Order-like structure of monotonically normal spaces

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Abstract. For a compact monotonically normal space X we prove: (1) X has a dense set of points with a well-ordered neighborhood base (and so X is co-absolute with a compact orderable space); (2) each point of X has a well-ordered neighborhood π-base (answering a question of Arhangel'skii); (3) X is hereditarily paracompact iff X has countable tightness. In the process we introduce weak-tightness, a notion key to the results above and yielding some cardinal function results on monotonically normal spaces.

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Introduction

A space X is called *orderable* provided there is a linear ordering of X whose induced order topology is the topology of X. All closed continuous images of metric or orderable spaces satisfy a strong version of normality (see [Gr], [HLZ], or [Bo]): A space X is called *monotonically normal* (or MN for brevity) provided that for each pair (x, G) consisting of a point and its neighborhood G, there is an open set $\mu(x, G)$ satisfying two conditions:

MN1. $x \in \mu(x, G) \subseteq G$;

MN2. if $\mu(x, G) \cap (y, H) \neq \emptyset$, then either $x \in H$ or $y \in G$.

We call μ the *monotone operator* for X. See [BR] and [WZ] for recent and related results on MN spaces.

Recently at the 8th Prague Topological Symposium, Mary Ellen Rudin [Ru3] announced that *each compact MN separable space is the continuous image of a compact orderable space*. This partially answered a question of J. Nikiel [Ni]: *Is each compact MN space the continuous image of a compact orderable space*? In addition, Rudin conjectured that Nikiel's problem has a negative solution in general. The underlying theme of this paper is to show how similar compact MN spaces are to compact orderable spaces. The results we exhibit here (and presented at the 7th Prague Topological Symposium in 1991) are all known or very easy to prove for continuous images of compact orderable spaces. Our main results are:

1.7 and 3.1. *Each point of a compact MN space has a well-ordered neighborhood π-base of cardinality both its π-character and the cofinality of its character*. (This answers Arhangel'skii question.)

2.5. *A compact MN space has a point with a linear ordered neighborhood base.*

2.8. *A compact MN almost-P space has a P-point.*

3.5. *A compact MN space is hereditarily paracompact iff it has countable tightness.*

4.1. *A compact MN space has a dense orderable subspace.*

4.8. *An MN space has a tree p-base iff each of its points has a pairwise-disjoint neighborhood π-base.*

Notation and preliminaries

All spaces are assumed Hausdorff and infinite. We use *nbhd* to abbreviate "neighborhood" and *bdry* for "boundary". The closure and interior operators are denoted by $\mathrm{cl}(\cdot)$ and $\mathrm{int}(\cdot)$. A *cellular family* in a space is a family of pairwise-disjoint non-empty sets. A *π-base* is a cofinal subset of $(\mathrm{TOP}(X)\setminus\{\emptyset\}, \supseteq)$. A *nbhd π-base* at a point x in a space X is a subset $\mathcal{B}$ of $\mathrm{TOP}(X) \setminus \{\emptyset\}$ such that each nbhd of x contains a member of $\mathcal{B}$.

For a set X, $|X|$ denotes the cardinality of X. In a space X, $x \in X$ is a *complete accumulation point* of a set $A \subseteq X$ if $|A \cap N| = |A|$ for each nbhd N of x. For a function $f : X \to Y$, $\mathrm{rng}(f)$ denotes the range of f.

We use the standard (see [Ho]) cardinal functions c (for cellularity = sup cardinality of cellular families), d (for density), L (for Lindelöf degree), χ (for character), t (for tightness), πw (for π-weight = least cardinality of a π-base), and $\pi\chi$ (for π-character). A prefix of "h" in front of a cardinal function is the "hereditary" version. An affix "(p, A)" behind a cardinal function is the local version with respect to A. So $\pi\chi(p, A)$ is the least cardinality of an A-nbhd π-base at $p \in A$.

1. Cardinal functions and applications

There are numbers of cardinal function results such as $d(X) \le c(X)^+$, known for the class of orderable spaces which hold for MN spaces (see 4.7). The following result, whose countable version appeared in [Os] and generalized by P. Moody, is a key to the entire paper.

1.1 Theorem. *For an MN space X, $hL(X) \le hc(X) = c(X)$.*

In any space $c(X) \le d(X)$. In any compact space, $\chi(X) \le hL(X)$ (see [Ho, 7.1]). Thus, 1.1 yields the next lemma.

1.2 Lemma. *For a compact MN space X, $\chi(X) \le hL(X) \le c(X) \le d(X)$.* □

We define a useful cardinal function — weak tightness. Suppose x is a point in a space X. Let $\mathrm{wt}(x, X) = \aleph_0 \cdot \min\{|A| : x \in \mathrm{cl}(A) \setminus A\}$ and let $\mathrm{wt}(X) = \sup\{\mathrm{wt}(x, X) : x \in X\}$. The following two results are either obvious or routine.

