^* A_{22} U) = A_{11} + A_{11}^* - U^*(A_{22} + A_{22}^*)U$. By Lemma 1 and Lemma 2, (i) $\Leftrightarrow$ (iii).

Similarly, (i) $\Leftrightarrow$ (ii) may be proved. $\square$

Lemma 3. *Let*

$$A = \begin{array}{l} \begin{pmatrix} A_{11} & A_{21}^* & X_{31}^* \\ A_{21} & A_{22} & A_{32}^* \\ X_{31} & A_{32} & A_{33} \end{pmatrix} \begin{matrix} r_1 \\ r_2 \\ n-r_1-r_2 \end{matrix} \\ \quad \begin{matrix} r_1 & \quad r_2 & \quad n-r_1-r_2 \end{matrix} \end{array}$$

be Hermitian. Then there exists $X_{31} \in \mathbb{C}^{(n-r_1-r_2)\times r_1}$ such that A is nnd if and only if both

$$\begin{pmatrix} A_{11} & A_{12}^* \\ A_{21} & A_{22} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} A_{22} & A_{32}^* \\ A_{32} & A_{33} \end{pmatrix}$$

are nnd.

Proof: "Necessity" is obvious by Lemma 2. Now we prove the "Sufficiency". By Lemma 2, we may assume that U_1 (respectively U_2) is an arbitrary solution of $A_{22}X = A_{21}$ (respectively $A_{33}X = A_{32}$) for X. Taking $X_{31} = A_{32}U_1$ and

$$P = \begin{pmatrix} I_{r_1} & O & O \\ -U_1 & I_{r_2} & O \\ O & -U_2 & I_{n-r_1-r_2} \end{pmatrix},$$

we get that

$$P^*AP = \operatorname{diag}(A_{11} - U_1^* A_{22} U_1, A_{22} - U_2^* A_{33} U_2, A_{33}).$$

By Lemma 2, A is nnd. □

Theorem 2. *Suppose*

$$A = \begin{array}{l} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ X_{31} & A_{32} & A_{33} \end{pmatrix} \begin{matrix} r_1 \\ r_2 \\ n-r_1-r_2 \end{matrix} \\ \quad \begin{matrix} r_1 & \quad r_2 & \quad n-r_1-r_2 \end{matrix} \end{array} \in \mathbb{C}^{n\times n}.$$

Then there exists $X_{31} \in \mathbb{C}^{(n-r_1-r_2)\times r_1}$ such that $A \in E^n$ if and only if

$$A_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in E^{r_1+r_2}, \quad A_2 = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} \in E^{n-r_1}.$$

Proof: Assume $B_{11} = A_{11} + A_{11}^*$, $B_{21} = A_{21} + A_{12}^*$, $B_{31} = X_{31} + A_{13}^*$, $B_{22} = A_{22} + A_{22}^*$, $B_{32} = A_{32} + A_{23}^*$, $B_{33} = A_{33} + A_{33}^*$. Then

$$2H(A_1) = \begin{pmatrix} B_{11} & B_{21}^* \\ B_{21} & B_{22} \end{pmatrix}, \quad 2H(A_2) = \begin{pmatrix} B_{22} & B_{32}^* \\ B_{32} & B_{33} \end{pmatrix},$$

$$2H(A) = \begin{pmatrix} B_{11} & B_{21}^* & B_{31}^* \\ B_{21} & B_{22} & B_{32}^* \\ B_{31} & B_{32} & B_{33} \end{pmatrix}.$$

Hence, the theorem follows immediately from Lemma 3 and Lemma 1. □

3. Re-nnd solutions to the matrix equation (1)

Now we consider the Re-nnd solutions of (1) where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times q}$, $C \in \mathbb{C}^{m\times q}$ are given and $X \in \mathbb{C}^{n\times n}$ is unknown.

