Some types of implicative ideals

Ladislav Beran

Abstract. This paper studies basic properties for five special types of implicative ideals (modular, pentagonal, even, rectangular and medial). The results are used to prove characterizations of modularity and distributivity.

Keywords: implicative ideals, modular ideals, pentagonal ideals, even ideals, rectangular ideals, medial ideals, modularity, distributivity

Classification: 06B10, 06C99, 06D99

1. Introduction

In this paper, we study lattice ideals that are implicative analogs of semiprime ideals. In particular, we focus upon a complete description of all possible inclusion relations between the corresponding classes. We also exhibit examples showing that the considered concepts define different classes. Since semiprime ideals occur as a natural tool for a description of Boolean algebras, it is not surprising that also these five classes of ideals play an interesting role in the theory of Boolean algebras (see [3] and the techniques of [5]). As a by-product we obtain new characterizations of modularity and distributivity.

First we note that the Rav's definition of a semiprime ideal ([4]) can be given in a slightly modified form: An ideal I of a lattice L is *semiprime* if

$$(1.1) \qquad \forall\, a,b,c,d \in L \;\; (a \wedge b) \vee (a \wedge c) \in I \;\; \Rightarrow \;\; a \wedge (b \vee c) \in I.$$

The class of all semiprime ideals will be denoted by **Sp**.

The new approach to the semiprimeness suggests a definition of a more general class of ideals in the following way: An ideal I of L is called *implicative*, if there exist two lattice polynomials $p(x_1, x_2, \cdots, x_n)$ and $q(x_1, x_2, \cdots, x_n)$ such that

$$\forall\, a_1, a_2, \cdots, a_n \in L \;\; p(a_1, a_2, \cdots, a_n) \in I \;\; \Rightarrow \;\; q(a_1, a_2, \cdots, a_n) \in I.$$

As usual, for any a, b, c of a lattice L, the *upper median* $\overline{\mathrm{med}}(a,b,c)$ is the element $(a\vee b)\wedge(a\vee c)\wedge(b\vee c)$; the *lower median* is defined dually by $\underline{\mathrm{med}}(a,b,c) :=$ $= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$.

2. Five classes of implicative ideals

To introduce the first class, we will need the following lemma.

Lemma 2.1. *The following are equivalent for an ideal I of a lattice L.*

$$\forall a,b,c \in L \ \ [a \vee (b \wedge c) \in I \ \ \& \ \ a \leq c] \ \Rightarrow \ (a \vee b) \wedge c \in I; \tag{2.1}$$

$$\forall a,b,c \in L \ \ (a \wedge c) \vee [b \wedge (a \vee c)] \in I \ \ \Rightarrow \ [(a \wedge c) \vee b] \wedge (a \vee c) \in I; \tag{2.1$'$}$$

$$\forall a,b,c \in L \ \ (a \wedge b) \vee (a \wedge c) \in I \ \ \Rightarrow \ a \wedge [b \vee (a \wedge c)] \in I. \tag{2.1$''$}$$

Proof: Immediate. □

An ideal I of L is said to be *modular* if it satisfies one of the conditions (2.1)–(2.1$''$). The class of modular ideals will be denoted by **M**.

An ideal I of L is called *pentagonal*, if

$$\forall a,b,c \in L \ \ a \vee (b \wedge c) \in I \ \ \Rightarrow \ (a \vee b) \wedge (a \vee c) \in I. \tag{2.2}$$

The class of all such ideals will be denoted by **Pe**.

An ideal I of L is said to be *even*, if

$$\forall a,b,c \in L \ \ [a \wedge (b \vee c)] \vee [b \wedge (a \vee c)] \in I \ \ \Rightarrow \ \overline{\mathrm{med}}(a,b,c) \in I. \tag{2.3}$$

We will use the letter **E** to denote the class of even ideals.