1.3 Lemma. *The following hold for a space X.*

(1) $\forall\, x \in X$, $\mathrm{wt}(x, X) \le \pi\chi(x, X)$ $(\le \chi(x, X))$.
(2) $\forall\, x \in X$, $t(x, X) = \sup\{\mathrm{wt}(x, A) : x \in \mathrm{cl}(A) \setminus A\}$.
(3) $\mathrm{wt}(X) \le t(X)$. □

1.4 Proposition. *The following hold for a subspace X of an orderable space.*

(1) *Suppose $\mathcal{N}$ is an infinite family of neighborhoods of $x \in X$ such that $|\mathcal{N}| < \mathrm{wt}(x, X)$. Then $\bigcap \mathcal{N}$ is a nbhd of x.*
(2) *$x \in X$ has a linear ordered nbhd base iff $\chi(x, X) = \mathrm{wt}(x, X)$.*
(3) *$x \in X$ has a linear ordered nbhd π-base of cardinality $\mathrm{wt}(x, X)$ and has a cellular nbhd π-base of cardinality $\mathrm{wt}(x, X)$.*
(4) *If K is a nowhere dense subspace of X, then $\chi(K) \le \chi(K, X)$.* □

1.5 Lemma. *Suppose X is an MN space and $\mathcal{N}$ is an infinite family of neighborhoods of $x \in X$ such that $|\mathcal{N}| + L(X) < \mathrm{wt}(x, X)$. Then $\bigcap \mathcal{N}$ is a nbhd of x.*

Proof: Without loss of generality, we may assume that the members of $\mathcal{N}$ are open and $\forall\, N \in \mathcal{N}$, $u(x, N) \in \mathcal{N}$. $\forall\, N \in \mathcal{N}$, $\exists\, F_N \subseteq X \setminus \mathrm{cl}(\mu(x, N))$ with $|F_N| \le L(X)$ such that $X \setminus N \subseteq \bigcup\{\mu(y, X \setminus \mathrm{cl}(\mu(x, N))) : y \in F_N\}$. Thus, $X \setminus \bigcap \mathcal{N} \subseteq \bigcup\{\mu(y, X) \setminus \mathrm{cl}(\mu(x, N)) : F_N, N \in \mathcal{N}\}$. Let $G = X \setminus \mathrm{cl}(\bigcup\{F_N : N e \mathcal{N}\})$. Since $|\mathcal{N}| + L(X) < \mathrm{wt}(x, X)$, $x \in G$. So $\forall\, N \in \mathcal{N}$ $\forall\, y \in F_N$, $\mu(x, G) \cap \mu(y, X \setminus \mathrm{cl}(\mu(x, N))) = \emptyset$. Therefore, $\mu(x, G) \subseteq \bigcap \mathcal{N}$. □

1.6 Corollary. *A point x in a Lindelöf MN space X has a linear ordered nbhd base iff $\chi(x, X) = \mathrm{wt}(x, X)$.* □

A space is said to be *radial* (see [Ar1] and [Ar2]) provided each boundary point of a subset is the limit of a well-ordered sequence in the set. A.V. Arhangel'skii asked in 1986 whether a compact MN space is radial. If each point of each closed subset of a space has a linear ordered relative neighborhood π-base, then the space is clearly radial, so 1.7 shows the answer to Arhangel'skii's question is yes.

1.7 Theorem. *Each point x in a compact MN space X has a linear ordered nbhd π-base of cardinality $\mathrm{wt}(x, X)$.*

Proof: Suppose $x \in \mathrm{cl}(A) \setminus A$ and $|A| = \mathrm{wt}(x, X) = \kappa$. From 1.2, $\chi(x, A) \le \kappa$. So $\chi(x, A) = \kappa$. According to 1.5, there is a family $\{B_\alpha : \alpha \in \kappa\}$ of nbhds of x such that $\{\mathrm{cl}(A) \cap B_\alpha : \alpha \in \kappa\}$ is a $\mathrm{cl}(A)$-nbhd base at x and such that the following holds:

$$(*) \qquad \beta < \alpha \text{ implies } \mathrm{cl}(B_\alpha) \subsetneq B_\beta \text{ and } A \cap B_\beta \setminus \mathrm{cl}(B_\alpha) \neq \emptyset.$$

$\forall\, \alpha \in \kappa$ let $C_\alpha = B_\alpha \setminus \mathrm{cl}(B_{\alpha+1})$ and choose $x_\alpha \in A \cap C_\alpha$. So $\mathcal{C} = \{\mu(x_\alpha, C_\alpha) : \alpha \in \kappa\}$ is a cellular family in X. Suppose G is an open nbhd of x and let $I = \{\alpha \in \kappa : \mu(x_\alpha, C_\alpha) \setminus G \neq \emptyset\}$. $\forall\, \alpha \in I$ choose $y_\alpha \in \mu(x_\alpha, C_\alpha) \setminus G$. Since

X is compact and $\mathcal{C}$ is cellular, $\{y_\alpha : \alpha \in I\}$ has a complete accumulation point $y \in X \setminus (G \cup (\bigcup \mathcal{C}))$.