We decompose the matrices A and B^* using the generalized singular value decomposition (GSVD) [9]

$$UAP = \Big[\underbrace{\textstyle\sum_A}_{k}, O_{n-k}\Big], \quad VB^*P = \Big[\underbrace{\textstyle\sum}_{k}, O_{n-k}\Big], \tag{2}$$

where

$$\sum_A = \begin{pmatrix} I_r & & \\ & S_A & \\ & & O \end{pmatrix}, \quad \sum = \begin{pmatrix} O & & \\ & S & \\ & & I_{k-r-s} \end{pmatrix}, \tag{3}$$
$$S_A = \mathrm{diag}(\alpha_{r+1}, \dots, \alpha_{r+s}), \quad S = \mathrm{diag}(\beta_{r+1}, \dots, \beta_{r+s}),$$

$\alpha_i^2 + \beta_i^2 = 1$, $i = r+1, \dots, r+s$, $1 > \alpha_{r+1} \ge \dots \ge \alpha_{r+s} > 0$, $0 < \beta_{r+1} \le \dots \le \beta_{r+s} < 1$, $k = \mathrm{rank}\binom{A}{B^*}$, $r = k - \mathrm{rank}\, B$, $s = \mathrm{rank}\, A + \mathrm{rank}\, B - k$, $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$ are unitary and $P \in GL_n$.

Remark. Proofs, properties of the GSVD and a numerically stable algorithm for the computation of the GSVD can be found in [9]–[10].

Let

$$P^{-1}XP^{-*} = \begin{array}{cc} \begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} \\ X_{21} & X_{22} & X_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix} & \begin{matrix} r \\ s \\ k-r-s \\ n-k \end{matrix} \\ \begin{matrix} \quad r & \quad s & \quad k-r-s & \quad n-k \end{matrix} & \end{array}, \tag{4}$$

$$UCV^* = \begin{array}{cc} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} & \begin{matrix} r \\ s \\ m-r-s \end{matrix} \\ \begin{matrix} n-k+r & \quad s & \quad k-r-s \end{matrix} & \end{array}. \tag{5}$$

Lemma 4. *Consider the matrix equation* (1). *Let* $P^{-1}XP^{-*}$, UCV^* *be as in* (4) *and* (5), *respectively. Then* (1) *is consistent if and only if* C_{i1} $(i = 1, 2, 3)$ *and* C_{3j} $(j = 2, 3)$ *vanish, in which case the general solution is*

$$X = P \begin{pmatrix} X_{11} & C_{12}S^{-1} & C_{13} & X_{14} \\ X_{21} & S_A^{-1}C_{22}S^{-1} & S_A^{-1}C_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{pmatrix} P^*, \tag{6}$$

where X_{i1}, X_{i4} $(i = 1,2,3,4)$, X_{3j}, X_{4j} $(j = 2,3)$ are arbitrary complex matrices whose orders are given by (4).

PROOF: Obviously, the matrix equation (1) is equivalent to

$$UAPP^{-1}XP^{-*}P^*BV^* = UCV^*.$$

Hence by (2)–(5), (1) is equivalent to

$$\begin{pmatrix} O & X_{12}S & X_{13} \\ O & S_AX_{22}S & S_AX_{23} \\ O & O & O \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}. \tag{7}$$

Accordingly, the lemma follows from (7). □

Now we give the main result of the present paper.

Theorem 3. *Under the conditions of Lemma* 4, *the matrix equation* (1) *has a Re-nnd solution if and only if C_{i1} $(i = 1,2,3)$ and C_{3j} $(j = 2,3)$ vanish, and $S_A^{-1}C_{22}S^{-1}$ is Re-nnd. In that case, the general Re-nnd solution of* (1) *is*

$$X = P\begin{pmatrix} M & N \\ -N^* + T^*(M + M^*) & D + T^*MT \end{pmatrix} P^*, \tag{8}$$

where

$$M = \begin{pmatrix} D_2 + T_2^*(S_A^{-1}C_{22}S^{-1})T_2 & C_{12}S^{-1} & C_{13} \\ F & S_A^{-1}C_{22}S^{-1} & S_A^{-1}C_{23} \\ X_{31} & G & D_1 + T_1^*S_A^{-1}C_{22}S^{-1}T_1 \end{pmatrix},$$