Remark 2.2. The ideal $(s]$ of the lattice L_9 pictured in Figure 1 (see [6, p. 192]) is modular. It is not even: If $a = r$, $b = u$ and $c = t$, then $[a \wedge (b \vee c)] \vee [b \wedge (a \vee c)] = 0$ and $\overline{\mathrm{med}}(a,b,c) = t$. Note that the same argument applies to the ideal $(0]$ of L_9. The ideal $(s]$ is not pentagonal, since $s \vee (u \wedge w) = s$ and $(s \vee u) \wedge (s \vee w) = x$.

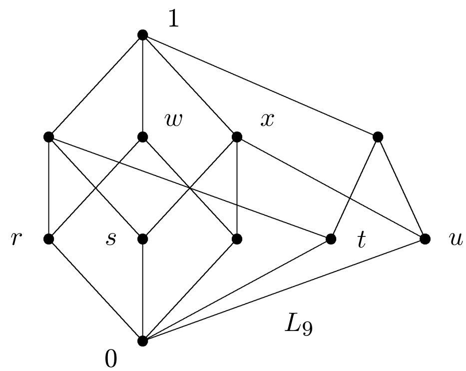

Figure 1

Remark 2.3. As noted above, the ideal $(0]$ of the lattice L_9 is not even. It can be verified that it is pentagonal. Hence the class **Pe** is not a subclass of the class **E**.

Remark 2.4. The ideal $(e]$ of the lattice L_7 (see Figure 2) is even and it is not pentagonal: Here $a \vee (b \wedge c) = e$ and $(a \vee b) \wedge (a \vee c) = 1$. Therefore, the class **E** is not a subclass of **Pe**.

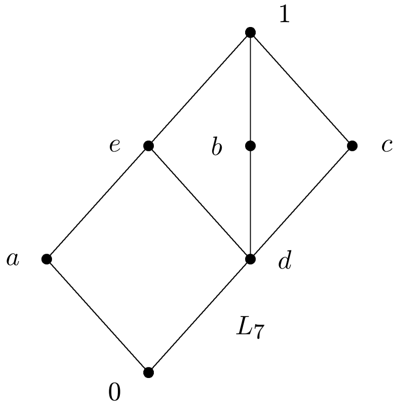

Figure 2

Theorem 2.5. *Let I be an ideal of a lattice L. Then*

(i) *if I is pentagonal, it is modular;*
(ii) *if I is even, it is modular.*

Proof: (i) Combine the definition of a pentagonal ideal with (2.1).

(ii) Suppose $(x \wedge y) \vee (x \wedge z) \in I$. Putting $a := x \wedge z$, $b := y$, $c := x$, we get $[a \wedge (b \vee \vee c)] \vee [b \wedge (a \vee c)] = (x \wedge y) \vee (x \wedge z) \in I$, and, by (2.3), $I \ni \overline{\mathrm{med}}(a, b, c) = = [(x \wedge z) \vee y] \wedge x$. Thus I is modular by (2.1$''$). □

Theorem 2.6. *Let L be a lattice. Then the following conditions are equivalent.*

(i) *The lattice L is modular.*
(ii) *Every ideal of L is modular.*
(iii) *Every ideal of L is even.*

Proof: (i) ⇒ (iii): Let $s := [a \wedge (b \vee c)] \vee [b \wedge (a \vee c)]$ be an element of an ideal I. By modularity,

$$I \ni s = \{[b \wedge (a \vee c)] \vee a\} \wedge (b \vee c) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c).$$

(iii) ⇒ (ii): This follows from Theorem 2.5.

(ii) ⇒ (i): Suppose $a \leq c$. Since $J := (a \vee (b \wedge c)]$ is modular for any $b \in L$, $(a \vee b) \wedge c \in J$ by (2.1). Therefore, $(a \vee b) \wedge c \leq a \vee (b \wedge c)$ and so $a \vee (b \wedge c) = (a \vee b) \wedge c$. □

An ideal I of a lattice L is called *rectangular*, if

$$(2.4) \qquad \forall\, a, b, c \in L \;\; (a \wedge c) \vee [b \wedge (a \vee c)] \in I \quad \Rightarrow \; \overline{\mathrm{med}}(a, b, c) \in I.$$

The class of all rectangular ideals will be denoted by **Re**.