Suppose H is an arbitrary nbhd of y missing x. Applying MN2 shows $y_\alpha \in \mu(y, H)$ implies $x_\alpha \in H$. So y is a complete accumulation point of $\{x_\alpha : \alpha \in I\}$. Since $x \neq y$, (*) implies $|I| < \kappa$. Therefore, $\{\bigcup_{\alpha > \beta} \mu(x_\alpha, C_\alpha) : \beta \in \kappa\}$ is a cellular nbhd π-base at x. □

In general for compact or for MN spaces, there is no relationship between $t(x, X)$ and $\pi\chi(x, X)$.

1.8 Corollary. *For each point x in a compact MN space X, $\mathrm{wt}(x, X) = \pi\chi(x, X) \leq t(x, X)$ and $\pi\chi(x, X)$ is a regular cardinal.*

Proof: From 1.7, there is a linear ordered nbhd π-base at x of cardinality the cofinality of $\mathrm{wt}(x, X)$. □

2. On neighborhood bases and π-bases

In this section, we strengthen some of the results in the last.

2.1 Lemma. *Suppose K is a closed subset of a countably compact hereditarily normal space X. If $\mathcal{R}$ is a family of open sets of X locally-finite in $\bigcup \mathcal{R}$ and such that $\forall R \in \mathcal{R}\ \ R \cap \mathrm{bdry}(K) \neq \emptyset$, then $|\mathcal{R}| \leq \chi(K, X)$.*

Proof: Let $\mathcal{N}$ be a nbhd base at K such that $|\mathcal{N}| = \chi(K, X) = \kappa$. Assume $|\mathcal{R}| = \lambda > \kappa$. $\forall R \in \mathcal{R}$ choose an open set C_R with $\mathrm{cl}(C_R) \subsetneq R$ and such that $\forall R \in \mathcal{R},\ \ C_R \cap \mathrm{bdry}(K) \neq \emptyset$. Let $\mathcal{C} = \{C_R : R \in \mathcal{R}\}$. For $N \in \mathcal{N}$ define $D_N = \bigcup\{\mathrm{cl}(C) \setminus N : C \in \mathcal{C}\}$ and $A_N = \mathrm{cl}(D_N) \setminus D_N$. If $x \in \bigcup \mathcal{R}$, then there is a nbhd G of x meeting just finitely many $C \in \mathcal{C}$. Since each $\mathrm{cl}(C) \setminus N$ is closed, $\bigcup \mathcal{R} \cap A_N = \emptyset$.

Let $A = \bigcup\{A_N : N \in \mathcal{N}\}$. Thus, $\bigcup \mathcal{R}$ is a nbhd of $T = \mathrm{bdry}(K) \cap \bigcup \mathcal{C}$ missing A. Since $\mathrm{int}(\mathrm{bdry}(K)) = \emptyset$, $X \setminus \mathrm{cl}(T)$ is a nbhd of A missing T. Since X is hereditarily normal, there is an open set V such that $T \subseteq V$ and $\mathrm{cl}(V) \cap A = \emptyset$. As $\mathcal{N}$ is a nbhd base at K, we may choose $\forall C \in \mathcal{C}$, $N_C \in \mathcal{N}$ such that $C \cap V \setminus \mathrm{cl}(N_C) \neq \emptyset$. Since $\kappa < \lambda$, $\exists N \in \mathcal{N}$ such that $N = N_C$ for infinitely many $C \in \mathcal{C}$. So V meets infinitely many sets $C \setminus \mathrm{cl}(N)$. Since X is countably compact, $\mathrm{cl}(V) \cap A \neq \emptyset$ — a contradiction. □

The following result strengthens 1.7.