with $F = -S^{-1}C_{12}^* + (S_A^{-1}C_{22}S^{-1} + S^{-1}C_{22}^*S_A^{-1})T_2$,
$G = -C_{23}^*S_A^{-1} + T_1^*(S_A^{-1}C_{22}S^{-1} + S^{-1}C_{22}^*S_A^{-1})$,
$X_{31} \in \{X_{31} \in \mathbb{C}^{(k-r-s)\times r} \mid M \in E^k\}$, $D_1 \in E^{k-r-s}$, $D_2 \in E^r$, $D \in E^{n-k}$, $T_1 \in \mathbb{C}^{s\times(k-r-s)}$, $T_2 \in \mathbb{C}^{s\times r}$, $T \in \mathbb{C}^{k\times(n-k)}$, $N \in \mathbb{C}^{k\times(n-k)}$ *are all arbitrary.*

PROOF: If the matrix equation (1) has a solution $X \in E^n$, then by Lemma 4 C_{i1} $(i = 1,2,3)$ and C_{3j} $(j = 2,3)$ vanish and X has the form of (6). Hence

$$\left(\begin{array}{ccc:c} X_{11} & C_{12}S^{-1} & C_{13} & X_{14} \\ X_{21} & S_A^{-1}C_{22}S^{-1} & S_A^{-1}C_{23} & X_{24} \\ X_{31} & X_{32} & X_{33} & X_{34} \\ \hdashline X_{41} & X_{42} & X_{43} & X_{44} \end{array}\right) \overset{\text{def.}}{=\!=} \left(\begin{array}{c:c} M & N \\ \hdashline N_1 & X_{44} \end{array}\right) \in E^n.$$

By Theorem 1, M and

$$X_{44} - T^*MT \overset{\text{def.}}{=\!=} D \tag{9}$$

are all Re-nnd where T is an arbitrary solution of the matrix equation

$$(M + M^*)X = N_1^* + N. \tag{10}$$

By Theorem 2,

$$\begin{pmatrix} X_{11} & C_{12}S^{-1} \\ X_{21} & S_A^{-1}C_{22}S^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} S_A^{-1}C_{22}S^{-1} & S_A^{-1}C_{23} \\ X_{32} & X_{33} \end{pmatrix}$$

are all Re-nnd. Hence by Theorem 1, on the one hand, both $S_A^{-1}C_{22}S^{-1}$ and

$$X_{11} - T_2^*S_A^{-1}C_{22}S^{-1}T_2 \overset{\text{def.}}{=\!=} D_2 \tag{11}$$

are all Re-nnd where T_2 is an arbitrary solution of the matrix equation

$$(S_A^{-1}C_{22}S^{-1} + S^{-1}C_{22}^*S_A^{-1})X = S^{-1}C_{12}^* + X_{21}. \tag{12}$$

On the other hand,

$$X_{33} - T_1^*S_A^{-1}C_{22}S^{-1}T_1 \overset{\text{def.}}{=\!=} D_1 \tag{13}$$

is also Re-nnd where T_1 is any solution of the matrix equation

$$(S_A^{-1}C_{22}S^{-1} + S^{-1}C_{22}^*S_A^{-1})X = S_A^{-1}C_{23} + X_{32}. \tag{14}$$

Consequently, by (10)–(14), X has the form of (8).

Conversely, assume C_{i1} $(i = 1, 2, 3)$ and C_{3j} $(j = 2, 3)$ vanish and $S_A^{-1}C_{22}S^{-1}$ is Re-nnd. Then by Theorem 1 and Theorem 2, there exists $X_{31} \in \mathbb{C}^{(k-r-s)\times r}$ such that

$$\begin{pmatrix} M & N \\ -N^* + T^*(M + M^*) & D + T^*MT \end{pmatrix}$$

is Re-nnd. Hence the matrix X of type (8) is Re-nnd. It is easy to verify that the matrix X of type (8) is a solution of the matrix equation (1). □

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(*Received* December 18, 1996, *revised* August 5, 1997)