Lemma 2.7. *Any rectangular ideal of a lattice is modular.*

Proof: Suppose $a \leq c$ and let $a \vee (b \wedge c) \in I$. Note that $a \leq c$ implies $a \wedge (b \vee c) = a$. Consequently, $I \ni a \vee (b \wedge c) = (b \wedge c) \vee [a \wedge (b \vee c)]$. Since I is rectangular, $I \ni \overline{\mathrm{med}}(a, b, c)$. However, $\overline{\mathrm{med}}(a, b, c) \geq (a \vee b) \wedge c$. Thus $(a \vee b) \wedge c \in I$ and I is modular by (2.1). □

Theorem 2.8. *Let I be a rectangular ideal of a lattice L. Then I is pentagonal and even.*

Proof: 1. Let $i \in I$ and $b \wedge c \in I$. Then $(b \wedge c) \vee [i \wedge (b \vee c)] \in I$ and, by the definition of a rectangular ideal, we have $(b \vee c) \wedge (b \vee i) \wedge (i \vee c) \in I$. Put $A := i \vee (b \wedge c)$, $B := b \vee c$ and $C := (b \vee i) \wedge (i \vee c)$ so that $A \in I$ and $B \wedge C \in I$. Clearly, $A \leq C$ and $A \vee (B \wedge C) \in I$. By Lemma 2.7, (2.1) and $A \vee B \geq C$ we can see that $I \ni (A \vee B) \wedge C = (b \vee i) \wedge (i \vee c)$. It follows that I is pentagonal.

2. Now suppose that $a \wedge (b \vee c)$ and $b \wedge (a \vee c)$ belong to I. A fortiori, $a \wedge c \in I$ and $b \wedge (a \vee c) \in I$. By the definition of a rectangular ideal we therefore have $\overline{\mathrm{med}}(a, b, c) \in I$ and we conclude that I is even. □

Remark 2.9. The ideal $(0]$ of the lattice L_6 shown in Figure 3 is even and pentagonal. It is not rectangular, since $(a \wedge c) \vee [b \wedge (a \vee c)] = 0$ and $\overline{\mathrm{med}}(a, b, c) = d$. It follows that the class **Re** is a proper subclass of the class **Pe** ∩ **E**.

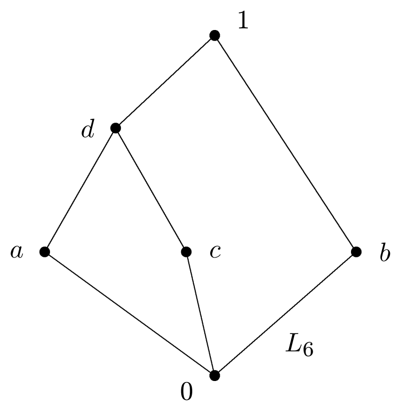

Figure 3

An ideal I of L is said to be *medial*, if the implication

$$\underline{\mathrm{med}}(a,b,c) \in I \quad \Rightarrow \quad \overline{\mathrm{med}}(a,b,c) \in I$$

is true for any $a, b, c \in L$. The class of medial ideals will be denoted by **Me**.

Theorem 2.10. *In any lattice,*

(i) *every medial ideal is rectangular;*
(ii) *every semiprime ideal is medial.*

PROOF: (i) Note that $\underline{\mathrm{med}}(a,b,c) \leq (a \wedge c) \vee [b \wedge (a \vee c)]$ holds for any $a, b, c \in L$.