2.2 Lemma. *Each point x in a compact MN space X has a cellular nbhd π-base of cardinality $\chi(x, X)$.*

Proof: Suppose X is a compact MN space and μ is a monotone operator on X. "Compactness" is unnecessary and trivial for points with a countable nbhd base. Thus, we assume the theorem is true for points y with $\chi(y, X) < \kappa$, and $x \in X$ has $\chi(x, X) = \kappa > \aleph_0$, say $\mathcal{B} = \{B_\alpha : \alpha \in \kappa\}$ is a nbhd base at x such that $\forall \alpha \in \kappa\ \ B_{\alpha+1} \subsetneq \mu(x, B_\alpha)$. Arbitrarily choose $x_0 \in B_0 \setminus \mathrm{cl}(B_1)$ and its nbhd G_0 with $\mathrm{cl}(G_0) \subseteq B_0 \setminus \mathrm{cl}(B_1)$. Suppose that $\lambda < \kappa$ and we have built a cellular family $\{\mu(x_\alpha, G_\alpha) : \alpha < \lambda\}$ satisfying $\forall\, \alpha < \lambda \;\; \mathrm{cl}(G_\alpha) \subseteq B_\alpha \setminus \{x\}$. Let $C = \mathrm{cl}(\{x_\alpha : \alpha < \lambda\}) \cup \{x\}$ and $\mathcal{D} = \{B_\alpha : \alpha \leq \lambda + 1\} \cup \{X \setminus \mathrm{cl}(G_\alpha) : \alpha < \lambda\}$.

Now suppose $\bigcap \mathcal{D} \subseteq C$. Since $\chi(C) \leq d(C) \leq |\lambda|$ and C is a compact MN space, there is a family $\mathcal{N} \subseteq \mathcal{B}$, $|\mathcal{N}| = |\lambda|$ such that $\{C \cap N : N \in \mathcal{N}\}$ is a C-nbhd base at x. Since X is compact, $\mathcal{N} \cup \mathcal{D}$ is a nbhd base at x; i.e. $\chi(x, X) < \kappa$ — a contradiction. Therefore, $(\bigcap \mathcal{D}) \setminus C \neq \emptyset$.

Choose $x_\lambda \in (\bigcap \mathcal{D}) \setminus C$ and its nbhd $G_\lambda \subseteq B_{\lambda+1} \setminus C$. Then $\forall\, \alpha < \lambda$, $x_\lambda \notin G_\alpha$ and $x_\alpha \notin G_\lambda$. So $\forall\, \alpha < \lambda$, $\mu(x_\alpha, G_\alpha) \cap \mu(x_\lambda, G_\lambda) = \emptyset$. In this fashion, we can build a cellular family $\{\mu(x_\alpha, G_\alpha) : \alpha \in \kappa\}$ as a nbhd π-base ar x. □

It is easy to see that for a closed nowhere dense subspace K of a compact MN space X, $\mathrm{wt}(K) \leq \chi(K, X)$. According to 1.3(1), the following lemma strengthens this result.

2.3 Theorem. *Suppose K is a closed nowhere dense subspace of a compact MN space X. Then $\chi(K) \leq \chi(K, X)$.*

PROOF: Consider $x \in K$. According to 2.2, there is a K-open cellular family $\mathcal{B}$, $|\mathcal{B}| = \chi(x, X)$, such that $\mathcal{B}$ is a nbhd π-base at x. $\forall\, B \in \mathcal{B}$ choose $x_B \in B$ and an X-open set G_B such that $G_B \cap K = B$. Then $\mu(x_B, G_B) \cap \mu(x_C, G_C) \neq \emptyset$ implies either $x_C \in G_B \cap K$ or $x_B \in G_C \cap K$, and hence $C = B$. Thus, $\{\mu(x_B, G_B) : B \in \mathcal{B}\}$ is a cellular X-open family. 2.1 shows $\chi(x, X) = |\{\mu(x_B, G_B) : B \in \mathcal{B}\}| \leq \chi(K, X)$. □

2.4 Lemma. *Suppose κ is a regular cardinal and $\mathcal{N} = \{N_\alpha : \alpha \in \kappa\}$ is a well-ordered family of open sets in a compact MN space X subject to the two conditions below:*

(1) $\mathrm{int}(\bigcap \mathcal{N}) = \emptyset$;
(2) $\beta < \alpha \in \kappa$ *implies* $\mathrm{cl}(N_\alpha) \subsetneq N_\beta$.

Then each point of $\bigcap \mathcal{N}$ possesses a well-ordered nbhd base of cardinality κ.

PROOF: Let $K = \bigcap \mathcal{N}$. Then (1) implies K is nowhere dense. Since X is compact and (2) holds, K is non-empty. Suppose $x \in K$. According to 2.3, $\mathrm{wt}(x, X) \leq \chi(K) \leq \kappa$. From 1.8 there is a nbhd π-base $\mathcal{B}$ at x of cardinality $\mathrm{wt}(x, X)$. If $|\mathcal{B}| < \kappa$, then (2) finds $B \in \mathcal{B}$ with $B \subseteq \bigcap \mathcal{N}$ — contradicting (1). Hence, $\mathrm{wt}(x, X) = \kappa$. From 1.6, each point of K has a well-ordered nbhd base of cardinality κ. □

2.5 Theorem. *A compact MN space contains a dense set of points with a well-ordered nbhd base.*

PROOF: Since the space is regular, in any open set one can recursively construct a family $\mathcal{N}$ satisfying the hypothesis of 2.4. □

2.6 Corollary. *If each point of a compact MN space X has countable π-character, then X has a dense first countable subspace.*

PROOF: Use the dense set guaranteed by 2.5. □

2.7 Corollary. *Suppose κ is a regular uncountable cardinal and $\mathcal{G} = \{G_\alpha : \alpha \in \kappa\}$ is a well-ordered family of open sets in a compact MN space X such that $\forall\, \alpha \in \kappa\ \mathrm{cl}(G_{\alpha+1}) \subsetneq G_\alpha$. Then $\mathrm{bdry}(\bigcap \mathcal{G})$ is finite.*

Proof: Let $K = \mathrm{bdry}(\bigcap \mathcal{G})$ and $Y = X \setminus \mathrm{int}(\bigcap \mathcal{G})$. Then for the compact MN space Y, $\{G_\alpha \setminus \mathrm{int}(\bigcap \mathcal{G}) : \alpha \in \kappa\}$ satisfies the conditions of 2.4. Thus, each point in K has a well-ordered Y-nbhd base of order type κ. Therefore, no countably infinite subset of K has a limit point — hence K is finite. □

E.K. van Douwen has called a space X an *almost-P* space provided each non-empty G_δ-set has non-empty interior. A point in a space is called a <u>P-point</u> if it is in the interior of every G_δ-set containing it. R. Levy [Le] has shown that each compact orderable almost-P space has a P-point. S. Watson [Wa] has found a compact almost-P space with no P-points. The following result improves Levy's theorem.

2.8 Theorem. *A compact MN almost-P space has a P-point.*

Proof: Notice that any non-P-point in a space X has $\mathrm{wt}(x, X) = \omega$. From 2.6, if X is also a compact MN space, then X has a dense first countable space. Each element of the dense set shows X is not an almost-P space. □

3. More on neighborhood π-bases and tightness

Our first result in this section is yet another strengthening of 1.7.

3.1 Theorem. *Each point x of a compact MN space X has a well-ordered nbhd π-base of cardinality $\mathrm{cf}(\chi(x, X))$.*

Proof: This is true if x has a countable nbhd base. Suppose $\kappa = \chi(x, X)$ is uncountable and $\{B_\alpha : \alpha \in \kappa\}$ is a nbhd base at x. We will assume each $\mathrm{cl}(B_{\alpha+1}) \subsetneq B_\alpha$. For each ordinal $\lambda \leq \kappa$, let $L_\lambda = \bigcap_{\alpha<\lambda} B_\alpha$ and η be the first ordinal such that $x \notin \mathrm{cl}(\mathrm{int}(L_\eta))$. Clearly η is a limit ordinal and L_η is closed. Let $Y = X \setminus \mathrm{int}(L_\eta)$ and $K = L_\eta \setminus \mathrm{int}(L_\eta)$. Since $x \notin \mathrm{cl}(\mathrm{int}(L_\eta))$, $\chi(x, Y) = \kappa$. On the other hand, $\chi(K, Y) \leq |\eta|$. Since K is closed nowhere dense in Y, 2.3 shows $\eta = \kappa$. Thus, $\{\mathrm{int}(L_\alpha) : \alpha \in \kappa\}$ is a nbhd π-base at x. □

Recall that a set S is <u>stationary</u> in an ordinal κ with $\mathrm{cf}(\kappa) > \omega$, provided it intersects each closed unbounded subset of κ. The following generalizes the well known result that $\beta\kappa = \kappa + 1$ for such an ordinal.

3.2 Lemma. *Suppose S is a stationary set of a regular ordinal κ densely embedded into a compact space X. Then $X \setminus S$ has at most one complete accumulation point x of S and $t(x, X) \geq \kappa$.*

Proof: Suppose $x \in X \setminus S$ is a complete accumulation point of S. If $y \in X \setminus S$ is another, choose nbhds G and H of x and y with disjoint closures. Let W be the set of all $\alpha \in \kappa$ such that both $G \cap \alpha$ and $H \cap \alpha$ are cofinal in α. Since x and y are complete accumulation points, W is clearly closed and unbounded in κ.