(ii) Let $I \in \mathbf{Sp}$ and suppose that $\underline{\mathrm{med}}(a,b,c) \in I$. Now, $\underline{\mathrm{med}}(a,b,c) = \overline{\mathrm{med}}(a,b,c)$ in any distributive lattice. By [1, Lemma 2.1], we can see that $(\underline{\mathrm{med}}(a,b,c), \overline{\mathrm{med}}(a,b,c)) \in \hat{C}(L)$ where $\hat{C}(L)$ denotes the smallest congruence of L such that $L/\hat{C}(L)$ is distributive (see [2]).

We claim that $\overline{\mathrm{med}}(a,b,c) \in I$. Were this false, we would have an allele p/i such that $i \in I$ and $p \notin I$. Consequently, by [1, Main Theorem], the ideal I is not semiprime, and a contradiction ensues. □

Remark 2.11. The ideal (0] of the lattice M_5 given in Figure 4 is rectangular. It is not medial, since $\underline{\mathrm{med}}(a,b,c) = 0$ and $\overline{\mathrm{med}}(a,b,c) = 1$.

Note that the ideal (0] in the lattice L_7 represented in Figure 2 is medial. However, it is not semiprime, since $(a \wedge b) \vee (a \wedge c) = 0$ and $a \wedge (b \vee c) = a$.

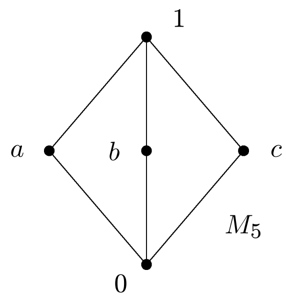

Figure 4

The theorems and the remarks mentioned above lead to a complete description of all inclusion relations between the studied classes of ideals, as indicated in Figure 5.

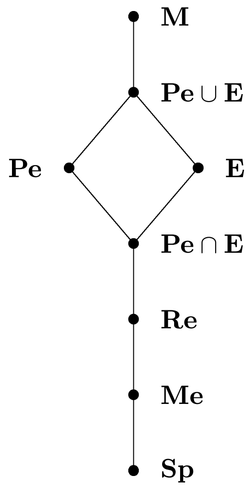

Figure 5

The following result can be viewed as alternative characterizations of distributivity.

Theorem 2.12. *For any lattice L, the following are equivalent:*

(i) *the lattice L is distributive;*
(ii) *any ideal of L is medial;*
(iii) *any ideal of L is rectangular;*
(iv) *any ideal of L is pentagonal.*

Proof: Using the preceding theorems, one can easily see that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ $\Rightarrow$ (iv).

Assume that L satisfies (iv). First note that there is no sublattice of L isomorphic to the lattice N_5 of Figure 6. Indeed, were this false, let $I = (a]$. Clearly $a \vee (b \wedge c) \in I$ and, by assumption, $I \ni (a \vee b) \wedge (a \vee c) = c$, a contradiction.

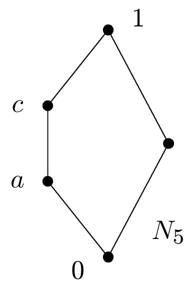

Figure 6

We are now reduced to proving that there is no sublattice M_5 (see Figure 4) in L. If this were false, then let $I = (a]$ and, similarly as above, we would have a contradiction. Thus L is distributive. □

References

[1] Beran L., *On semiprime ideals in lattices*, Journ. Pure Appl. Algebra **64** (1990), 223–227.

[2] Beran L., *Orthomodular Lattices (Algebraic Approach)*, Reidel, Dordrecht, 1985.

[3] Beran L., Salvati S., *Boolean algebras revisited*, Boll. Unione Matem. Italiana, ser. VII, **9** (sez. B) (1997), 895–901.

[4] Rav Y., *Semiprime ideals in general lattices*, Journ. Pure Appl. Algebra **56** (1989), 105–118.

[5] Salvati S., *A characterization of Boolean algebras*, Ricerche di Matematica **43** (1994), 357–363.

[6] Zassenhaus H., *The Theory of Groups,* 2nd ed., Vandenhoeck & Ruprecht, Göttingen, 1958.

Department of Algebra, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

(*Received* January 29, 1997)