Hence, $W \cap S \neq \emptyset$. Each nbhd of $s \in W \cap S$, intersects both $\mathrm{cl}(G)$ and $\mathrm{cl}(H)$. Therefore, $s \in \mathrm{cl}(G) \cap \mathrm{cl}(H)$ — a contradiction.

Suppose $t(x, X) < \kappa$. Since x is a complete accumulation point of S, we may choose $\forall\, \alpha \in S\ \exists\, \alpha^* \in S$ with $\alpha^* > \alpha$ and $x \in \mathrm{cl}(S \cap [\alpha+1, \alpha^*])$. Let W be the set of all $\lambda \in \kappa$ such that $\alpha < \lambda$ implies $\alpha^* < \lambda$. Since x is a complete accumulation point of S, W is closed and unbounded in κ. So $W \cap S \neq \emptyset$. As each nbhd of $s \in W \cap S$ includes $S \cap [\alpha+1, \alpha^*]$, $x = s$ — a contradiction. □

3.4 Theorem ([BR]). *An MN space is paracompact iff it does not embed, as a closed subspace, some stationary set in an ordinal.*

3.5 Theorem. *A compact MN space X is hereditarily paracompact iff $t(X) = \omega$.*

Proof: Suppose Y is a non-paracompact subspace of X. From 3.4, a stationary set embeds into $\mathrm{cl}(Y)$. From 3.3, $t(\mathrm{cl}(Y) > \omega)$.

Conversely, suppose $t(X) > \omega$. Then there is $A \subseteq X$ such that $x \in \mathrm{cl}(A) \setminus A$ and $\mathrm{wt}(x, \mathrm{cl}(A)) = \kappa > \omega$. We can assume $\mathrm{cl}(A) = X$. From 1.5, there is a family $\{N_\alpha : \alpha \in \kappa\}$ nbhds of x satisfying $\beta < \alpha \in \kappa$ implies $\mathrm{cl}(N_\alpha) \subsetneq N_\alpha$. So $\bigcup_{\alpha \in \omega_1} X \setminus N_\alpha$ is a countably compact non-closed subspace of X. Therefore, X is not hereditarily paracompact. □

4. Trees

In this section we prove the following result.

4.1 Theorem. *A compact MN space has a dense orderable subspace.*

According to [Wi], 4.1 implies: For each compact MN space X, there is a compact space and a compact linear ordered space L such that both X and L are perfect irreducible images of E.

4.2 Theorem ([Wi]). *If a space X has a dense set of points with a linear ordered nbhd base, then X has a dense orderable space iff it has a tree π-base.*

A *tree* is a partially ordered set by in which each element has a well-ordered set of predecessors. All of our trees will be contained in the partially ordered set $(\mathrm{TOP}(X) \setminus \{\emptyset\}, \supseteq)$ for some space X, and their elements S and T are unrelated iff they are disjoint. We call such a tree, a *tree in X*.

Suppose $\mathcal{T}$ is a tree in X. For each ordinal α, let $\mathcal{T}_\alpha$ denote those members $T \in \mathcal{T}$ whose set $\uparrow T = \{S \in \mathcal{T} : T \subsetneq S\}$ of predecessors has order type α, and let $\mathcal{T} \,|\, \alpha = \bigcup_{\beta<\alpha} \mathcal{T}_\alpha$. The *height* $h(\mathcal{T})$ of $\mathcal{T}$ is the least ordinal α with $\mathcal{T} \,|\, \alpha = \mathcal{T} \,|\, \alpha + 1$. A *branch* in $\mathcal{T}$ is a maximal chain. $\mathrm{Br}(\mathcal{T})$ is the set of all branches of *ct*. Let $\mathrm{bot}(\mathcal{T}) = \{x \in X : \exists\, b \in \mathrm{Br}(\mathcal{T}), x \in \bigcap b\}$.

We remind the reader of well-known linear order induced by trees. First, given $T \in \mathcal{T}$, let $\mathcal{M} = \{S \in \mathcal{T} : \uparrow S = \uparrow T\}$. Give $\mathcal{M}$ a linear order $\lhd$ (with no end-points if $\mathcal{M}$ is infinite). The *node order* is defined on $\mathcal{T}$ by $S < T$ iff $T \subsetneq S$ or $\uparrow S = \uparrow T$ and $S \lhd T$.

In a space X, one can recursively build an *unbounded tree* ([Wi]); i.e. a tree $\mathcal{T}$ in $(\mathrm{TOP}(X) \setminus \{\emptyset\}, \subsetneq)$ satisfying:

UT1. X is the largest member of $\mathcal{T}$;

UT2. if $T \in \mathcal{T}$ has a proper open subset, then the immediate successors to T is a cellular (infinite if T is infinite) open family $\mathcal{D}$ such that $T \subseteq \mathrm{cl}(\bigcup \mathcal{D})$ and $\forall\, S \in \mathcal{D},\ \mathrm{cl}(S) \subseteq T$;

UT3. if $\mathcal{C}$ is a chain in $\mathcal{T}$ with $\mathrm{int}(\bigcap \mathcal{C}) \neq \emptyset$, then $\mathrm{int}(\bigcap \mathcal{C}) \in \mathcal{T}$.

4.3 Lemma ([Wi]). *Suppose X is an MN space. Each tree π-base for X contains a tree π-base satisfying UT1, UT2, and UT3.*

4.4 Lemma. *Suppose X is an MN space with monotone operator μ. Then X has a tree π-base satisfying UT1, UT2, UT3 and*

UT4. *In UT2, $\forall\, S \in \mathcal{D}$, there is open $G \subseteq T$ and $x \in G$ such that $\mu(x, G) = S$.*

PROOF: Obvious from 4.3. □

Suppose $\mathcal{T}$ is an unbounded tree satisfying UT1–4 in a space X with a monotone operator μ, define

$$\mathcal{T}^* = \mathcal{T} \setminus \bigcup \{\mathcal{T}_\lambda : \lambda \in h(\mathcal{T}) \ \text{ is a limit ordinal}\}.$$

Applying UT4, we have

4.5 Lemma. *Suppose $\mathcal{T}$ is an unbounded tree satisfying UT1–4 in an MN space X with monotone operator μ. Then there is a pair $(f, g) = (f_{\mathcal{T}}, g_{\mathcal{T}})$ of injections $f : \mathcal{T}^* \to X$ and $g : \mathcal{T}^* \to \mathrm{top}(X)$ such that $\forall\, T \in \mathcal{T}^*,\ g(T) \subseteq \bigcap \uparrow T$ and $\mu(f(T), g(T)) = T$.* □

4.6 Lemma. *Suppose $\mathcal{T}$ is an unbounded tree in an MN space X. Then for f as defined in 4.5, $\mathrm{rng}(f)$ is dense in X and has a coarser order topology.*

PROOF: Suppose that $x \in G = X \setminus \mathrm{cl}(\mathrm{rng}(f))$. Since any branch of $\mathcal{T}$ has either an open singleton or nowhere dense intersection, UT1 finds a first $\alpha \in h(\mathcal{T})$ such that $\mu(x, G) \not\subset T \in \mathcal{T}_\alpha$. From UT2 and UT3, $\mu(x, G)$ meets two members $\mu(f(R), g(R)), \mu(f(S), g(S)) \in \mathcal{T}_{\alpha+1}$, where $g(R)$ and $g(S)$ are, by UT2, disjoint. From UT4 and MN2, $x \in g(R) \cap g(S)$ — a contradiction. Therefore, $\mathrm{rng}(f)$ is dense. The order on $\mathrm{rng}(f)$ induced by the node order on $\mathcal{T}$ induces a topology coarser than the subspace topology. □

4.7 Corollary. *Suppose X is an MN space. Then $d(X) \leq c(X)^+$.*

PROOF: From UT2 and UT3, the tree $\mathcal{S}$ has at most $c(X)^+$ elements. So the result follows from 4.6. □

4.8 Theorem. *An MN space X has a tree π-base iff each point of X has a cellular nbhd π-base.*

Proof: We only prove the non-trivial implication. For $\forall\, x \in X$, let $\mathcal{B}(x)$ be a cellular nbhd π-base at x. We can easily build an unbounded tree $\mathcal{T}$ satisfying the following extension of UT4.

UT5. If $T = \mu(f(T), g(T)) \in \mathcal{T}_\alpha$, then $\forall\, B \in \mathcal{B}(f(T))\ \exists\, S \in \mathcal{T}_{\alpha+1}$ such that $S \subseteq B \cap T$.

From UT5, $\{\mathrm{rng}(f) \cap T : T \in \mathcal{T}\}$ is a π-base for $\mathrm{rng}(f)$. From 4.5, $\mathrm{rng}(f)$ is dense in X. So $\mathcal{T}$ is a π-base for X. □

4.9 Corollary. *If each point of an MN space X has a linear ordered nbhd base, then X has a dense orderable subspace.*

Proof: Any point with a linear ordered nbhd π-base has a cellular nbhd π-base. So the theorem follows from 4.2. □

Proof of Theorem 4.1: Use 2.5 and 4.9. □

5. Remarks

5.1. The question of radiality was first solved, six months prior to our work, by G. Gruenhage (unpublished) for a class strictly including all compact MN spaces; namely compact hereditarily collectionwise normal, hereditarily weakly collectionwise Hausdorff spaces where $\chi(F) \leq d(F)$ for each closed subset F. On the other hand, the proofs of 1.5 and 1.7, together, are different from and considerably shorter than Gruenhage's proof.

5.2. Related to radial is the notion of *biradial* (each convergent ultrafilter contains a convergent chain), a property satisfied orderable spaces and known to be preserved by perfect maps. At Mary Ellen Rudin retirement conference, Boris Shapirovskii asked the following question for which for which we have no answer: *Is each compact MN space biradial*?

5.3. From 3.1, one might conjecture that for a point x in a compact MN space X, $\chi(x, X)$ is regular (a fact true for any orderable space). For a compact extremely normal (see [WZ]) counter-example, consider the one-point compactification of the discrete space of cardinality $\aleph_0$.

5.4. R.W. Heath has shown that the space of all eventually 0 sequences of rationals with the box topology is a countable stratifiable (and hence MN) group G with each $\pi\chi(x, G)$ uncountable (also see [Bo2]). So no point of G has a linear ordered nbhd π-base. Clearly G has no MN compactification. Since any cellular family in G must be countable, G does not even have a tree π-base. Note M. Rudin [Ru2] has constructed a locally compact MN space with no MN compactification.

5.5. The proof of 4.1 also shows that a compact extremely normal space has countable tightness and hence (by 3.5) it is hereditarily paracompact. However, in [WZ] we proved that all extremely normal spaces are hereditarily paracompact.

5.6. The following simple result may be of independent interest to the reader: suppose $f : X \to Y$ is continuous and irreducible. If $x \in X$ has a linear ordered (by $\subseteq$) nbhd π-base, then $f(x)$ has a linear ordered (by $\subseteq$) nbhd π-base.

5.7. The following questions related to our work here are also of interest.

1. [**UH** problem 210] Is every compact MN space supercompact? J. Nikiel has announced a "yes" for continuous images of orderable spaces.

2. [**UH** problem 211] Is every compact MN space the image of a zero-dimensional compact MN space?

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References

[Ar1] Arhangel'skii A.V., *Structure and classification of topological spaces and cardinal invariants*, Russian Math. Surv. **33** (1977), 33–95.

[Ar2] Arhangel'skii A.V., *Some properties of radial spaces*, Math. Zametki **27** (1980), 95–104.

[BR] Balogh Z., Rudin M.E., *Monotone Normality*, Topology and Appl. **47** (1992), 115–127.

[Bo1] Borges C.J.R., *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1–16.

[Bo2] Borges C.J.R., *Direct sums of stratifiable spaces*, Fundamenta Math. **100** (1978), 95–99.

[En] Engelking R., *General Topology*, Polish Scientific Publishers, 1977.

[Gr] Gruenhage G., *Generalized metric spaces*, Handbook of Set-theoretic Topology, North Holland, 1984, pp. 423–502.

[HLZ] Heath R.W., Lutzer D.J., Zenor P.L., *Monotonically normal spaces*, Trans. Amer. Math. Soc. **155** (1973), 481–494.

[Ho] Hodel R., *Cardinal Functions I*, Handbook of set-theoretic Topology, North Holland, 1984, pp. 1–61.

[Ju] Juhasz I., *Cardinal Functions in Topology*, Math. Centre Tracts 34, 1971.

[Ni] Nikiel J., *Some problems on continuous images of compact ordered spaces*, Quest. & And. in Gen. Top. **4** (1986), 117–128.

[NP] Nyikos P.J., Purisch S., *Monotone normality and paracompactness in scattered spaces*, Annals on the New York Acad. Sci. 552, 1989, pp. 124–137.

[Os] Ostacewski A., *Monotone normality and G_δ-diagonals in the class of inductively generated spaces*, Coll. Math. Soc. Janos Bolyai, Topology **23** (1978), 905–930.

[Ru1] Rudin M.E., *Dowker spaces*, Handbook of Set-theoretic Topology, North Holland, 1984, pp. 761–780.

[Ru2] Rudin M.E., *Monotone normality and compactness*, to appear in Topology and Appl. volume of Ehime Conference Proceedings, 1995.

[Ru3] Rudin M.E., *Compact monotonically normal spaces*, invited lecture, 8th Prague Topological Symposium, August, 1996.

[St] Stephenson R.M., Jr., *Initially k-compact and related spaces*, Handbook of Set-theoretic Topology, North Holland, 1984, pp. 603–602.

[UH] The University of Houston Problem Book, 6/30/90.

[Wi] Williams S.W., *Trees, Gleason spaces, and co-absolutes of βN-N*, Trans. Amer. Math. Soc. **271** (1982), 83–100.

[WZ] Williams S.W., Zhou H., *Strong versions of normality*, Proceedings of the New York Conference on Topology and its Applications.

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